Lect 21 Path integral for quantum mechanics

§1 propagator of a point particle.

Consider $H = \frac{p^2}{2m} + V(x)$, and thus the time evolution operator is $\mathcal{U}(t_b, t_a) = e^{-i(t_b - t_a)H}$. Define the propagator

$$i\, G(x_b t_b;\, x_a t_a) \equiv \langle x_b | \mathcal{U}(t_b, t_a) | x_a \rangle,$$ the it satisfies the following Schrödinger Eq as

$$i\partial_t G(xt;\, x_a t_a) = \langle x | i\partial_t\, e^{-i(t-t_a)H} | x_a \rangle = \langle x | H\, \mathcal{U}(t, t_a) | x_a \rangle.$$

H in the coordinate Rep is a function of x and ∂_x, thus

$$\langle x | H\, \mathcal{U}(t, t_a) | x_a \rangle = \int dx'\, \delta(x - x')\, H(x', \partial_{x'})\, \mathcal{U}(t, t_a)\, \delta(x' - x_a)$$

$$= H(x, \partial_x) \int dx'\, \delta(x - x')\, \mathcal{U}(t, t_a)\, \delta(x' - x_a) = H(x, \partial_x) \langle x | \mathcal{U}(t, t_a) | x_a \rangle$$

$$\Rightarrow \boxed{i\partial_t G(xt, x_a t_a) = H(x, \partial_x)\, G(xt, x_a t_a)}$$

Ex: for 1 D free space, with the initial condition $G(x t_a; x_a t_a) = -i\delta(x - x_a)$, we have $G(x_b t; x_a t_a) = (-i) \left(\frac{m}{2\pi i t}\right)^{1/2} \exp\left[\frac{i\, m (x_b - x_a)^2}{2t}\right]$.

Hint: Solve the differential Eq. $i\partial_t G(xt, x_a t_a) = \frac{-\hbar^2 \partial_x^2}{2m} G(xt, x_a t_a)$.

§2 Path integral representations of the propagator

$$\mathcal{U}(t_b, t_a) = \mathcal{U}(t_b\, t)\, \mathcal{U}(t, t_a) \Rightarrow i\, G(x_b t_b; x_a t_a) = \int dx\; i\, G(x_b t_b; x t)\, i\, G(x t; x_a t_a)$$

let us divide the time interval $[t_b, t_a]$ into N equal segments

$$iG(x_b t_b ; x_a t_a) = \int dx_1 \cdots dx_{N-1}\, iG(x_b t_b ; x_{N-1} t_{N-1}) \cdots \quad iG(x_1 t_1 ; x_a t_a)$$

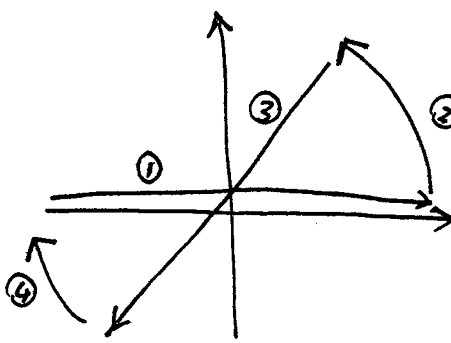
$$= A^N \int \prod_{i=1}^{N-1} dx_i \ \exp\left[i \sum \Delta t\, L\left(t_j, \frac{x_j + x_{j-1}}{2}, \frac{x_j - x_{j-1}}{\Delta t}\right)\right]$$

$$iG(x_i t_i, x_{i-1} t_{i-1}) = \langle x_i | \, e^{-i \frac{p^2}{2m} \Delta t} \, e^{-iV(x)\Delta t} \, | x_{i-1} \rangle$$

$$= \int dp_i \ \langle x_i | \, e^{-\frac{i p^2}{2m}\Delta t} | p_i \rangle \langle p_i | \, e^{-iV(x)\Delta t} \, | x_{i-1} \rangle$$

$$= \int_{-\infty}^{+\infty} \frac{dp_i}{2\pi} \ e^{-i\frac{p^2}{2m}\Delta t - iV(x)\Delta t} \ e^{+ip_i(x_i - x_{i-1})}$$

using Gauss integral $\displaystyle\int_{-\infty}^{+\infty} dx \, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$



then what's the vaule of $\displaystyle\int_{-\infty}^{+\infty} dx \, e^{-iax^2} = ?$

For the countour $\underbrace{\oint}_{e^{-az^2}dz} = \int_1 + \int_2 + \int_3 + \int_4 = 0$, the contribution from ② and ④ $\longrightarrow 0$

$$\int_{-\infty}^{+\infty} dx\, e^{-ax^2} = \int_{-\infty e^{i\pi/4}}^{+\infty e^{i\pi/4}} dz \ e^{-az^2} = \int_{-\infty}^{+\infty} dy \ e^{-ia\,y^2} \cdot e^{i\pi/4}$$

$$\Rightarrow \boxed{\int_{-\infty}^{+\infty} dy\, e^{-iay^2} = e^{-i\pi/4} \sqrt{\frac{\pi}{a}} = \sqrt{\frac{\pi}{ai}}}$$

$$\Rightarrow iG(x_i t_i, x_{i-1} t_{i-1}) = \left(\frac{m}{2\pi i \Delta t}\right)^{1/2} \exp\left[i \left(\frac{m}{2} \frac{(x_i - x_{i-1})^2}{(\Delta t)^2} - V\left(\frac{x_i + x_{i-1}}{2}\right)\right) \underline{\Delta t} \right]$$

$$\Rightarrow \quad A = \left(\frac{m}{2\pi i \Delta t}\right)^{1/2} \quad \text{and} \quad \mathcal{L}(t, x, \dot{x}) = \frac{m}{2}\dot{x}^2 - V(x)$$

i.e.

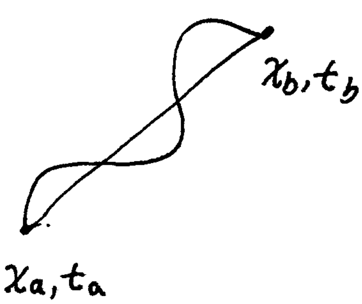
$$i\,G(x_b t_b, x_a t_a) = \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} \int dx_1 \cdots dx_{N-1}\; e^{\,i\int_{t_a}^{t_b} dt\left[\frac{m}{2}\dot{x}^2 - V(x)\right]}$$

§ Evalution of the path integral:

We can first find the saddle point solution, which corresponds to the solution of the classic path, and then evaluate the fluctuation parts. Let us consider the free space propagator as an example.

The classic path

$$X_c(t) = X_a + \frac{X_b - X_a}{t_b - t_a}(t - t_a)$$



The action of this part $\int_{t_a}^{t_b} \mathcal{L}\, dt$

$$= \frac{m}{2}\dot{x}^2 (t_b - t_a) = \frac{m}{2}\frac{(X_b - X_a)^2}{t_b - t_a}$$

The fluctuations $\delta X_j = X_j - X_c(t_j)$, $\quad \begin{cases} X_0 = X_a \\ X_N = X_b \end{cases} \quad \delta X_0 = \delta X_N = 0$

$$i\,G(x_b t_b; x_a t_a) = \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} \int dx_1 \cdots dx_{N-1} \prod_{i=1}^{N} \exp\left[i\frac{m}{2}\frac{(X_i - X_{i-1})^2}{\Delta t}\right]$$

$$\frac{(X_i - X_{i-1})^2}{\Delta t} = \frac{\left(X_c(t_i) - X_c(t_{i-1})\right)^2}{\Delta t} + \frac{\left(\delta X_i - \delta X_{i-1}\right)^2}{\Delta t} + 2\left(\delta X_i - \delta X_{i-1}\right)\frac{X_b - X_a}{t_b - t_a}$$

Add together, the linear term of δX_i vanishes,

$$\Rightarrow iG(x_b t_b ; x_a t_a) = \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} e^{iS_c} \int dx_1 \cdots dx_{N-1} \prod_{i=2}^{N-1} e^{i\frac{m}{2}\frac{(\delta x_i - \delta x_{i-1})^2}{\Delta t}} \underbrace{\cdot\, e^{i\frac{m}{2}\left[(\delta x_1)^2 + (\delta x_{N-1})^2\right]}}$$

$$\exp\left[i \sum_{j,k} \delta x_j \, M_{jk} \, \delta x_k \right]$$

with $M_{jk} = \frac{m}{2\Delta t}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & & \ddots & \\ & & -1 & 2 \end{bmatrix}$

$$\int dx_1 \cdots dx_{N-1} \exp\left[i \sum_{\substack{j,k \\ 1,\cdots N-1}} \delta x_j \, M_{jk} \, \delta x_k \right] = \frac{(\sqrt{\pi})^{N-1}}{\sqrt{\mathrm{Det}(-iM)}} \quad \text{(gaussian fluctation)}$$

tricks to calculate determinaint of $M_N = \begin{pmatrix} 2\cosh(u) & -1 & & \\ -1 & 2\cosh u & -1 & \\ & & \ddots & \\ & & -1 & 2\cosh u \end{pmatrix}$

$$\begin{cases} \mathrm{Det}\, M_N = 2\cosh u \, \mathrm{Det}\, M_{N-1} - \mathrm{Det}\, M_{N-2} \\ \mathrm{Det}\, M_1 = 2\cosh u, \quad \mathrm{Det}\, M_2 = 4\cosh^2 u - 1 \end{cases}$$

can be solved by the ansatz $\mathrm{Det}\, M_N = a e^{nu} + b e^{-nu}$

$$\Rightarrow \mathrm{Det}\, M_N = \frac{\sinh (N+1)u}{\sinh u} \longrightarrow N+1 \text{ as } u \to 0$$

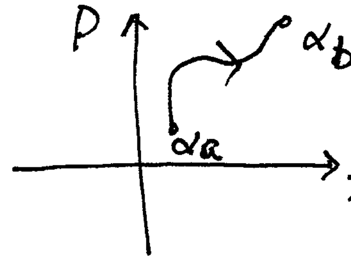
$$\Rightarrow \mathrm{Det}(-i M_{jk}) = \left(\frac{m}{2\Delta t i}\right)^{N-1} \times N$$

$$\Rightarrow iG(x_b t_b, x_a t_a) = \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} \left(\frac{m}{2\pi i \Delta t}\right)^{-\left(\frac{N-1}{2}\right)} N^{-1/2} e^{iS_{cl}}$$

$$\boxed{iG(x_b t_b, x_a t_a) = \left(\frac{m}{2\pi i (t_b - t_a)}\right)^{1/2} e^{iS_{cl}}}$$

§ Coherent state path integral

Consider a harmonic oscilator $H = \omega a^\dagger a$, and we define the coherent state $|\alpha\rangle$ satisfying $a|\alpha\rangle = \alpha|\alpha\rangle$. The propogator in the coherent state Rep is even simpler.



$$iG(\alpha_b t_b;\alpha_a t_a) = \langle\alpha_b|\,\mathcal{U}(t_b t_a)\,|\alpha_a\rangle$$

Resolution identity

$$\boxed{\int \frac{d\,\mathrm{Re}\,\alpha\; d\,\mathrm{Im}\,\alpha}{\pi}\,|\alpha\rangle\langle\alpha| = 1}$$

Plug in
$$\boxed{\begin{aligned}|\alpha\rangle &= e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \\ &= N_\alpha e^{\alpha a^\dagger}|0\rangle\end{aligned}}$$

Proof: $$LHS = \int\frac{d\,\mathrm{Re}\,\alpha\, d\,\mathrm{Im}\,\alpha}{\pi}\, e^{-|\alpha|^2}\sum_{nn'}\frac{\alpha^n\alpha^{*n'}}{\sqrt{n!n'!}}\,|n\rangle\langle n'|$$

$$= \int\frac{d\theta}{\pi}\,|\alpha|\,d|\alpha|\; e^{-|\alpha|^2}\sum_n\frac{|\alpha|^{2n}}{n!}\,|n\rangle\langle n|$$

For those $n \neq n'$, they vanish after $\int d\theta$

$$= \sum_n \underbrace{\int_0^\infty d|\alpha|^2\,\frac{e^{-|\alpha|^2}\,(|\alpha^2|)^n}{n!}}_{\Gamma\text{-function}}\,|n\rangle\langle n| = \sum_n |n\rangle\langle n| = 1$$

Inner product

$$\langle\alpha|\alpha'\rangle = e^{-\frac{|\alpha|^2}{2}}\,e^{-\frac{|\alpha'|^2}{2}}\,\langle 0|\,e^{\alpha^*\hat{a}}\,e^{\alpha'\hat{a}^\dagger}\,|0\rangle$$

$$e^{\alpha^*\hat{a}}\,e^{\alpha'\hat{a}^\dagger} = e^{\alpha'\hat{a}^\dagger}\,e^{\alpha^*\hat{a}}\;e^{\alpha^*\alpha'[\hat{a},\hat{a}^\dagger]}$$

$$\Rightarrow \boxed{\langle\alpha|\alpha'\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\alpha'|^2)+\alpha^*\alpha'}}$$

$$iG(\alpha_b t_b;\alpha_a t_a) = \int \frac{d^2a_1 \cdots d^2a_{N-1}}{\pi^{N-1}} \; iG(\alpha_b t_b, \alpha_{N-1} t_{N-1}) \cdots iG(\alpha_1 t_1;\alpha_a t_a)$$

$$iG(a_i t_i, a_{i-1} t_{i-1}) = \langle \alpha_i | \, e^{-i\Delta t\, \omega \hat{a}^\dagger a} \, | \alpha_{i-1} \rangle = e^{-i\Delta t\, \alpha_i^* \alpha_{i-1} \omega} \langle \alpha_i | \alpha_{i-1} \rangle$$

$$= e^{-i\Delta t\, \omega\, \alpha_i^* \alpha_{i-1} - \frac{1}{2}|\alpha_i|^2 - \frac{1}{2}|\alpha_{i-1}|^2 + \alpha_i^* \alpha_{i-1}}$$

$$= e^{-i\Delta t\, \omega\, \alpha_i^* \alpha_{i-1} - \frac{1}{2}\alpha_i^*(\alpha_i - \alpha_{i-1}) + \frac{1}{2}\alpha_{i-1}(-\alpha_{i-1}^* + \alpha_i^*)}$$

$$= e^{i\Delta t\, [-\omega\, \alpha_i^* \alpha_i + \frac{i}{2}(\alpha_i^* \dot{\alpha}_i - \alpha_i \dot{\alpha}_i^*)]}$$

$$\Rightarrow iG(\alpha_b t_b;\alpha_a t_a) = \int \frac{\prod_{i=1}^{N-1} d^2\alpha_i}{\pi^{N-1}} \; e^{i\int_{t_a}^{t_b} dt \left[\frac{i}{2}(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) - \omega\alpha^*\alpha\right]}$$

$$\longrightarrow \int D[\alpha(t)] \; e^{i\int_{t_a}^{t_b} dt\, \mathcal{L}}$$

where $\mathcal{L} = \frac{i}{2}(\alpha^*\dot{\alpha} - \alpha\dot{\alpha}^*) - \omega\alpha^*\alpha = p\dot{x} - H.$

§ Path integral Rep of partition function

$$Z(\beta) = \operatorname{tr} e^{-\beta H} = \int dx \; \mathcal{G}(x,x,\beta) \quad \text{where } \beta = \frac{1}{k_B T}$$

$$\mathcal{G}(x_b\, x_a;\beta) \equiv \langle x_b | \, e^{-\beta H} \, | x_a \rangle$$

$$= \left(\frac{m}{2\pi\Delta\tau}\right)^{N/2} \int D x(\tau) \; e^{-\int_0^\beta d\tau \, \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)}$$

$$\Rightarrow Z(\beta) = \int D x(\tau) \; e^{-\oint d\tau \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right]} \quad \text{for closed path } x(0) = x(\beta).$$

$$\mathcal{G}(x_b, x_a, \tau)\Big|_{\tau = it} = i\, G(x_b, x_a; t)$$

imaginary time path integral

Lect 2: Path integral — instantons (warm up).

§1. Consider a single particle potential $H = \frac{p^2}{2m} + V(x)$.

We use the imaginary time path integral

$$\boxed{\begin{aligned}\langle x_f | e^{-HT/\hbar} | x_i \rangle &= A \int [Dx(t)]\, e^{-\frac{S}{\hbar}} \\ &= A \int Dx(t) \exp\left\{ - \int_{-T/2}^{T/2} \frac{d\tau}{\hbar} \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) \right] \right\}.\end{aligned}}$$

A_N: Const from integral measure.

Search for classic paths with the following boundary condition

$$x\left(\frac{T}{2}\right) = x_f \quad \text{and} \quad x\left(-\frac{T}{2}\right) = x_i$$

$\Rightarrow$ classic path $\boxed{\bar{x}(\tau): \quad \frac{\delta S_c}{\delta \bar{x}} = -m \frac{d^2 \bar{x}}{d\tau^2} + \frac{d}{dx} V(x)\Big|_{x = \bar{x}(t)} = 0}$

Consider small fluctuations. $x(\tau) = \bar{x}(\tau) + \delta\bar{x}(\tau)$

with $\delta \bar{x}(\tau) = 0$ at $\tau = -T/2$, and $T/2$

$$S = \int \left[\frac{m}{2} \left(\frac{dx}{d\tau}\right)^2 + V(x) \right] d\tau = \int \left(\frac{m}{2} \frac{d\bar{x}}{d\tau} + V(\bar{x}) \right) d\tau + \int d\tau \frac{m}{2} \left(\frac{d\delta\bar{x}}{d\tau} \right)^2 + \frac{1}{2} V''(\bar{x}(\tau)) (\delta\bar{x})^2$$

where $V'' = \frac{d^2 V}{dx^2}$

$$= S_c(\bar{x}(t)) + \int_{-T/2}^{T/2} d\tau \frac{1}{2}\, \delta x(\tau) \left[-m \frac{d^2}{d\tau^2} + V''(\bar{x}(\tau)) \right] \delta x(\tau)$$

$$\Rightarrow \langle x_f | e^{-HT/\hbar} | x_i \rangle \simeq A\, e^{-\frac{S(\bar{x}(\tau))}{\hbar}} \cdot \left\{ \det\left[-m\frac{d^2}{d\tau^2} + V''(\bar{x}(t)) \right] \right\}^{-1/2}$$

Suppose we can solve the eigen spectra and orth-normal eigenfunction

$$\left\{ -m\frac{d^2}{d\tau^2} + V''(\bar{x}(\tau)) \right\} x_n(\tau) = \lambda_n x_n(\tau) \text{ with } x_n(-\tfrac{T}{2}) = x_n(\tfrac{T}{2}) = 0$$

then we express $\delta x(\tau) = \sum_n C_n\, x_n(\tau)$

$$\Rightarrow \int_{-T/2}^{T/2} d\tau\, \frac{1}{2}\, \delta x(\tau) \left[-m\frac{d^2}{d\tau^2} + V'' \right] \delta x_n(\tau) = \frac{1}{2} \sum_n \lambda_n C_n^2$$

$$\rightarrow \int Dx(t)\, e^{-\delta S/\hbar} \simeq \prod_n \int_{-\infty}^{+\infty} dC_n\, e^{-\frac{1}{2}\lambda_n C_n^2} \simeq \boxed{\prod_n \lambda_n^{-1/2} = \left\{ \det\left[-m\frac{d^2}{d\tau^2} + V'' \right] \right\}^{-1/2}}$$

where we drop all the ~~anstants~~ constants.

Example: consider a harmonic potential $V(x) = \frac{1}{2} m\omega^2 x^2$, and $x_i = x_f = 0$. Calculate $\boxed{\langle 0 | e^{-HT/\hbar} | 0 \rangle = \left[\frac{m\omega}{2\pi\hbar \sinh \omega T} \right]^{1/2}}$.

It's obvious that $\bar{x}(\tau) = 0$. For fluctuations

$$\left\{ -m\frac{d^2}{d\tau^2} + m\omega^2 \right\} x_n(\tau) = \lambda_n x_n(\tau)$$

$$\Rightarrow x_n(\tau) = \begin{cases} \cos\left(\frac{n\pi}{T}\tau\right)\frac{1}{\sqrt{2T}}, & n = 1, 3, \cdots \\ \sin\left(\frac{n\pi}{T}\tau\right)\frac{1}{\sqrt{2T}}, & n = 2, 4, \cdots \end{cases} \qquad \Rightarrow \lambda_n = m\left[\omega^2 + \left(\frac{n\pi}{T}\right)^2 \right]$$

$$\Rightarrow \langle 0| e^{-HT/\hbar} |0\rangle = A \prod_{n=1}^{\infty} \left[m \left(\omega^2 + \left(\frac{n\pi}{T}\right)^2\right)\right]^{-1/2} = A' \prod_{n=1}^{\infty} \left[1 + \left(\frac{\omega T}{n\pi}\right)^2\right]^{-1/2}$$

we do not know A' but it can be calibrated through the free space result:

$$\underbrace{\langle 0| e^{-iHt/\hbar} |0\rangle}_{\text{Set } \omega \to 0} = \left(\frac{m}{2\pi i t\hbar}\right)^{1/2} \longrightarrow \left(\frac{m}{2\pi T\hbar}\right)^{1/2} = A'_N$$

$$\Rightarrow \langle 0| e^{-HT/\hbar} |0\rangle = \left(\frac{m}{2\pi T\hbar}\right)^{1/2} \left[\prod_{n=1}^{\infty} \left(1 + \left(\frac{\omega T}{n\pi}\right)^2\right)\right]^{-1/2} = \left(\frac{m}{2\pi T\hbar}\right)^{1/2} \left(\frac{\sinh \omega T}{\omega T}\right)^{-1/2}$$

$$= \left[\frac{m\omega}{2\pi\hbar \sinh \omega T}\right]^{1/2}.$$

$$\boxed{\text{Lemma to calculate } \det\left[-m\frac{d^2}{d\tau^2} + V''\right]:}$$

Suppose $\left[-m\frac{d^2}{d\tau^2} + W(t)\right]\psi = \lambda\psi$, where ψ satisfies the boundary condition

$$\begin{cases} \psi(-T/2) = 0 \\ \frac{d}{d\tau}\psi\Big|_{\tau=-T/2} = 1 \end{cases}.$$

Let $W_1(t)$ and $W_2(t)$ be two functions of t, and $\psi_{1,2}(t)$ be the corresponding solutions for the above boundary condition; then we have

$$\boxed{\frac{\det[-m\partial_\tau^2 + W_1(t) - \lambda]}{\det[-m\partial_\tau^2 + W_2(t) - \lambda]} = \frac{\psi_{1,\lambda}(\frac{T}{2})}{\psi_{2,\lambda}(\frac{T}{2})}}.$$

Proof: ~~treat~~ λ as a continous variable. Both sides are semi-analytical functions of λ. As $\lambda \to \lambda_{1n}$, $\psi_{1,\lambda_{1n}}(\frac{T}{2}) \to 0$, and $\lambda \to \lambda_{2,n}$, $\psi_{2,\lambda_{2,n}}(\frac{T}{2}) \to 0$ as assumed.

In these cases, $\det[-m\partial_\tau^2 + W_{1,2}(t) - \lambda] \longrightarrow 0$ for $\lambda \to \lambda_{1n}$ and λ_{2n}, respectively. Then both sides have the same pattern of zeros and poles, thus they must be equal up to a constant.

Then as $|\lambda| \to \infty$, both sides $\to 1$, $\Rightarrow$ LHS = RHS.

Exercise: use this method, calculate $\langle 0| e^{-HT/\hbar} |0\rangle$ for $V = \frac{1}{2} m\omega^2 x^2$ again.

$$\langle 0| e^{-HT/\hbar} |0\rangle = A \left[\det\left[-m\frac{d^2}{d\tau^2} + m\omega^2\right]\right]^{-1/2}.$$

$$\left[\frac{\det \frac{m}{\hbar}(-\frac{d^2}{d\tau^2} + \omega^2)}{\det \frac{m}{\hbar}(-\frac{d^2}{d\tau^2})}\right] = \frac{\psi_{\omega,\lambda=0}(T/2)}{\psi_{\omega=0,\lambda=0}(T/2)}.$$

For $\omega = 0$, $\lambda = 0 \Rightarrow \psi_{\omega=0,\lambda=0}(\tau) = \tau + T/2$

ω, $\lambda = 0 \Rightarrow (\frac{d^2}{d\tau^2} - \omega^2)\psi(\tau) = 0 \Rightarrow \psi_{\omega,\lambda=0}(\tau) = \frac{\sinh(\omega(\tau + T/2))}{\omega}$

$$\Rightarrow \frac{\det m(-\frac{d^2}{d\tau^2} + \omega^2)}{\det m(-\frac{d^2}{d\tau^2})} = \left.\frac{\sinh \omega(\tau + T/2)}{\omega(\tau + T/2)}\right|_{\tau = T/2} = \frac{\sinh \omega T}{\omega T}$$

For $\det\left[-\frac{m}{\hbar}\frac{d^2}{d\tau^2}\right]$:
$$\langle 0|e^{-HT/\hbar}|0\rangle_{\omega=0} = A\det\left[-\frac{m}{\hbar}\frac{d^2}{d\tau^2}\right]$$
$$= \int_{-\infty}^{+\infty}\frac{dp}{2\pi\hbar}\cdot\langle 0|p\rangle\langle p|0\rangle\, e^{-p^2T/2m\hbar} = \frac{1}{2\pi\hbar}\left(\frac{\pi}{T/2m\hbar}\right)^{1/2}.$$
$$= \left(\frac{m}{2\pi\hbar T}\right)^{1/2}$$

$$\Rightarrow \langle 0|e^{-HT/\hbar}|0\rangle = A\det\left[-\frac{m}{\hbar}\frac{d^2}{d\tau^2}\right]\frac{\det\left[\cdot\frac{m}{\hbar}\left(-\frac{d^2}{d\tau^2}+\omega^2\right)\right]}{\det\left[\frac{m}{\hbar}\left(-\frac{d^2}{d\tau^2}\right)\right]}$$
$$= \left(\frac{m}{2\pi\hbar T}\right)^{1/2}\left(\frac{\sinh\omega T}{\omega T}\right)^{-1/2} = \left[\frac{m\omega}{2\pi\hbar\sinh\omega T}\right]^{1/2}$$
$$\xrightarrow[T\to\infty]{} \frac{m\omega}{\pi\hbar}\, e^{-\omega T/2}.$$

Compare with $\langle 0|e^{-HT/\hbar}|0\rangle \longrightarrow |\langle 0|G\rangle|^2 e^{-E_0T/\hbar} = \left\{\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\right\}^2 e^{-\omega T/2}.$

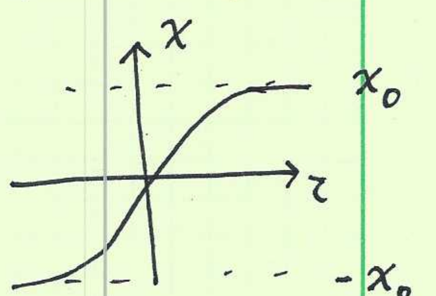
Instantons: double well problem

§ Consider the potential $V(x) = \frac{mg}{4}(x^2 - x_0^2)^2$

at $x = \pm x_0$, the local frequency $\omega_0 \simeq g x_0^2$.

Now we calculate $\langle x_0 | e^{-HT/\hbar} | -x_0 \rangle = A \int Dx(\tau)\, e^{-\int_{-T/2}^{T/2} d\tau \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right]}$

① figure out the classic path → motion in the potential of $-V(x)$



$$\frac{dx}{d\tau} = \sqrt{\frac{2V(x)}{m}} \quad \Rightarrow \int d\tau = \int \frac{dx}{\sqrt{2V(x)/m}}$$

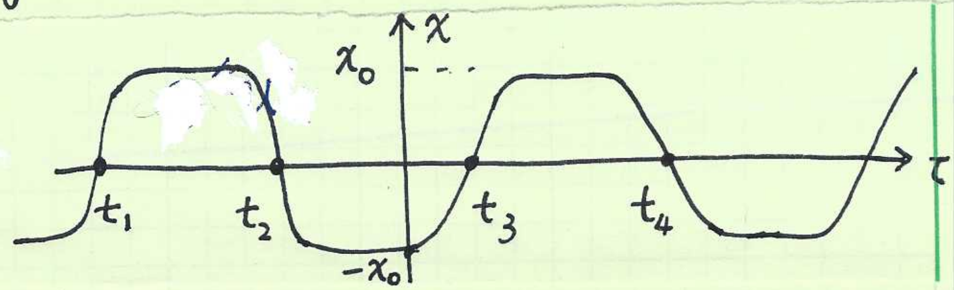
$$\Rightarrow \tau = -T/2 + \int_{-x_0}^{x} dx' \left(\frac{2V(x')}{m}\right)^{-1/2}$$

$$\left\{ \; x(-T/2) = -x_0 \right.$$

classic action $S_0 = \int_{-T/2}^{T/2} \frac{d\tau}{\hbar}\, \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x) = \int_{-T/2}^{T/2} \frac{d\tau}{\hbar}\, m \left(\frac{dx}{d\tau}\right)^2$

$$= \int_{-x_0}^{x_0} dx \left(\frac{dx}{d\tau}\right) \frac{m}{\hbar} = \frac{m}{\hbar} \int_{-x_0}^{x_0} dx \sqrt{\frac{2mV(x)}{\hbar}}$$

At large $T \to +\infty$, we can have other classic paths:



a configuration with n-instanton / anti-instanton

The leading order contribution $\quad S = nS_0 \leftarrow$ contribution around $t \simeq t_1, t_2, \ldots t_4$.

turning point contribution

At other time, i.e $t \neq t_1 t_2, \cdots$, the particle is mainly at $\pm x_0$ where $V''(x) = m\omega_0^2$. This gives the contribution $\left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\omega T/2}$

Now let's integrate out all the possible location of centers.

$$\int_{-T/2}^{T/2} d\tau_1 \int_{-T/2}^{\tau_1} d\tau_2 \cdots \int_{-T/2}^{\tau_{n-1}} d\tau_n = \frac{T^n}{n!} \quad \Rightarrow$$

$$\langle x_0 | e^{-HT/\hbar} | -x_0 \rangle = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{n \in odd} \frac{\left(K e^{-S_0/\hbar} T\right)^n}{n!}$$

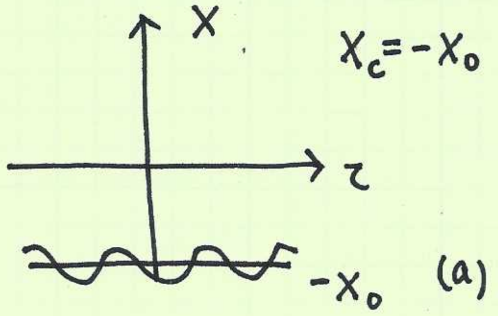
Similarly $\langle -x_0 | e^{-HT/\hbar} | -x_0 \rangle = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \sum_{n \in even} \frac{\left(K e^{-S_0/\hbar} T\right)^n}{n!}$

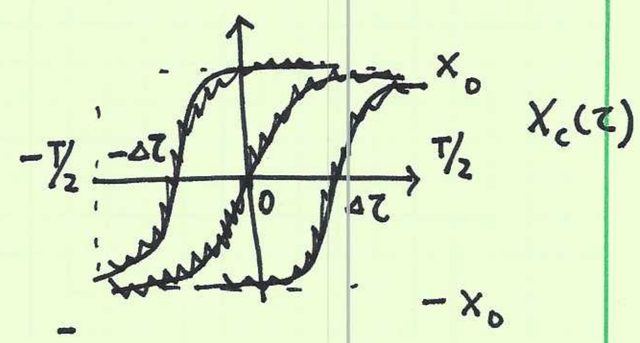
$$\Rightarrow \langle \pm x_0 | e^{-HT/\hbar} | -x_0 \rangle = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/2} e^{-\omega T/2} \begin{cases} \cosh\left(KT e^{-S_0/\hbar}\right) \\ \sinh\left(KT e^{-S_0/\hbar}\right) \end{cases}.$$

Extract the time dependent part $\Rightarrow E_{1,2} = \frac{\hbar\omega_0}{2} \mp \hbar K e^{-S_0/\hbar}$.

§ Evaluation of K (K carry unit of frequency)

Compare the sector of zero instanton, and one instanton





instanton can occur at any time between $-T/2$ and $T/2$

define $$K T e^{-S_0/\hbar} = \frac{\int_{x_c} Dx \; e^{-S/\hbar}}{\int_{x \simeq -x_0} Dx \; e^{-S/\hbar}}$$ ← around (b), ← config around (a)

then LHS and RHS both dimension less.

¶ For RHS, we only need to choose an instanton configuration located at $\tau = 0$. The other instantons are not independent, but can be viewed as zero mode fluctuations, i.e if

$X_c(\tau)$ is a classic solution, so does $X_c(\tau + \Delta\tau)$

$$\Rightarrow \quad X_c(\tau + \Delta\tau) = X_c(\tau) + \Delta\tau \frac{d}{d\tau} X_c(\tau)$$

Since $x_c(\tau+\Delta\tau)$ and $x_c(\tau)$ give the same minimal action $\Rightarrow$

$\frac{d}{d\tau}x_c(\tau)$ is a zero mode. We can prove explicitly

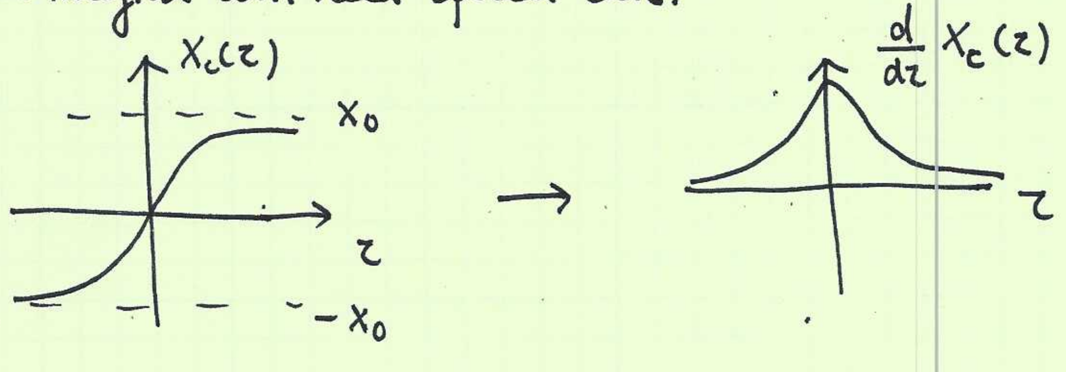
the classic solution satisfies $\quad -m\frac{d^2x_c(\tau)}{d\tau^2}+V'(x_c(\tau))=0$

$$\Rightarrow \boxed{-m\frac{d^2}{d\tau^2}\left(\frac{d}{d\tau}x_c(\tau)\right)+V''(x_c(\tau))\left(\frac{d}{d\tau}x_c(\tau)\right)=0}$$

Now near $x=x_c(\tau)$, $\quad S=S_0+\int_{-T/2}^{T/2}d\tau\,\delta x\left[-\frac{m}{2}\frac{d^2}{d\tau^2}+\frac{V''}{2}(x_c(\tau))\right]\delta x$

$$\Rightarrow \boxed{\int_{x_c}Dx\,e^{-S/\hbar}=e^{-S_0/\hbar}\int_{x_c}D\delta x\;e^{-\frac{1}{\hbar}\int d\tau\,\delta x\left(\underline{-\frac{m}{2}\frac{d^2}{d\tau^2}+\frac{V''}{2}(x_c(\tau))}\right)\delta x}}$$

Since $\frac{d}{d\tau}x_c(\tau)$ is a zero mode of $-\frac{m}{2}\frac{d^2}{d\tau^2}+V''(x_c(\tau))$, the path integral will need special care.



Let us discretize the functional integral to have a better understanding.

$$\int_{x_c}D\delta x\,e^{-\frac{1}{\hbar}\int d\tau\,\delta x\left(-\frac{m}{2}\frac{d^2}{d\tau^2}+V''\right)\delta x}=A\int\prod_{i=1}^{N}d\,\delta x_i\,\exp\left[-\frac{\Delta\tau}{2\hbar}\sum_{i,j=1}^{N}\delta x_i\left(-m(\Delta\tau)^{-2}\Delta+V''\right)\delta x_j\right.$$

$$= A\left(\sqrt{\frac{2\pi}{\Delta\tau}}\right)^N \Big/ \left\{\mathrm{Det}\left[-m(\Delta\tau)^{-2}\Delta + V''\right]\right\}^{1/2}$$ (Δ is the discretized laplacian)

Now let's expand δX_i in terms of orth normal basis of eigenstates of $-m\frac{d^2}{d\tau^2} + V''$

$$\delta X_i = c_1 \varphi_1(\tau_i) + \sum_{n=2}^{N} c_n \varphi_n(\tau_i)$$ and $\varphi_n(\tau_i)$ is normalized

as $\sum_i \Delta\tau \varphi_j(\tau_i)\varphi_{j'}(\tau_i) = \delta_{ij}$ $\Rightarrow$ $\det\left[(\Delta\tau)^{1/2}\varphi_j(\tau_i)\right] = 1$

orth-normal

or $\det[\varphi_j(\tau_i)] = (\Delta\tau)^{-\frac{N}{2}}$

The Jaccbian change from $\prod_{i=1}^{N} dx_i \rightarrow \prod_{i=1}^{N} dC_n$

$$\prod_{i=1}^{N} dX_i = \prod_{i=1}^{N} dC_i \left|\frac{\partial X_i}{\partial C_j}\right| = \prod_{i=1}^{N} dC_i \det[\varphi_j(\tau_i)]$$

$$= \prod_{i=1}^{N}\left[dC_i (\Delta\tau)^{-1/2}\right]$$

$$\Rightarrow \int_{x_c} D\delta x \, e^{-S} = \underbrace{A}_{e^{-S_0/\hbar}} \int \prod_{i=1}^{N}\left[dC_i(\Delta\tau)^{-1/2}\right] \exp\left[-\sum_{j=1}^{N}\frac{\Delta\tau}{2}\lambda_j C_j^2\right]$$

$$= \underbrace{A}_{e^{-S_0/\hbar}} \int dc_1 (\Delta\tau)^{-1/2} \int \prod_{i=2}^{N} dc_i (\Delta\tau)^{-1/2} \exp\left[-\sum_{j=2}^{N}\frac{\Delta\tau}{2}\lambda_j C_j^2\right]$$

$$= \underbrace{A}_{e^{-S_0/\hbar}} \int dc_1 (\Delta\tau)^{-1/2} \left(\frac{2\pi}{\Delta\tau}\right)^{\frac{N-1}{2}} \mathrm{Det}'\left[-m\frac{d^2}{d\tau^2} + V''(x_c)\right]$$ zero mode excluded

$$\Rightarrow K\,T\,e^{-S_0/\hbar} = \frac{A e^{-S_0/\hbar} (\Delta\tau)^{\frac{-N}{2}} \left(\frac{2\pi}{\hbar}\right)^{\frac{N-1}{2}} \left[\mathrm{Det}'\left[-m\frac{d^2}{d\tau^2}+V''(x_c)\right]\right]^{-1/2} \int dC_1}{A(\Delta\tau)^{-N/2}(2\pi\hbar)^{\frac{N}{2}}\left\{\mathrm{Det}\left[-m\frac{d^2}{d\tau^2}+V''(-x_0)\right]\right\}^{-1/2}}$$

$$KT = \int \frac{dC_1}{\sqrt{2\pi\hbar}} \left\{\frac{\mathrm{Det}'\left[-m\frac{d^2}{d\tau^2}+V''(x_c)\right]}{\mathrm{Det}\left[-m\frac{d^2}{d\tau^2}+V''(-x_0)\right]}\right\}^{-1/2}$$

now we need to figure out the relation $\int dC_1$ and $\int_{-T/2}^{T/2} d\tau = T$.

Let's normalize $\varphi_1(\tau) \propto \dot{x}_c(\tau) \leftarrow$ (zero mode)

$$S_0 = \int d\tau\ \frac{1}{2}m\dot{x}_c^2 + V(x_c) = \int d\tau\ m\dot{x}_c^2 = m\Delta\tau \sum_{i=1}^{N}\left(\frac{\Delta x_c}{\Delta\tau}\right)^2$$

$$\Rightarrow \Delta\tau \sum_{i=1}^{N}\left(\frac{\dot{x}_c(\tau_i)}{\sqrt{S_0/m}}\right)^2 = 1 \quad \Rightarrow \varphi_1(\tau) = \frac{\dot{x}_c(\tau)}{\sqrt{S_0/m}}$$

For $\delta x(\tau) = \dfrac{C_1}{\sqrt{S_0/m}}\dot{x}_c(\tau)$, $\rightarrow$ correspond a shift

$$x_c(\tau) \rightarrow x_c(\tau+\Delta\tau) \text{ with}$$

$$\Delta\tau = \frac{C_1}{\sqrt{S_0/m}}$$

Since the interval for $\Delta\tau \in [-T/2, T/2] \Rightarrow C_1$'s interval

$$-\sqrt{S_0/m}\,\frac{T}{2} \le C_1 \le \sqrt{S_0/m}\,\frac{T}{2} \quad \Rightarrow \int dC_1 = \sqrt{\frac{S_0}{m}}\int d\tau = \sqrt{\frac{S_0}{m}}\,T$$

$$\Rightarrow \quad K T = \sqrt{\frac{S_0}{2\pi m\hbar}}\, T \lim_{N\to\infty} \left\{ \frac{\mathrm{Det}'\left[-m\frac{d^2}{d\tau^2} + V''(x_c)\right]}{\mathrm{Det}\left[-m\frac{d^2}{d\tau^2} + V''(-x_0)\right]} \right\}^{-1/2}$$

$$\boxed{K = \sqrt{\frac{S_0}{2\pi\hbar}} \lim_{N\to\infty} \left\{ \frac{\mathrm{Det}'\left[-\frac{d^2}{d\tau^2} + \frac{1}{m}V''(x_c)\right]}{\mathrm{Det}\left[-\frac{d^2}{d\tau^2} + \frac{1}{m}V''(-x_0)\right]} \right\}^{-1/2}}$$

please note the absorbtion of "m".

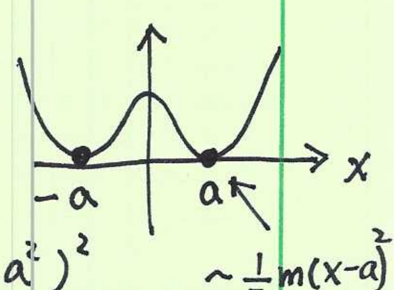
Check unit: eigenvalues of Det has unit T^{-2}, Det' has one less eigenvalues ~~that~~ then Det, $\Rightarrow [K] = (T^{-2})^{-1\times(-1/2)} = T^{-1}$

K carries frequency's unit, correct!

We have shown that eigenvalues of $-\frac{d^2}{d\tau^2} + \frac{1}{m}V''(-x_0)$ are all positive definite. How about $-\frac{d^2}{d\tau^2} + \frac{1}{m}V''(x_c)$?

Since its zero mode $\varphi_1(\tau)$ is nodeless, it must be the mode with lowest eigenvalue, so all other modes' eigenvalue must be positive

$$\Rightarrow \boxed{K \text{ is real.}}$$



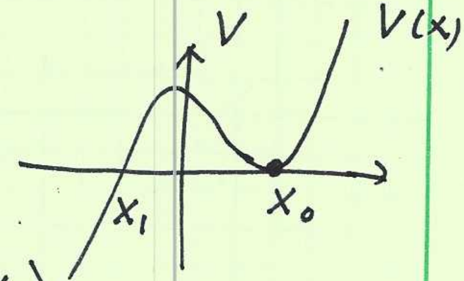
Consider a concrete potential $V(x) = \frac{m\omega^2(x^2-a^2)^2}{8a^2}$.

it can be calculated in J. Zinn-Justin (QFT and critical phenomena 1993)

$$\left\{ \begin{array}{l} \dfrac{\det'\left[-\frac{d^2}{d\tau^2}+V''(x_{cl})\right]}{\det\left[-\frac{d^2}{d\tau^2}+\omega^2\right]} = \dfrac{1}{12}\,\omega^{-2} \\[2ex] S_0 = \dfrac{2}{3}m\omega a^2 \end{array} \right.$$

$$\Rightarrow \boxed{K = \sqrt{\frac{2m\omega a^2}{3\pi\hbar\cdot 2}}\,\sqrt{12}\,\omega = 2\omega\sqrt{\frac{m\omega a^2}{\hbar}}}$$

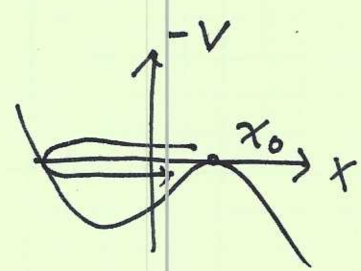
§ Decay of a false vaccum



Consider the potential $V(x)$, at put particle at x_0. We calculate $\langle x_0|e^{-HT}|x_0\rangle$

$$\langle x_0|e^{-HT}|x_0\rangle = A\int D[x(\tau)]\,\exp\left[-\int_{-T/2}^{T/2}\frac{d\tau}{\hbar}\left(\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2+V(x)\right)\right]$$

The classic path $m\dfrac{d^2x(\tau)}{d\tau^2} = V'(x)$



Again we have

$$\langle x_0|e^{-HT}|x_0\rangle = e^{-T\omega/2}\sum_n\int_{-T/2}^{T/2}d\tau_1\int_{\tau_1}^{T/2}d\tau_2\cdots\int_{\tau_{n-1}}^{T/2}d\tau_n\,(Ke^{-S_0})^n$$

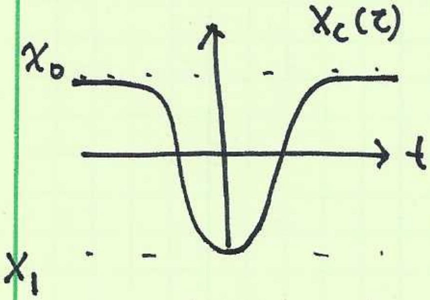
$$= \exp\left[KTe^{-S_0}\right]e^{-T\omega_0/2}$$

However, the "K" here is not real, but imaginary. If we go back to real time, we have $T\to iT$,

$$\langle x_0|e^{-iHT}|x_0\rangle = e^{-iT\frac{\omega_0}{2}\underbrace{-T|K|e^{-S_0}}}$$

the decay probability.

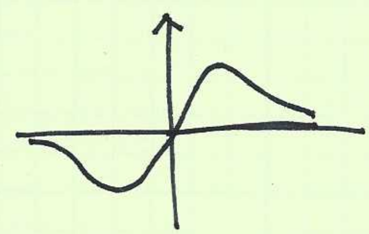
The difference from the double well solution: we don't have an instanton, but a "bounce" solution for e^{-S_0}



the particle cannot stop at x_1 and must come back.

The zero mode of $-\frac{d^2}{d\tau^2} + V''(x_c)$

behaves like:



$\varphi_1(\tau) = \frac{d}{d\tau} x_c(\tau)$

Since $\varphi_1(\tau)$ has one node, there must be an eigenmode with lower energy (i.e. negative). As a result, $\left[\mathrm{Det}' \left[-\frac{d^2}{d\tau^2} + \frac{1}{m} V''(x_c) \right] \right]^{-1/2}$

becomes imaginary. $\rightarrow$ $K = |K| i$

path integral fir dissipative system

Consider a QM system $H_p = \frac{p^2}{2m} + V(q)$. which furthur couples to an environment. The environment is described by a bounch of harmonic oscilattors $H_B[q_\alpha] = \sum_{\alpha=1}^{N} \left(\frac{\hat{p}_\alpha^2}{2m_\alpha} + \frac{m_\alpha}{2} \omega_\alpha^2 q_\alpha^2 \right)$, and the coupling between them is linear to q_α. as $H_c = - \sum_{\alpha=1}^{N} f_\alpha(q) q_\alpha$.

Then consider the survival probability of a particle confined to a metastable minimum at $q=a$

$$\langle a | e^{-i\hat{H}t} | a \rangle = \int_{q(0)=q(t)=a} Dq \; e^{i S_p[q]} \int Dq_\alpha \, e^{i S_B[q_\alpha] + i S_c[q, q_\alpha]}$$

where $S_p[q] = \int_0^t dt \left(\frac{m}{2} \dot{q}^2 - V(q) \right)$, $S_B = \int_0^t dt \sum_\alpha \frac{m_\alpha}{2} \left(\dot{q}_\alpha^2 - \omega_\alpha^2 q_\alpha^2 \right)$

$$S_c[q] = - \int_0^t dt \sum_\alpha f_\alpha(q) q_\alpha$$

Now we change to Eucledian Rep, and integrate out environment degrees of freedom

$$\int Dq_\alpha \; e^{- \int_0^\beta d\tau \left[\left(\frac{m}{2} \frac{dq_\alpha}{d\tau} \right)^2 + \omega_\alpha^2 q_\alpha^2 + f_\alpha(q) q_\alpha \right]}$$

Peform Fourier transform $q(\tau) = \sum_{i\omega_n} q(i\omega_n) e^{-i\omega_n \tau}$

$$f_\alpha(q) = \sum_{i\omega_n} f_{\alpha, \omega_n}(q) \, e^{-i\omega_n \tau}$$

②

Then, the above Eq $= \int Dq_\alpha \, e^{-\beta \sum_{\omega_n} \left[\frac{m}{2}(\omega_n^2 + \omega_\alpha^2)\, q_\alpha(i\omega_n)\, q_\alpha(-i\omega_n) + f_{\alpha,-\omega_n}(q)\, q_\alpha(i\omega_n) \right]}$

$$= \int Dq_\alpha \exp\left[-\beta \sum_{i\omega_n} \frac{m(\omega_n^2+\omega_\alpha^2)}{2} \left[q_\alpha(i\omega_n) + \frac{f_{\alpha,\omega_n}}{m(\omega_n^2+\omega_\alpha^2)} \right] \left[q_\alpha(-i\omega_n) + \frac{f_{\alpha,-\omega_n}}{m(\omega_n^2+\omega_\alpha^2)} \right] \right]$$

$$\cdot \exp\left[\beta \sum_{\omega_n} \frac{f_{\alpha,\omega_n}(q) f_{\alpha,-\omega_n}(q)}{2 m_\alpha (\omega_n^2 + \omega_\alpha^2)} \right]$$

The integral over q_α is still Gaussian, which gives a q-independent result. Let us consider a simple case, that f_α is a linear function

$$f_\alpha[q(\tau)] = C_\alpha q(\tau) \longrightarrow f_{\alpha,\omega_n} = C_\alpha q(i\omega_n)$$

$$\Rightarrow S_{eff}[q] = S_p(q) - \frac{\beta}{2} \sum_{i\omega_n,\alpha} \frac{C_\alpha^2 q(i\omega_n) q(-i\omega_n)}{m_\alpha(\omega_n^2+\omega_\alpha^2)}$$

we would like to subtract a frequency independent term

$$\frac{1}{\omega_n^2+\omega_\alpha^2} \to \frac{1}{\omega_n^2+\omega_\alpha^2} - \frac{1}{\omega_\alpha^2} = \frac{-\omega_n^2}{\omega_\alpha^2(\omega_n^2+\omega_\alpha^2)}$$

The coupling shift the potential

$$V(q) \to V(q) + \left(\frac{1}{2} \sum_\alpha \frac{C_\alpha^2}{\omega_\alpha^2 m_\alpha} \right) q^2 ,$$

C carries the unit of $m\omega^2$.

This shift is non-interesting, we neglect it. then we arive at

$$\boxed{\begin{gathered} S_{eff}[q] = S_p[q] + \beta \sum_{i\omega_n} q(i\omega_n) K(i\omega_n) q(-i\omega_n), \text{ where} \\ K(i\omega_n) = \frac{1}{2} \sum_\alpha \frac{C_\alpha^2 \omega_n^2}{m_\alpha \omega_\alpha^2 (\omega_\alpha^2 + \omega_n^2)} \end{gathered}}$$

Define spectra density $J(\omega) = \frac{\pi}{2} \sum_\alpha \frac{C_\alpha^2}{m_\alpha \omega_\alpha} \delta(\omega - \omega_\alpha)$

then $K(i\omega) = \int_0^{+\infty} \frac{d\omega}{\pi} J(\omega) \frac{\omega_n^2}{\omega(\omega^2+\omega_n^2)}$. Consider a weight of

Ohmic damping $J(\omega) = \eta\,\omega$. $\boxed{J\text{'s unite} = [m\omega^2]\ [\eta] = m\omega}$.

$$\boxed{K(i\omega_n) = \int_0^{+\infty} \frac{d\omega}{\pi}\, \eta \frac{\omega_n^2}{\omega^2+\omega_n^2} = \frac{|\omega_n|\eta}{\pi} \int_0^{\infty} dx \frac{1}{x^2+1} = \frac{\eta}{2}|\omega_n|},$$

then

go back to time-domain

$$q(i\omega_n) = \frac{1}{\beta}\int_0^\beta d\tau\, q(\tau) e^{i\omega_n \tau} \quad . \quad K(i\omega_n) = \frac{1}{\beta}\int_0^\beta d\tau'\, K(\tau') e^{-i\omega_n \tau'}$$

$$\rightarrow \beta \beta^{-3} \int_0^\beta d\tau \int_0^\beta d\tau' \int_0^\beta d\tau''\, q(\tau) q(\tau') \sum_{\omega_n} e^{i\omega_n(\tau-\tau'-\tau'')}\ K(\tau'')$$

$$\boxed{\overset{(*)}{\frac{1}{\beta}} \sum_{\omega_n} e^{i\omega_n(\tau-\tau'-\tau'')} = \sum_m \delta(\tau-\tau'-\tau''+m\beta)}$$

$$= \beta^{-1} \int_0^\beta d\tau \int_0^\beta d\tau'\, q(\tau) K(\tau-\tau') q(\tau')$$

$$\omega_n = \frac{2n\pi}{\beta}$$

(~~we neglect~~ ($K(\tau)$ is also a periodical function of τ

$\tau-\tau'$ may go outside $[0, \beta]$, which is equivalent

to put $\tau-\tau'$ inside $[0,\beta]$. but keep all possible $m \neq 0$ in $(*)$]

$$\Rightarrow \boxed{S_{eff}[q] = S_p[q] + \beta^{-1} \int_0^\beta d\tau \int_0^\beta d\tau'\, q(\tau) K(\tau-\tau') q(\tau')}$$

K's unit: $[K] = [m\omega^2]$.

Now we calculate $K(\tau-\tau')$:

$$K(\tau) = \sum_n K(\omega_n) e^{i\omega_n \tau} = \frac{\eta}{2} \sum_n |\omega_n| e^{i\omega_n \tau} = \eta \sum_{n=1}^{\infty} \frac{2n\pi}{\beta} e^{in\frac{2\pi}{\beta}(\tau+i0^+)}$$

set $x = \frac{2\pi}{\beta}$,

$$\sum_{n=1}^{\infty} nx\, e^{inx(\tau+0^+)} = -i\frac{\partial}{\partial\tau} \sum_{n=1}^{\infty} e^{inx(\tau+i0^+)}$$

$$= -i\frac{\partial}{\partial\tau} \sum_{n=0}^{\infty} e^{inx(\tau+i0^+)} = -i\frac{\partial}{\partial\tau} \frac{1}{1-e^{ix(\tau+0^+i)}}$$

$$= -i\frac{x(-)(-)ie^{ix\tau}}{(1-e^{ix\tau})^2} = \frac{x}{(e^{-i\frac{x\tau}{2}} - e^{\frac{i}{2}x\tau})^2} = \frac{x}{-4\sin^2\frac{x\tau}{2}}$$

$$\Rightarrow K(\tau) = \eta\frac{2\pi}{\beta}\frac{1}{-4\sin^2\frac{\pi}{\beta}(\tau+i0^+)} = -\frac{\pi}{2}\frac{\eta}{\beta}\frac{1}{\sin^2\frac{\pi(\tau+i0^+)}{\beta}}$$

at $\tau \ll \beta \Rightarrow K(\tau) = \frac{-\eta\beta}{2\pi(\tau+i0^+)^2}$

Further $q(\tau)K(\tau-\tau')q(\tau') = \left\{q^2(\tau) + q^2(\tau') - \frac{1}{2}[q(\tau)-q(\tau')]^2\right\}K(\tau-\tau')$

The first two terms again after integration only renormalize to $V(q)$, and will be droped. Then we arrive at

$$\boxed{S_{eff}[q] = \int_0^\beta d\tau\, \frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 + V(q) + \frac{\eta}{4\pi}\int_0^\beta d\tau \int_0^\beta d\tau' \frac{(q(\tau)-q(\tau'))^2}{(\tau-\tau'+i0^+)^2}}$$

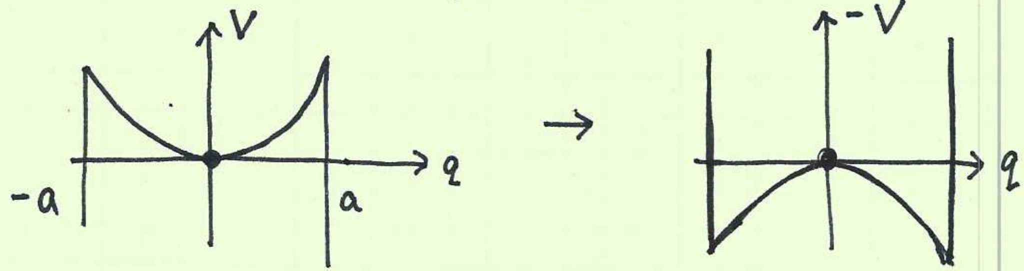
I

change back to Minkowski space $\tau \to it$

$$i S_{eff}[q(t)] = -S_{eff}(\tau \to it) = i\int dt\left[\frac{m}{2}\left(\frac{dq}{dt}\right)^2 - V(q)\right] - \underbrace{\int dt dt'}_{\text{damping}} \frac{\eta}{4\pi}\frac{(q(t)-q(t'))^2}{(t-t')^2}$$

$$\boxed{S[q(t)] = \int dt\, \frac{m}{2}\left(\frac{dq}{dt}\right)^2 - V(q) + i\int dt dt' \frac{\eta}{4\pi}\frac{(q(t)-q(t'))^2}{(t-t')^2}}$$

§2 Now let us come back to the survival probability



Consider the bounce solution $q(0) = q(\beta) \to 0$, at $\tau = \beta/2$. the particle bounce back. The classic equation of motion

$$-m\ddot{q} + m\omega_c^2 q + \frac{\eta}{\pi}\int_0^\beta d\tau' \frac{q(\tau)-q(\tau')}{(\tau-\tau')^2} = A\,\delta(\tau - \beta/2)$$

A is the coefficent to represent the dis continous change of the velocity

LHS: $\sum_n m(\omega_n^2 + \omega_c^2)\, q(\omega_n)\, e^{-i\omega_n\tau} - \frac{2}{\beta}\int_0^\beta d\tau' \left[\sum_{n''} K(i\omega_{n''})\, e^{i\omega_{n''}(\tau-\tau')}\right] \cdot$

$$\left\{\sum_n q(\omega_n)\left(e^{-i\omega_n\tau} - e^{-i\omega_n\tau'}\right)\right\}$$

$$-2\int_0^\beta d\tau' \frac{1}{\beta}\sum_{nn''} K(i\omega_{n''})\, q(\omega_n)\,(-)\, e^{i\omega_{n''}(\tau-\tau') - i\omega_n\tau'} = \eta\sum_n |\omega_n|\, q(\omega_n)\, e^{-i\omega_n\tau}$$

$$\int_0^\beta d\tau' \sum_{nn''} q(\omega_n) e^{-i\omega_n\tau} |\omega_{n''}| e^{-i\omega_{n''}(\tau-\tau')} = 0$$

$$\Rightarrow \text{LHS} = \sum_n \left\{ m(\omega_n^2+\omega_c^2) + \eta|\omega_n| \right\} q(\omega_n) e^{-i\omega_n\tau}$$

$$\text{RHS} = \frac{1}{\beta}\sum_n A e^{-i\omega_n(\tau-\beta/2)}$$

$$\boxed{(m(\omega_n^2+\omega_c^2)+\eta|\omega_n|)\, q(\omega_n) = \frac{A}{\beta} e^{i\omega_n\frac{\beta}{2}}} \leftarrow$$ Solution of classic equation.

$$\sum_n q(\omega_n) e^{-i\omega_n\frac{\beta}{2}} = \frac{A}{\beta}\sum_n \frac{1}{m(\omega_n^2+\omega_c^2)+\eta|\omega_n|}$$

$$a = q\left(\frac{\beta}{2}\right) = Af \text{ , where } f \triangleq \frac{1}{\beta}\sum_n \frac{1}{m(\omega_n^2+\omega_c^2)+\eta|\omega_n|}$$

$\uparrow$
boundry condition of bouncing action.

or $A = \dfrac{a}{f}$, $[f] = \left[\dfrac{1}{m\omega^2}\right]$

The action of the bouncing solutions

$$S_0 = \beta\sum_n \frac{m}{2}|q(\omega_n)|^2(\omega_n^2+\omega_c^2) + \beta^{-1}\int_0^\beta d\tau\int_0^\beta d\tau' \sum_n q(\omega_n) e^{-i\omega_n\tau} \sum_{n'} K(\omega_{n'}) e^{i\omega_{n'}(\tau-\tau')} \sum_{n''} q(\omega_{n''}) e^{-i\omega_{n''}\tau'}$$

$$= \beta\sum_n \frac{m}{2}|q(\omega_n)|^2(\omega_n^2+\omega_c^2)$$

$$+ \beta\sum |q(\omega_n)|^2 K(\omega_n) \leftarrow \frac{\eta}{2}|\omega_n|$$

$$\Rightarrow S_{\text{bounce}} = \frac{\beta}{2}\sum_n \left[m(\omega_n^2+\omega_c^2)+\eta|\omega_n|\right] |q_n|^2$$

plug in $q(\omega_n) = \frac{A}{\beta} e^{i\omega_n \frac{\beta}{2}} \Big/ \left(m(\omega_n^2+\omega_c^2) + \eta|\omega_n|\right)$

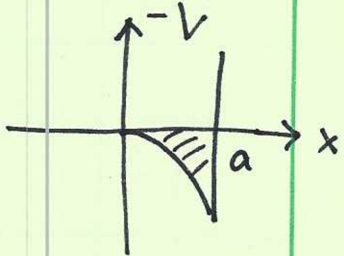
$$\Rightarrow S_{bounce} = \frac{A^2}{2\beta}\sum_n \frac{1}{(\cdots)} = \frac{A^2}{2} f = \boxed{\frac{a^2}{2f} = S_{bounce}}.$$

⊛ and the tunneling rate $\boxed{\Gamma \sim e^{-S_{bounce}}}$.

① Consider the limit $\eta\to 0$ and $\beta\to\infty$: zero temperature, no dissipation

$$f = \frac{1}{\beta}\sum_n \frac{1}{m(\omega_n^2+\omega_c^2)} = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{1}{m(\omega^2+\omega_c^2)} = \frac{1}{2\pi m\omega_c}\int_{-\infty}^{+\infty} dx\, \frac{1}{x^2+1}$$

$$= \frac{1}{2m\omega_c} \Rightarrow S_{bounce} = m\omega_c a^2$$



$$S = 2\int_0^a dx \left(\frac{dx}{d\tau}\right) m = 2\int_0^a dx\, \omega_c x\, m = m\omega_c a^2$$

bound back and forth

so $\Gamma \sim e^{-m\omega_c a^2/\hbar}$ is controlled by the barrier height $\frac{m\omega_c^2 a^2}{2}$

at the attempt frequecy ω_c.

The classic solution

$$q(\tau) = \sum_{\omega_n} q(\omega_n) e^{-i\omega_n\tau} = \cancel{\beta}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{A}{\cancel{\beta}}\frac{e^{-i\omega(\tau-\beta/2)}}{m(\omega^2+\omega_c^2)+\eta|\omega_n|} \leftarrow \eta=0$$

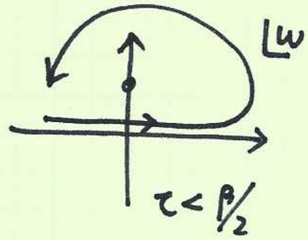
$$= \frac{a}{mf}\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega(\tau-\beta/2)}}{\omega^2+\omega_c^2} = \frac{-2\pi i}{2\pi}\frac{a}{mf}\,\mathrm{Res}\left(\frac{1}{\omega-\omega_c i}\, e^{-i\omega(\tau-\beta/2)}\right)\Big|_{\omega\to-\omega_c i}$$

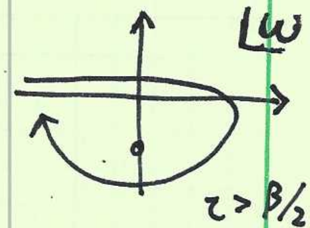
$\boxed{\text{at } \tau > \beta/2}$

$$q(\tau) = -\frac{2\pi i}{2\pi}\,\frac{a}{mf}\,\frac{1}{-2\omega_c i}\;e^{-\omega_c|\tau-\beta/2|}$$ ← combine results at $\tau<\beta/2$ from residual theory

$$= \frac{a}{2\pi m f\omega_c}\,e^{-\omega_c|\tau-\beta/2|}$$

$$= a\,e^{-\omega_c|\tau-\beta/2|}$$





② consider $\beta\to+\infty$, and overdamped reigon $\eta \gg m\omega_c$

$$f = \frac{1}{\beta}\sum_n \frac{1}{m(\omega_n^2+\omega_c^2)+\eta|\omega_n|} = \int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\,\frac{1}{m(\omega^2+\omega_c^2)+\eta|\omega|}$$

in the regin: $\left.\begin{array}{l}\eta|\omega| \gg m\omega^2 \\ \eta|\omega| \gtrsim m\omega_c^2\end{array}\right\} \Rightarrow \frac{m\omega_c^2}{\eta} \ll |\omega| \ll \eta/m$

the contribution ~~the comes from~~ $\cancel{\frac{1}{\eta|\omega|}}$

$$\Rightarrow f_1 \simeq \frac{2}{2\pi\eta}\int_{\frac{m\omega_c^2}{\eta}}^{\eta/m} d\omega\,\frac{1}{\omega} = \frac{1}{\pi\eta}\ln\frac{\eta/m}{m\omega_c^2/\eta} = \frac{2}{\pi\eta}\ln\frac{\eta}{m\omega_c}$$

At $|\omega| \gg \eta/m$, we have $f' = \frac{1}{\pi m}\int_{\eta/m}^{+\infty} d\omega\,\frac{1}{\omega^2} = \frac{1}{\pi m}\,\frac{m}{\eta} \simeq \frac{1}{\pi\eta}$

$|\omega| \ll \frac{m\omega_c^2}{\eta} \rightarrow f'' = \frac{1}{\pi m}\int_0^{\frac{m\omega_c^2}{\eta}} \frac{1}{\omega_c^2} \simeq \frac{1}{\pi\eta}$

combine together, the contribution mainly from $\frac{m\omega_c^2}{\eta} \ll |\omega| \ll \eta/m$

$$\Rightarrow f \simeq \frac{2}{\pi\eta}\left[\ln\frac{\eta}{m\omega_c} + 1\right] \simeq \frac{2}{\pi\eta}\ln\frac{\eta}{m\omega_c}$$

then $\boxed{S_{bounce} = \frac{\pi \eta a^2}{4 \ln \eta / m\omega_c}}$, at $\eta >> m\omega_c$, this soultion

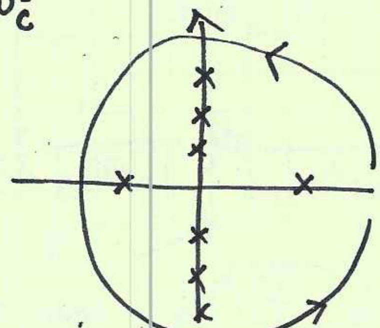
is much larger than the coherent case $S_{bounce} = m\omega_c a^2$.

$\Rightarrow$ tunneling probability is exponentially suppressed.

$\longrightarrow$ The particle becomes classic, - no tunnely.

③ $\eta \to 0$ but $\beta \neq 0$, thermal : $f = \frac{1}{m\beta} \sum_n \frac{1}{\omega_n^2 + \omega_c^2}$

define $\oint \frac{dz}{2\pi i} \frac{1}{e^{\beta z} - 1} \frac{1}{m(-\omega^2 + \omega_c^2)}$



$$0 = \frac{1}{m\beta} \sum_n \frac{1}{\omega_n^2 + \omega_c^2} + \sum_i \text{Res} \frac{1}{e^{\beta z} - 1} \frac{1}{m(-\omega^2 + \omega_c^2)} \Bigg|_{z = \pm\omega_c}$$

$$\Rightarrow f = + \left[\frac{1}{e^{\beta\omega_c} - 1} \frac{1}{2\omega_c} + \frac{1}{e^{-\beta\omega_c} - 1} \left(- \frac{1}{2\omega_c} \right) \right] = \frac{1}{2\omega_c m} \frac{e^{\beta\omega_c} + 1}{e^{\beta\omega_c} - 1} = \frac{\coth \frac{\beta\omega_c}{2}}{2m\omega_c}$$

$$\Rightarrow S = \frac{a^2}{\cancel{4} \left(\frac{\coth \frac{\beta\omega_c}{2}}{m\omega_c} \right)} = m\omega_c a^2 \cdot \tanh \frac{\beta\omega_c}{2} \rightarrow \begin{cases} m\omega_c a^2 \text{ at } \beta \to \infty \\ \frac{\beta}{2} m\omega_c^2 a^2 \text{ at } \beta \to 0 \end{cases}$$

at $\beta \to 0$ (high T) $P \sim e^{\frac{m\omega_c^2 a^2}{2 k_B T}}$ (classic thermal activated behavior!)