path integral for quantum spin

§ Schwinger boson representation

$$\boxed{S_x = \frac{a_1^\dagger a_2 + a_2^\dagger a_1}{2}, \quad S_y = \frac{a_1^\dagger a_2 - a_2^\dagger a_1}{2i}, \quad S_z = \frac{a_1^\dagger a_1 - a_2^\dagger a_2}{2}}$$

the value of S is implemented through the constraint $a_1^\dagger a_1 + a_2^\dagger a_2 = 2S$.

The normalized spin eigenstate $|S.m\rangle$ is represented as

$$|Sm\rangle = \frac{(a_1^\dagger)^{S+m}}{\sqrt{(S+m)!}} \frac{(a_2^\dagger)^{S-m}}{\sqrt{(S-m)!}} |0\rangle .$$

Q): how Schwinger bosons transform under spatial rotation?

Eulerian angle representation of SU(2) rotation

$$R[\phi, \theta, \chi] = e^{-i\phi S_z} e^{-i\theta S_y} e^{-i\chi S_z}$$

1) rotation around z-axis at the angle of ϕ.

2) rotation around the new position of y-axis at the angle of θ

3) rotation around the new position of z-axis at the angle of χ.

Ex: prove that

$$e^{-i\chi \vec{S}\cdot\hat{e}_{z''}} e^{-i\theta \vec{S}\cdot\hat{e}_y'} e^{-i\phi S_z} = e^{-i\phi S_z} e^{-i\theta S_y} e^{-i\chi S_z}$$

where $\hat{e}_y'$ is the position of y-axis after rotation 1) and

$\hat{e}_{z''}$ is the position of z-axis after rotation 2).

The transformation of Schwinger bosons are defined as

$$\begin{cases} a_1'^{\dagger} = R\, a_1^{\dagger}\, R^{-1} \\ a_2'^{\dagger} = R\, a_2^{\dagger}\, R^{-1} \end{cases}$$

in order to derive $a_1'^{\dagger}$ and $a_2'^{\dagger}$. ① we first calculate $e^{-i\phi S_z} \begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger} \end{pmatrix} e^{i\phi S_z}$.

Define $f(\phi) = e^{-i\phi S_z} \begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger} \end{pmatrix} e^{i\phi S_z}$

$$\Rightarrow i \frac{d}{d\phi} f(\phi) = e^{-i\phi S_z} \begin{pmatrix} [S_z, a_1^{\dagger}] \\ [S_z, a_2^{\dagger}] \end{pmatrix} e^{i\phi S_z} = \frac{1}{2} e^{-i\phi S_z} \begin{pmatrix} a_1^{\dagger} \\ -a_2^{\dagger} \end{pmatrix} e^{i\phi S_z}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} f(\phi)$$

Considering $f(\phi=0) = \begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger} \end{pmatrix}$, we can integrate it out as

or $\boxed{e^{-i\phi S_z} \begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger} \end{pmatrix} e^{i\phi S_z} = \begin{pmatrix} e^{-i\phi/2} & \\ & e^{i\phi/2} \end{pmatrix} \begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger} \end{pmatrix}}$

② then we calculate $g(\theta) = e^{-i\theta S_y} \begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger} \end{pmatrix} e^{i\theta S_y}$

$$\Rightarrow i \frac{d}{d\theta} g(\theta) = e^{-i\theta S_y} \begin{pmatrix} [S_y\, a_1^{\dagger}] \\ [S_y\, a_2^{\dagger}] \end{pmatrix} e^{i\theta S_y} = \begin{pmatrix} e^{-i\theta S_y} \cdot \frac{i a_2^{\dagger}}{2} e^{i\theta S_y} \\ e^{-i\theta S_y} \frac{-i a_1^{\dagger}}{2} e^{i\theta S_y} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} & i \\ -i & \end{pmatrix} g(\theta) \qquad \Rightarrow \begin{cases} \frac{d}{d\theta} g(\theta) = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g(\theta) \\ g(0) = \begin{pmatrix} a_1^{\dagger} \\ a_2^{\dagger} \end{pmatrix} \end{cases}$$

$$\Rightarrow g(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a_1'^\dagger \\ a_2'^\dagger \end{pmatrix} = R \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix} R^{-1} = e^{-i\phi S_z} e^{-i\theta S_y} \underbrace{e^{-i\chi S_z} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix} e^{i\chi S_z}} e^{i\theta S_y} e^{i\phi S_z}$$

$$= e^{-i\phi S_z} e^{-i\theta S_y} \begin{bmatrix} e^{-i\chi/2} & 0 \\ 0 & e^{i\chi/2} \end{bmatrix} \begin{bmatrix} a_1^\dagger \\ a_2^\dagger \end{bmatrix} e^{i\theta S_y} e^{i\phi S_z}$$

$$= \begin{bmatrix} e^{-i\chi/2} & 0 \\ 0 & e^{i\chi/2} \end{bmatrix} e^{-i\phi S_z} \underbrace{e^{-i\theta S_y} \begin{bmatrix} a_1^\dagger \\ a_2^\dagger \end{bmatrix} e^{i\theta S_y}} e^{i\phi S_z}$$

$$= \begin{bmatrix} e^{-i\chi/2} & 0 \\ 0 & e^{i\chi/2} \end{bmatrix} \begin{bmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix} \begin{bmatrix} a_1^\dagger \\ a_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} e^{-i\frac{\phi+\chi}{2}} & \sin\frac{\theta}{2} e^{i\frac{\phi-\chi}{2}} \\ -\sin\frac{\theta}{2} e^{i\frac{-\phi+\chi}{2}} & \cos\frac{\theta}{2} e^{i\frac{\phi+\chi}{2}} \end{bmatrix} \begin{bmatrix} a_1^\dagger \\ a_2^\dagger \end{bmatrix}$$

define $\begin{cases} u = \cos\frac{\theta}{2} e^{-i\phi/2} \\ v = \sin\frac{\theta}{2} e^{i\phi/2} \end{cases} \Rightarrow \begin{pmatrix} a_1^{\dagger\prime} \\ a_2^{\dagger\prime} \end{pmatrix} = \begin{pmatrix} u e^{-i\chi/2}, & v e^{-i\chi/2} \\ -v^* e^{i\chi/2} & u^* e^{i\chi/2} \end{pmatrix} \begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}$

or $$\boxed{(a_1^{\dagger\prime}, a_2^{\dagger\prime}) = (a_1^\dagger \; a_2^\dagger) \begin{pmatrix} u e^{-i\chi/2} & -v^* e^{i\chi/2} \\ v e^{-i\chi/2} & u^* e^{i\chi/2} \end{pmatrix}}$$

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§2 spin-coherent state path integral

$$|\hat{\Omega}\rangle = R(\chi,\theta,\phi)\,|SS\rangle = e^{-iS_z\phi}\, e^{-iS_y\theta}\, e^{-iS_z\chi}\,|SS\rangle$$

$|\hat{\Omega}\rangle$ is denoted as spin coherent state and $|SS\rangle = |\Omega = \hat{z}\rangle$.

Generally. $\hat{\Omega} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$.

what's $|\hat{\Omega}\rangle$ under the orth-normal basis?

$$|\hat{\Omega}\rangle = \frac{(a_1^{+\prime})^{S+S}}{\sqrt{2S!}}\,|SS\rangle = \left(e^{-i\chi/2}\right)^{2S} \frac{(u a_1^+ + v a_2^+)^{2S}}{\sqrt{2S!}}\,|0\rangle$$

$$= e^{-iS\chi} \sum_m \binom{2S}{S+m} \frac{u^{S+m} v^{S-m}}{\sqrt{2S!}} (a_1^+)^{S+m} (a_2^+)^{S-m}\,|0\rangle$$

$$= e^{-iS\chi} \sqrt{2S!} \sum_m \frac{u^{S+m} v^{S-m}}{\sqrt{(S+m)!}\sqrt{(S-m)!}}\,|Sm\rangle$$

Inner product

$$\langle\Omega|\Omega'\rangle = e^{iS(\chi-\chi')}\, 2S! \sum_m \frac{(u^* u')^{S+m} (v^* v')^{S-m}}{(S+m)!\,(S-m)!}$$

$$= e^{iS(\chi-\chi')} \left(u^* u' + v^* v'\right)^{2S}$$

the right hand side is the inner product on the S^3-sphere with $2S$ power.

$$u^* u' + v^* v' = \cos\frac{\theta}{2}\cos\frac{\theta'}{2}\, e^{i\frac{\phi-\phi'}{2}} + \sin\frac{\theta}{2}\sin\frac{\theta'}{2}\, e^{-i\frac{\phi-\phi'}{2}}$$

$$= \cos\frac{\theta-\theta'}{2}\cos\frac{\phi-\phi'}{2} + i\sin\frac{\theta+\theta'}{2}\sin\frac{\phi-\phi'}{2}$$

$$|u^*u' + v^*v'|^2 = \cos^2\frac{\theta}{2}\cos^2\frac{\theta'}{2} + \sin^2\frac{\theta}{2}\sin^2\frac{\theta'}{2} + 2\sin\frac{\theta}{2}\sin\frac{\theta'}{2}\cos(\phi-\phi')$$

$$= \frac{1+\cos\theta}{2}\,\frac{1+\cos\theta'}{2} + \frac{1-\cos\theta'}{2}\,\frac{1-\cos\theta}{2} + \frac{1}{2}\sin\theta\sin\theta'\cos(\phi-\phi')$$

$$= \frac{1}{2}\left[1+\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')\right]$$

$$= \frac{1}{2}\left[1+\hat{\Omega}\cdot\hat{\Omega}'\right]$$

$$\Rightarrow \boxed{\langle\Omega|\Omega'\rangle = \left(\frac{1+\hat{\Omega}\cdot\hat{\Omega}'}{2}\right)^{2S} e^{iS\psi}}$$

The phas $\psi = \chi - \chi' + 2\,\mathrm{arctg}\left[\tan\frac{\phi-\phi'}{2}\,\frac{\cos\frac{\theta+\theta'}{2}}{\cos\frac{\theta-\theta'}{2}}\right]$

Remark: (u, v) canbe viewed as a coordinate on the S^2-sphere of $\hat{\Omega}$.

$$\boxed{\hat{n} = (u^*, v^*)\,\vec{\sigma}\begin{pmatrix} u \\ v \end{pmatrix}}$$

~~or~~ certainly if we do $\begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} u\,e^{-i\chi/2} \\ v\,e^{-i\chi/2} \end{pmatrix}$, $\hat{n}$ doesnot change. Thus χ-angle becomes a gauge degree of freedom.

This expressin is called 1st Hopf map from $S^3 \longrightarrow S^2$.

resolution identity:

$$\boxed{\frac{2S+1}{4\pi}\int d\Omega\ |\hat{\Omega}\rangle\langle\hat{\Omega}| = 1}$$

Proof: $$\text{LHS} = \frac{2S+1}{4\pi}(2S)!\int d\Omega \sum_m \frac{|u|^{2S+2m}\,|v|^{2S-2m}}{(S+m)!\,(S-m)!}\,|Sm\rangle\langle Sm|$$

$$= \int\frac{d\Omega}{4\pi}\,(2S+1)!\sum_m \frac{\left(\frac{1+\cos\theta}{2}\right)^{S+m}\left(\frac{1-\cos\theta}{2}\right)^{S-m}}{(S+m)!\,(S-m)!}\,|Sm\rangle\langle Sm|$$

$$\int\frac{d\Omega}{4\pi}\left(\frac{1+\cos\theta}{2}\right)^{S+m}\left(\frac{1-\cos\theta}{2}\right)^{S-m} = \frac{1}{2}\int_{-1}^{1}dx\left(\frac{1+x}{2}\right)^{S+m}\left(\frac{1-x}{2}\right)^{S-m}$$

set $y = \frac{x+1}{2}$ $\Rightarrow$ $\Rightarrow$ $$= \int_0^1 dy\ y^{S+m}(1-y)^{S-m}$$

$$= \frac{(S+m)!\,(S-m)!}{(2S+1)!}$$

$$\Rightarrow \text{LHS} = \sum_m |Sm\rangle\langle Sm| = 1.$$

§ path integral Rep of partition function (a single spin)

$$Z = tr\left[e^{-\beta \hat{H}}\right] = \lim_{N_\epsilon \to \infty} tr \prod_{n=0}^{N_\epsilon - 1} (1 - \epsilon \hat{H}(\tau_n)]$$

$\hat{H}$ is Hamitonian for a single spin, say $\hat{H} = -\vec{B}\cdot\vec{S}$

Insert Resolution indentity $\Rightarrow$

$$Z = \lim_{N_\epsilon \to \infty} \int \prod_{\tau} d\Omega(\tau) \prod_{\tau=\epsilon}^{\beta} \langle \Omega(\tau) | 1 - \epsilon \hat{H}(\tau) | \Omega(\tau-\epsilon) \rangle$$

$$\langle \Omega(\tau) | \Omega(\tau-\epsilon) \rangle \xrightarrow[\text{keep } \epsilon\text{'s linear order}]{} e^{iS\psi} \leftarrow \left(\frac{1+\hat{\Omega}\cdot\hat{\Omega}'}{2}\right)^{2S} \approx 1$$

and $\psi \approx \epsilon \dot{\chi} + \dot{\phi} \epsilon \cos\theta$

$$\Rightarrow \langle \Omega(\tau) | \Omega(\tau-\epsilon) \rangle \simeq \exp\left[i\epsilon S (\dot{\chi} + \dot{\phi}\cos\theta) \right]$$

as for $\langle \Omega(\tau) | \hat{H}(\tau) | \Omega(\tau-\epsilon) \rangle$, we only need to keep to zeroth order of ϵ, thus $\rightarrow$ $\langle \Omega(\tau) | \hat{H}(\tau) | \Omega(\tau) \rangle = H(\hat{\Omega}_i)$

because $(\hat{\Omega}\cdot\vec{S})|\hat{\Omega}\rangle = S|\hat{\Omega}\rangle$

$$\Rightarrow \quad \mathcal{Z} = \oint D\hat{\Omega}(\tau) \exp\Big[\sum_{n=0}^{N\epsilon-1} i \epsilon S [(\dot{\chi} + \dot{\phi}\cos\theta) - \sum_{n=0}^{N\epsilon-1} \epsilon H(\hat{\Omega}_i) \Big]$$

we can organize as

$$\boxed{\begin{aligned} \mathcal{Z} &= \oint D\hat{\Omega}(\tau) \exp[-S(\hat{\Omega})], \text{ with} \\ S[\hat{\Omega}] &= i S \, \omega(\Omega) + \int_0^\beta d\tau \, H(\hat{\Omega}(\tau)) \end{aligned}}$$

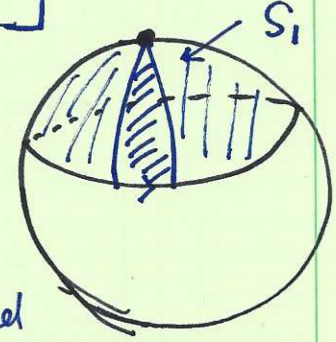
In order to have the geometric expression of $\omega(\Omega) = \int d\tau \, [-\dot{\chi} - \dot{\phi}\cos\theta]$, we need to choose the gauge of χ. Physically, θ and ϕ are the direction of $\hat{\Omega}$, and we have the boundry condition $\hat{\Omega}(\beta) = \hat{\Omega}(0)$. We would also let $\begin{pmatrix} u \\ v \end{pmatrix} e^{-i\chi/2} = \begin{pmatrix} \cos\frac{\theta}{2} \, e^{-i(\phi+\chi)/2} \\ \sin\frac{\theta}{2} \, e^{i(\phi-\chi)/2} \end{pmatrix}$ also periodic.

when the loop of $\hat{\Omega}$ enclose the north pole, we have $\varphi \to \varphi \pm 2\pi$, the factor $e^{i\phi/2}$ causes problem. (It changes sign). On the other hand, the choice of north pole is arbitary. We would like a geometric expression of $\omega(\Omega)$, which only depends on the loop of $\hat{\Omega}(\tau)$, but not on whether the north pole is enclosed or not.

(If we insist to use $\chi = 0$, then we need to take into acount the jump of ϕ at the boundary).

Then we can choose $\chi = -\phi$, then we have

$$\boxed{\omega(\Omega) = \oint d\tau \; \dot{\phi}(1-\cos\theta) = \oint d\phi \, (1-\cos\theta)} \leftarrow S_1$$



This is nothing but the area enclosed by the trajectory.

Remark:
Then a question appear: on the sphere, a closed loop could be interpret to enclose 2 areas. How do we choose?

The other choice is equivalent to the choice of $\chi = \phi$.

Then

$$\omega(\Omega) = -\oint d\tau \; \dot{\phi}(1+\cos\theta)$$

$$= -S_2$$

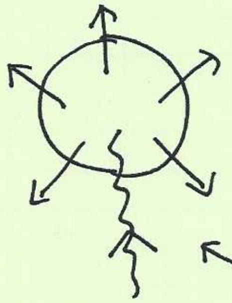
we have $S_1 + S_2 = 4\pi$, the spin value is intege or half integer, thus these two different choices do not lead trouble.

$$\boxed{\omega(\Omega) \text{ is the Berry phase} \leftarrow \text{area enclosed by the loop.}}$$

$$\omega = \int_0^\beta d\tau \; \vec{A}(\Omega) d\hat{\Omega} \quad \text{where} \quad (\nabla \times \vec{A}) \cdot \hat{\Omega} = 1$$

$\vec{A}$ is the vector potential for a magnetic monopole.

A standard choice is $\vec{A} = \frac{1-\cos\theta}{\sin\theta}\hat{e}_\phi = \tan\frac{\theta}{2}\hat{e}_\phi$



§ Equation of motion and large-S expansion

When in the real-time, we have

$$G(\Omega_t, \Omega_0; t) = \langle \Omega_t | T \exp \int_0^t dt' (-i H(t')) | \Omega_0 \rangle \leftarrow \tau \to it$$

$$= \int_{\Omega_0}^{\Omega_t} D\Omega(t') \exp[i S(\hat{\Omega})]$$ where $S[\hat{\Omega}]$ is the real time action

$$\boxed{S[\hat{\Omega}] = \int_0^t dt' \left[S \vec{A} \cdot \dot{\hat{\Omega}} - H(\Omega(t')) \right].}$$

We will find the saddle point equation

$$\frac{\delta}{\delta \hat{\Omega}} S[\hat{\Omega}] \Big|_{\Omega^{cl}} = 0,$$ where the boundary condition

$\hat{\Omega}^{cl}(0) = \hat{\Omega}_0$ and $\hat{\Omega}^{cl}(t) = \hat{\Omega}_t$.

$$\delta\left[\int_0^t \vec{A}\cdot\dot{\hat{\Omega}}\,dt'\right] = \int_0^t dt' \; \frac{\delta A^\alpha}{\delta \Omega^\beta}\delta\Omega^\beta\,\dot{\hat{\Omega}}^\alpha + A^\alpha \frac{d}{dt'}\delta\hat{\Omega}^\alpha$$

$$+\left(\frac{\partial A^\alpha}{\partial \Omega^\beta}\dot{\Omega}^\beta\delta\hat{\Omega}^\alpha - \frac{\partial A^\alpha}{\partial \Omega^\beta}\dot{\Omega}^\beta\delta\hat{\Omega}^\alpha\right)$$

this term is zero to rearange the first line

$$= \int_0^t dt' \; \frac{\partial A^\alpha}{\delta \Omega^\beta}\left[\dot{\Omega}^\alpha\delta\Omega^\beta - \dot{\Omega}^\beta\delta\Omega^\alpha\right) + \int_0^t dt' \; \underbrace{A^\alpha\frac{d}{dt'}\delta\Omega^\alpha + \frac{dA^\alpha}{dt'}\delta\Omega^\alpha}_{\frac{d}{dt'}(\vec{A}\cdot\delta\hat{\Omega})}$$

The last term $=0$ due to the boundary condition.

The first term $= \int_0^t dt' \; \frac{\partial A^\alpha}{\partial \Omega^\beta}[\dot{\Omega}^{\alpha'}\delta\Omega^{\beta'}][\delta_{\alpha\alpha'}\delta_{\beta\beta'} - \delta_{\alpha'\beta}\delta_{\alpha\beta'}]$

$$= \int_0^t dt' \; \frac{\partial A^\alpha}{\partial \Omega^\beta}[\dot{\Omega}^{\alpha'}\delta\Omega^{\beta'}]\;\epsilon^{\gamma\alpha\beta}\,\epsilon^{\gamma\alpha'\beta'}$$

$$= \int_0^t dt' \; \underbrace{\epsilon^{\gamma\alpha\beta}\frac{\partial A^\alpha}{\partial \Omega^\beta}}_{\text{field streigh} = \Omega^\gamma} \;\epsilon^{\gamma\alpha'\beta'}[\dot{\Omega}^{\alpha'}\delta\Omega^{\beta'}]$$

$$\Rightarrow \int_0^t dt' \; \hat{\Omega}\cdot[\dot{\hat{\Omega}}\times\delta\hat{\Omega}] = \int_0^t dt' \; \delta\hat{\Omega}\cdot(\hat{\Omega}\times\dot{\hat{\Omega}})$$

then $\delta S[\hat{\Omega}] = \int_0^t dt' \left[(\hat{\Omega}\times\dot{\hat{\Omega}})S - \frac{\partial H}{\partial \Omega}\right]\cdot\delta\Omega$

$$\Rightarrow \boxed{S\,\hat{\Omega}\times\dot{\hat{\Omega}} = \frac{\partial H}{\partial \Omega}[\hat{\Omega}]}$$

$$S\,\hat{\Omega}\times[\hat{\Omega}\times\dot{\hat{\Omega}}] = \hat{\Omega}\times\frac{\partial H}{\partial\Omega}(\hat{\Omega})$$

$$\rightarrow (\hat{\Omega}\cdot\dot{\hat{\Omega}})\hat{\Omega} - (\Omega\cdot\Omega)\dot{\hat{\Omega}} = -\dot{\hat{\Omega}}$$

$$\Big\}\Rightarrow S\dot{\hat{\Omega}} = \hat{\Omega}\times\left(-\frac{\partial H}{\partial\hat{\Omega}}\right)$$

if $H = -\vec{B}\cdot\vec{S} = -\vec{B}\cdot\hat{\Omega}\,S$

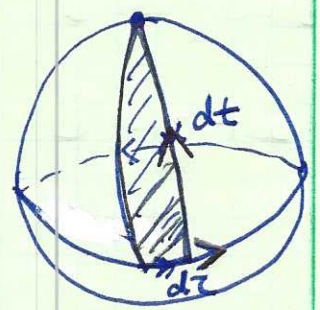
$$\Rightarrow \boxed{\dot{\hat{\Omega}} = \hat{\Omega}\times\vec{B}} \leftarrow \text{Lamor procession}$$

θ-term Haldane conjecture

- Wess-Zumino term for a single spin

$$Z = \oint D\hat{n}(\tau)\, e^{-S(\hat{n})} \quad , \text{where } S(\hat{n}) = i S\, \omega[\hat{n}(\tau)] + \int_0^\beta d\tau\, H(\hat{n})$$

$$\omega[\Omega(\tau)] = \oint d\phi\, (1-\cos\theta) = \oint d\hat{n}\cdot \vec{A}(\hat{n})$$



$$= \int_0^\beta d\tau \int_0^1 dt\ \hat{n}(\tau,t)\cdot\left(\partial_\tau \hat{n}(\tau,t)\times \partial_t \hat{n}(\tau,t)\right)$$

$$\text{with } \begin{cases} \vec{n}(\tau,0) = \hat{n}(\tau) \\ \vec{n}(\tau,1) = \hat{n}_0 \end{cases}$$

Note the WZ term does not change when back to the real time

$$S_M = S\,\omega[\hat{n}(\tau)] - \int_0^T d\cancel{\tau}_0\, H(\hat{n}) \longrightarrow Z = \int D\hat{n}\, e^{i S_M[\hat{n}]}$$

- Now we derive the action for a 1D spin chain (AF)

Due to the AF tendency, we separate slow mode and the fast mode

$$\hat{n}(j) = \hat{m}_j (-)^j + \vec{l}(j)\, a_0$$

← lattice constant

$$|\hat{m}|^2 = 1, \text{ and } \hat{m}\cdot\vec{l} = 0, \text{ with } |\vec{l}| \ll 1.$$

$$S_M(\hat n) = S\sum_{j=1}^{N} \omega[\hat n(j)] - \int_0^T dx_0 \sum_{j=1}^{N} JS^2\, \hat n(j,x_0)\cdot \hat n(j+1,x_0)$$

$$\rightarrow -\frac{JS^2}{2}\int_0^T dx_0 \sum_{j=1}^{N} (\hat n(j,x_0)+\hat n(j+1,x_0))^2$$

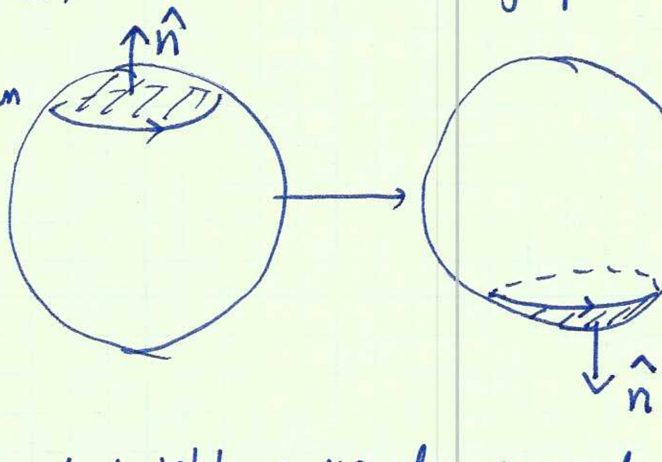
$$= -\frac{JS^2}{2}\int_0^T dx_0 \sum_{j=1}^{N} [\underbrace{(\hat m(j,x_0)-\hat m(j+1,x_0))(-)^l}_{(\partial_1 \hat m)\cdot a_0 (-)^l} + \underbrace{a_0(l(j,x_0)+l(j+1,x_0))}_{2a_0 \vec l}]^2$$

$$= -\frac{a_0 J S^2}{2}\int_0^T dx_0 \int dx\, [(\partial_1 \hat m)^2 + 4\vec l^2] \leftarrow \sum_j = \frac{1}{a_0}\int dx$$

> HW: Show that the cross term vanishes after suming over "j".
> Hint: $(\partial_1 \hat m)(-)^l$ and $\vec l$ belong to different sectors of momenta. One is staggered, the other is uniform.

Now check the W-Z term:

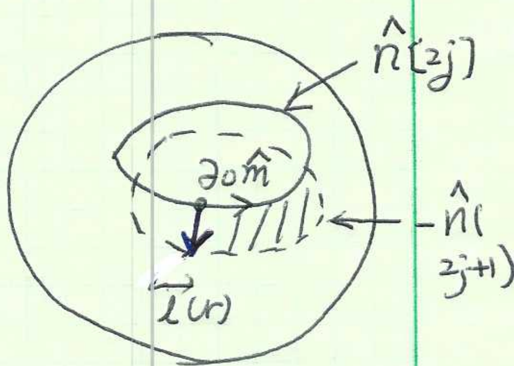
If we flip the direction of $\hat n$, the WZ term also flips the sign. The circulation orientation does not change, but the norm change, hence the areas surrounded have opposite sign.



Hence the WZ term from nearest neighbours nearly cancel.

$$S\sum_{j=1}^{N} W(n[j,\tau]) = S\sum_{j=1}^{N/2} \omega[n(2j)] - \omega[-n(2j+1)]$$

$\omega[n(2j)] - \omega[-n(2j+1)]$



$$\bar{=} S\int dx_0 \; [n(2j)+n(2j+1)] \cdot \underline{\underline{[\partial_0 \hat{n}(2j)\times \hat{n}(2j)]}}$$

$$= S\int dx_0 \; [\partial_1 m(j)\cdot a_0 + 2\vec{l}(j)\cdot a_0]\cdot(\partial_0\hat{m}(2j)\times\hat{m}(2j))$$

For term underlined, we could also use $(-\partial_0 \hat{n}(2j+1)\times(-\hat{n}(2j+1))$

$$\sim (\partial_0\hat{m} - \hat{l})\times(\partial_0\hat{m} - \hat{l})$$

while $\partial_0\hat{n}(2j)\times\hat{n}(2j) \sim (\partial_0\hat{m}+\hat{l})\times(\hat{m}+\hat{l})$

as an averge for high order accuracy, $\longrightarrow (\partial_0\hat{m}\times\hat{m})$

$$\Rightarrow S\sum_{j=1}^{N} \omega[n(j,\tau)] = S\sum_{j=1}^{N/2} \int dx_0 \; [\partial_1\hat{m}\cdot a_0 + 2\vec{l}\cdot a_0]\cdot(\partial_0\hat{m}\times\hat{m})$$

$$= \frac{S}{2}\int dx_0 dx_1 \; \hat{m}\cdot(\partial_0\hat{m}\times\hat{m}) - S\int dx_0 dx_1 \; \vec{l}\cdot(\hat{m}\times\partial_0\hat{m})$$

$$\sum_{j=1}^{N/2}\cdot \rightarrow \int\frac{dx}{2a_0}$$

$$\Rightarrow \mathcal{L}_M(\vec{m},\vec{l}) = -2a_0JS^2 l^2 - \frac{a_0JS^2}{2}(\partial_1\hat{m})^2$$

$$- S\vec{l}\cdot(\hat{m}\times\partial_0\hat{m}) + \frac{S}{2}\hat{m}\cdot(\partial_0\hat{m}\times\hat{m})$$

$$Z_M = \int D\hat{m}\, Dl \; e^{i\int dt \, \mathcal{L}_M[\hat{m},\vec{l}]}$$

l term: $-2a_0JS^2\left[\vec{l} + \dfrac{\hat{m}\times\partial_0\hat{m}}{4a_0JS}\right]^2 + \dfrac{1}{8a_0J}\left[\hat{m}\times\partial_0\hat{m}\right]^2$

$$\searrow \frac{1}{8a_0J}\left(\partial_0\hat{m}\right)^2$$

$$\Rightarrow \mathcal{L}_M(\hat{m}) = \frac{1}{8a_0J}(\partial_0\hat{m})^2 - \frac{a_0JS^2}{2}(\partial_1\hat{m})^2 + \frac{S}{2}\hat{m}\cdot(\partial_0\hat{m}\times\partial_1\hat{m})$$

define $g = 2/S$, $v_s = 2a_0JS$, $\theta = 2\pi S$

$$\Rightarrow \mathcal{L}_M = \frac{1}{2g}\left[\frac{1}{v_s}(\partial_0\hat{m})^2 - v_s(\partial_1\hat{m})^2\right] + \theta\, \frac{\varepsilon_{\mu\nu}\,\hat{m}\cdot(\partial_\mu\hat{m}\times\partial_\nu\hat{m})}{8\pi}$$

→ Euclidean action

$$\mathcal{L}_E = \frac{1}{2g}\left[\frac{1}{v_s}(\partial_0\hat{m})^2 + v_s(\partial_1\hat{m})^2\right] + \frac{i\theta}{8\pi}\varepsilon_{ij}\,\hat{m}\cdot(\partial_i\hat{m}\times\partial_j\hat{m})$$

$$\int dx_0\int dx_1 \,\frac{\varepsilon_{\mu\nu}}{8\pi}\,\hat{m}\cdot(\partial_\mu\hat{m}\times\partial_\nu\hat{m}) = \text{Pontryagian } \#$$

$$\Rightarrow Z_M = \sum_n \int D[\hat{m}] \; e^{i2\pi S\cdot n}\; e^{-\int_0^\beta d\tau\, \frac{1}{2g}\left[\frac{1}{v_s}(\partial_0\hat{m})^2 + v_s(\partial_x\hat{m})^2\right]}$$

- for half-integer S

$$e^{i\pi 2Sn} = (-)^n \Rightarrow Z_M = \sum_n \int D[\hat{m}]\, (-)^n\, e^{-\int_0^\beta d\tau \frac{1}{2g}[\partial_\mu \hat{m} \cdot \partial_\mu \hat{m})}$$

- ~~The s~~ for integer spin, the Θ-term does not make a difference!

☆ Haldan conjecture:

1. Spin 1/2 AFM Heisenberg model is gapless, and its non-linear σ model has non-trivial Θ-term.

2. The classic non-linear σ model is disordered.
2D

Conjecture: The gapless behavior of spin-1/2 AFM chain is due to effect of Θ-term. Hence, Haldane conjectured that half integer spin chains are all gapless. However, for integer AFM spin chain, they remain gapped!

But spin-1 AFM chain $\neq$ ~~to~~ classic 2D non-linear sigma model, it has boundary modes.

notes on non-linear σ model

$$\Omega = \eta_i \, \hat{n}(x) \sqrt{1-\frac{L^2}{S^2}} + \frac{L^2}{S^2} \qquad \vec{L}\cdot\vec{n} = 0$$

"J" only appears in the velocity, not in the coupling constant

$$\mathcal{L} = \chi_0 (\partial_t n)^2 - \rho_s (\partial_x n)^2 \qquad n^2 = 1$$

by definition $\rho_s = J S^2 a^{2-d}$ check dimension

but χ comes from $J \cdot L^2 a^{-d} \Rightarrow \chi \propto \frac{1}{J a^d}$

$$\Rightarrow \left.\begin{array}{l} \underline{v = \sqrt{\chi_0^{-1} \rho_s} \propto a J S} \\ \underline{g = \sqrt{\chi_0 \rho_s} \propto S a} \end{array}\right\}$$

$\boxed{\text{coupling constant only contains } S\text{, no } J}$

"J" effects on the temproal & spacial are opposite:

Large "J" means difficult to get ferro-moment, i.e. small susceptibility.

④ Spin-1 models (1D) – AKLT

$$H_{AKLT} = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + 1/3 \left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2 + 2/3$$

$$\vec{S}_i \cdot \vec{S}_{i+1} = \frac{1}{2}\left(\vec{S}_i + \vec{S}_{i+1}\right)^2 - 2 = \begin{cases} 1 & \text{for } S_{tot} = 2 \\ -1 & \quad = 1 \\ -2 & \quad\ \ 0 \end{cases}$$

$$\rightarrow \quad \vec{S}_i \cdot \vec{S}_{i+1} = P_2(i,i+1) - P_1(i,i+1) - 2P_0(i,i+1)$$

$$(\vec{S}_i \cdot \vec{S}_{i+1})^2 = P_2(i,i+1) + P_1(i,i+1) + 4P_0(i,i+1)$$

with $P_2(i,i+1) + P_1(i,i+1) + P_0(i,i+1) = 1$

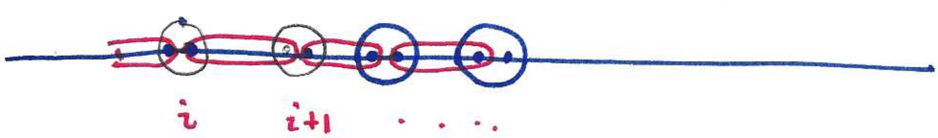
$$\Rightarrow \boxed{H_{AKLT} = J \sum_i P_2(i,i+1)}$$

For spin-1, it can written as 2-Schwinger bosons

$$S_z = 1 \qquad \frac{1}{\sqrt{2}}(a^+)^2 \,|0\rangle$$

$$0 \qquad a^+ b^+ \,|0\rangle \qquad\qquad a^+a + b^+b = 2$$

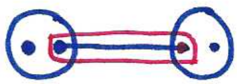
$$-1 \qquad \frac{1}{\sqrt{2}}(b^+)^2 \,|0\rangle$$



Consider state $|\psi\rangle = \prod_i \left(a_i^+ b_{i+1}^+ - b_i^+ a_{i+1}^+\right)|0\rangle$

Then $|\psi\rangle$ is a ground state.

Consider a bond $i, i+1$, If we trace out the states of all other sites, we are left the configuration



i $\quad$ $i+1$

$$(a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger) \otimes [\text{one boson on } i \text{ and one boson on } i+1]$$

$$a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger = (a_i^\dagger \;\; b_i^\dagger) \underbrace{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}}_{\text{charge conjugation matrix } R} \begin{pmatrix} a_{i+1}^\dagger \\ b_{i+1}^\dagger \end{pmatrix}$$

$R \begin{pmatrix} a^\dagger \\ b^\dagger \end{pmatrix}$ under rotation, transforms the same as $\begin{pmatrix} a \\ b \end{pmatrix}$

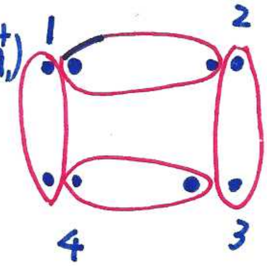
Excercise prove that actually $(a_i^\dagger b_{i+1}^\dagger - b_i^\dagger a_{i+1}^\dagger)$ is invariant under rotation, i.e. form singlet. $\Rightarrow$ the total spin of sites $i, i+1$ only comes from one boson on "i" and one boson on "$i+1$". The total spin can only be 0 or 1. $\Rightarrow$ $\boxed{P_2(i, i+1)\,|\psi\rangle = 0.}$

$\Rightarrow$ Since P_2 is a non-negative operator, $\Rightarrow$ $|\psi\rangle$ is a ground state of H.

(I do not know how to prove the uniqueness of the ground state. I guess it is also true).

Let me use a 4-site ring to illustrate the WF

$$|\psi\rangle = (a_1^+ b_2^+ - b_1^+ a_2^+)(a_2^+ b_3^+ - b_2^+ a_3^+)(a_3^+ b_4^+ - b_3^+ a_4^+)(a_4^+ b_1^+ - b_4^+ a_1^+)$$



$$= a_1^+ b_2^+ a_2^+ b_3^+ a_3^+ b_4^+ a_4^+ b_1^+ \quad - \quad a_1^+ b_2^+ a_2^+ b_3^+ a_3^+ b_4^+ b_4^+ a_1^+$$

L L L L   L L L R

$$- a_1^+ b_2^+ a_2^+ b_3^+ b_3^+ a_4^+ a_4^+ b_1^+ \quad + \quad a_1^+ b_2^+ a_2^+ b_3^+ b_3^+ a_4^+ b_4^+ a_1^+$$

L L R L   L L R R

$$- a_1^+ b_2^+ b_2^+ a_3^+ a_3^+ b_4^+ a_4^+ b_1^+ \quad + \quad a_1^+ b_2^+ b_2^+ a_3^+ a_3^+ b_4^+ b_4^+ a_1^+$$

L R L L   L R L R

$$+ a_1^+ b_2^+ b_2^+ a_3^+ b_3^+ a_4^+ a_4^+ b_1^+ \quad - \quad a_1^+ b_2^+ b_2^+ a_3^+ b_3^+ a_4^+ b_4^+ a_1^+$$

L R R L   L R R R

$$- b_1^+ a_2^+ a_2^+ b_3^+ a_3^+ b_4^+ a_4^+ b_1^+ \quad + \quad b_1^+ a_2^+ a_2^+ b_3^+ a_3^+ b_4^+ b_4^+ a_1^+$$

R L L L   R L L R

$$+ b_1^+ a_2^+ a_2^+ b_3^+ b_3^+ a_4^+ a_4^+ b_1^+ \quad - \quad b_1^+ a_2^+ a_2^+ b_3^+ b_3^+ a_4^+ b_4^+ a_1^+$$

R L R L   R L R R

$$+ b_1^+ a_2^+ b_2^+ a_3^+ a_3^+ b_4^+ a_4^+ b_1^+ \quad - \quad b_1^+ a_2^+ b_2^+ a_3^+ \overset{a_3^+ b_4^+}{\cancel{b_3^+ a_4^+}} b_4^+ a_1^+$$

R R L L   R R L R

$$- b_1^+ a_2^+ b_2^+ a_3^+ b_3^+ a_4^+ a_4^+ b_1^+ \quad + \quad b_1^+ a_2^+ b_2^+ a_3^+ b_3^+ a_4^+ b_4^+ a_1^+$$

R R R L   R R R R

$= 2|0000\rangle - \frac{1}{2}|100-1\rangle - \frac{1}{2}|00-11\rangle + \frac{1}{2}|10-10\rangle$

$-\frac{1}{2}|0-110\rangle + \frac{1}{4}|1-11-1\rangle + \frac{1}{2}|0-101\rangle - \frac{1}{2}|1-100\rangle$

$-\frac{1}{2}|-1100\rangle + \frac{1}{2}|010-1\rangle + \frac{1}{4}|-11-11\rangle - \frac{1}{2}|01-10\rangle$

$+\frac{1}{2}|-1010\rangle - \frac{1}{2}|001-1\rangle - \frac{1}{2}|-1001\rangle$

Pattern observed ① If we remove "0", ⇒ then we have

1 -1 or 1-1 1-1, or a Neel order.

The AKLT ground state can be viewed by adding zeros into the Neel configure.

② states with more "1" and "-1" have less weight

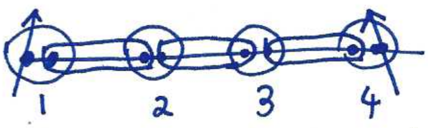
HW ① For the long chain, prove that for the ground state, in the S_z-representation, there does not exist two 1's or two -1's adjacent after 0's are removed.

② $$O_{string} = \lim_{|i-j|\to\infty} \langle S_i^\alpha \, e^{i\pi \sum_{k=i}^{j-1} S_k^\alpha} S_j^\alpha \rangle \qquad \alpha = x, y, z$$

~~This a non-local order develop nonzero expectation value.~~

$O_{string} \neq 0 \rightarrow$ non-local order

5 fractional edge excitations — open boundry condition



Two unpaired Schwinger bosons appearing on the edges. How to express in S_z-basis.

$$|\psi\rangle = a_1^\dagger (a_1^\dagger b_2^\dagger - b_1^\dagger a_2^\dagger)(a_2^\dagger b_3^\dagger - b_2^\dagger a_3^\dagger)(a_3^\dagger b_4^\dagger - b_3^\dagger a_4^\dagger)\, a_4^\dagger\, |0\rangle$$

$$= \Big(a_1^\dagger\ \underset{L}{a_1^\dagger b_2^\dagger}\ \underset{L}{a_2^\dagger b_3^\dagger}\ \underset{L}{a_3^\dagger b_4^\dagger}\ a_4^\dagger \;-\; a_1^\dagger (\underset{L}{a_1^\dagger b_2^\dagger}\ \underset{L}{a_2^\dagger b_3^\dagger}\ \underset{R}{b_3^\dagger a_4^\dagger})\, a_4^\dagger$$

$$- a_1^\dagger\ \underset{L}{a_1^\dagger b_2^\dagger}\ \underset{R}{b_2^\dagger a_3^\dagger}\ \underset{L}{a_3^\dagger b_4^\dagger}\ a_4^\dagger \;+\; a_1^\dagger\ \underset{L}{a_1^\dagger b_2^\dagger}\ \underset{R}{b_2^\dagger a_3^\dagger}\ \underset{R}{b_3^\dagger a_4^\dagger}\ a_4^\dagger$$

$$- a_1^\dagger\ \underset{R}{b_1^\dagger a_2^\dagger}\ \underset{L}{a_2^\dagger b_3^\dagger}\ \underset{L}{a_3^\dagger b_4^\dagger}\ a_4^\dagger \;+\; a_1^\dagger\ \underset{R}{(b_1^\dagger a_2^\dagger)}\ \underset{L}{a_2^\dagger b_3^\dagger}\ \underset{R}{b_3^\dagger a_4^\dagger}\ a_4^\dagger$$

$$+ a_1^\dagger\ \underset{R}{b_1^\dagger a_2^\dagger}\ \underset{R}{b_2^\dagger a_3^\dagger}\ \underset{L}{a_3^\dagger b_4^\dagger}\ a_4^\dagger \;-\; a_1^\dagger\ \underset{R}{b_1^\dagger a_2^\dagger}\ \underset{R}{b_2^\dagger a_3^\dagger}\ \underset{R}{b_3^\dagger a_4^\dagger}\ a_4^\dagger \Big) |0\rangle$$

$$= \frac{1}{\sqrt{2}}|1\,0\,0\,0\rangle - \frac{1}{\sqrt{8}}|1\,0\,{-1}\,1\rangle - \frac{1}{\sqrt{2}}|1\,{-1}\,1\,0\rangle + \frac{1}{2}|1\,{-1}\,0\,1\rangle$$

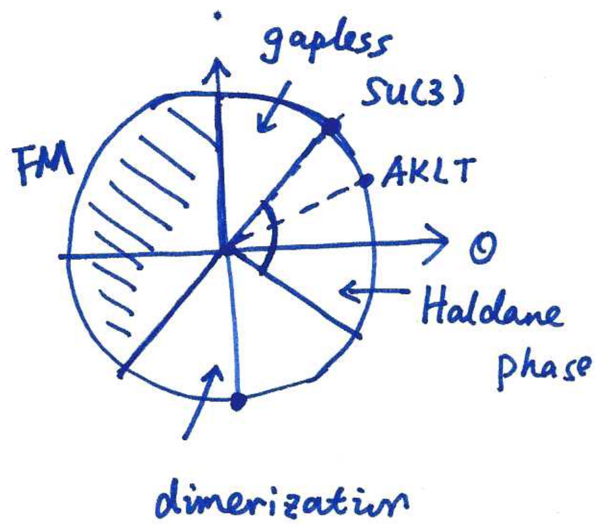
$$- \frac{1}{\sqrt{2}}|0\,1\,0\,0\rangle + \frac{1}{\sqrt{8}}|0\,1\,{-1}\,1\rangle + \frac{1}{\sqrt{2}}|0\,0\,1\,0\rangle - \frac{1}{\sqrt{2}}|0\,0\,0\,1\rangle$$

§ ~~AKLT state~~

Spin-1 mode $H = J \sum_i \cos\theta \, \vec{S}_i \cdot \vec{S}_{i+1} + \sin\theta \, (\vec{S}_i \cdot \vec{S}_{i+1})^2$

~~sin~~ $\theta = 0$. the pure Heisenberg model lies in the same phase of the AKLT one

		$S_i \cdot S_j$	$(S_i \cdot S_j)^2$	
singlet	0	-2	4	$-2\cos\theta + 4\sin\theta$
triplet	1	-1	1	$-\cos\theta + \sin\theta$
quintet	2	1	1	$\cos\theta + \sin\theta$



① at $\theta = 45^\circ \Rightarrow E_{singlet} = E_{quint}$

SU(3) symmetry $3 \times 3 = 3 + 6$

② at ~~$\theta = 45^\circ$~~ $\theta = -90^\circ$, $E_{triple} = E_{quint}$ $\square \times \square^* = 1 + 8$

SU(3), symmetry

§ other AKLT phase