Lect 4: operator formalism, response function,

§1: perturbation theory

Consider a system with Hamiltonian H, and we add an external perturbation $f(t)\, O_1$, (H may contain interaction).

$$H + f(t)\, O_1 .$$

We assum that at $t=-\infty$, $f(t)\to 0$, and then the eigenstates $|\psi_n\rangle$ correspond to H. At time t, we have

$$|\psi_n(t)\rangle = T\left[e^{-i\int_{-\infty}^{t} dt' \left[H + f(t)\, O_1\right]}\right] |\psi_n\rangle, \quad \text{where } T$$

is the time-ordering operator.

Assuming $f(t)$ is small, we expand

$$T\left[e^{-i\int_{-\infty}^{t} dt' (H + f(t) O_1)}\right] = T\left[e^{-i\int_{-\infty}^{t} dt' H} \left(1 + (-i)\int_{-\infty}^{t} dt'' f(t'')\, O_1\right)\right]$$

$$= e^{-i(t-t_{-\infty})H} - i\int_{-\infty}^{t} dt'' \, e^{-i\int_{t''}^{t} dt' H} \left[f(t'')\, O_1\right] e^{-i\int_{-\infty}^{t''} dt' H}$$

$$= e^{-i(t-t_{-\infty})H} - i\int_{-\infty}^{t} dt' \, e^{-iH(t-t')}\left[f(t')\, O_1\right] e^{-iH(t'-t_{-\infty})}$$

$$\Rightarrow |\psi_n(t)\rangle = e^{-i\int_{t_{-\infty}}^{t} dt' H} |\psi_n(t)\rangle + \delta |\psi_n(t)\rangle$$

with $\delta|\psi_n(t)\rangle = -i\int_{t_{-\infty}}^{t} dt'\, e^{-iH(t-t')} [f(t') O_1]\, e^{-iH(t'-t_{-\infty})} |\psi_n\rangle$

$$\boxed{\begin{aligned}&\delta|\psi_n(t)\rangle = -i\int_{t_{-\infty}}^{t} dt'\, f(t')\, e^{-iH(t-t_{-\infty})}\, O_1(t')\, |\psi_n\rangle \\ &\text{where } O_1(t) = e^{iH(t-t_{-\infty})}\, O_1\, e^{-iH(t-t_{-\infty})} \leftarrow \text{interaction picture.}\end{aligned}}$$

§. Calculation of Physical quantity O_2

$$\delta O_2(t) = \langle\psi_n(t)|\, O_2\, |\psi_n(t)\rangle - \langle\psi_n|\, e^{iH(t-t_{-\infty})}\, O_2\, e^{-iH(t-t_{-\infty})} |\psi_n\rangle$$

$$= -i\int_{t_{-\infty}}^{t} dt' \Big\{ \langle\psi_n|\, e^{i(t-t_{-\infty})H}\, O_2\, e^{-iH(t-t_{-\infty})}\, f(t')\, O_1(t')\, |\psi_n\rangle - \langle\psi_n|\, O_1(t') f(t')\, e^{iH(t-t_{-\infty})}\, O_2\, e^{-i(t-t_{-\infty})H} |\psi_n\rangle$$

$$= -i\int_{t_{-\infty}}^{t} dt'\, \langle\psi_n|\, [O_2(t), O_1(t')]\, |\psi_n\rangle\, f(t')$$

we define retarded Green's function

$$\boxed{D(t-t') = -i\,\Theta(t-t')\,\langle\, |\,[O_2(t), O_1(t')]\,|\,\rangle},$$

where $\langle|\cdots|\rangle$ means $\langle\psi_0|\cdots|\psi_0\rangle$ at zero temperature, and thermal average at finite T. We have $\boxed{\delta O_2(t) = \int_{-\infty}^{+\infty} dt'\, D(t,t')\, f(t')}$

A simple example: Hamonic oscillater in an external E field

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2, \quad \text{and} \quad H_1 = -exE$$

· Calculate polarizibility

① $|\psi_0\rangle = |0\rangle + \sum_n |n\rangle\langle n| H_1 |0\rangle / (E_0 - E_n)$

$$d = \langle\psi_0| ex |\psi_0\rangle = -2E \sum_{n>0} \frac{\langle 0| ex|n\rangle\langle n|ex|0\rangle}{E_0 - E_n} = 2e^2 E \frac{\langle 0|x^2|0\rangle}{\omega_0}$$

$$\Rightarrow \quad \chi = 2e^2 \frac{\langle 0|x^2|0\rangle}{\omega_0}.$$ — straight forward calculation.

② method of Green's function:

Set $H_1(t) = -exE\, e^{-\eta|t|}$, and $\eta \to 0^+$

$$d(t) = \int_{-\infty}^{+\infty} dt'\, e^2 D(t-t')\, e^{\eta t'} (-E)$$

$$D(t-t') = -i\Theta(t-t') \langle 0| [x(t), x(t')] |0\rangle$$

$$= -i\Theta(t-t') \{ \langle 0| e^{iHt} x e^{-iH(t-t')} x e^{-iHt'} |0\rangle - \langle 0| e^{iHt'} x e^{-iH(t'-t)} x e^{-iHt} |0\rangle$$

$$= -i\Theta(t-t') [\langle 0|x|1\rangle\langle 1|x|0\rangle (e^{-i(E_1-E_0)(t-t')} - e^{i(E_1-E_0)(t-t')})$$

$$= -2\Theta(t-t') \langle 0|x^2|0\rangle \sin \omega_0 (t-t')$$

The d at zero frequency $\boxed{d(\omega=0) = -e^2 D(\omega=0)\, E}$, we need to calculate $D(\omega)$.

$$D(\omega) = \int_{-\infty}^{+\infty} d(t-t')\, e^{i(\omega+i\eta)(t-t')} (-i)\,\Theta(t-t')\langle 0|x^2|0\rangle \left(e^{-i\omega_0(t-t')} - e^{i\omega_0(t-t')}\right)$$

$\omega+i\eta \leftarrow$ control the convergence of the integral

$$\Rightarrow D(\omega) = -i\int_0^{+\infty} dt\, \langle 0|x^2|0\rangle \left[e^{i(\omega-\omega_0+i\eta)t} - e^{i(\omega+\omega_0+i\eta)t}\right]$$

$$= \langle 0|x^2|0\rangle\, i\left[\frac{1}{i(\omega-\omega_0+i\eta)} - \frac{1}{i(\omega+\omega_0+i\eta)}\right]$$

$$= \langle 0|x^2|0\rangle\left[\frac{1}{\omega-\omega_0+i\eta} - \frac{1}{\omega+\omega_0+i\eta}\right] = \langle 0|x^2|0\rangle \frac{2\omega_0}{\omega^2-\omega_0^2+i\,\mathrm{sgn}(\omega)\,\eta}$$

$$\Rightarrow D(\omega=0) = \frac{-2\langle 0|x^2|0\rangle}{\omega_0}, \quad \text{and } d(\omega=0) = \frac{2e^2\langle 0|x^2|0\rangle}{\omega_0}E.$$

§ Time-ordered correlation function and path integral harmonic oscillator

First assume $t>t'$, the first term of the retarded Green's function

$$\langle 0|O_2(t)\,O_1(t')|0\rangle = \langle 0|\,U^\dagger(t,-\infty)\; O_2\; U(t,t')\; O_1\; U(t',-\infty)\,|0\rangle$$

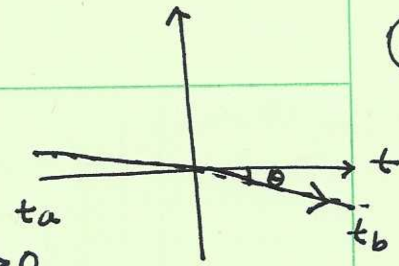
$$= \langle 0|\,U^\dagger(+\infty,-\infty)\; U(+\infty,t)\; O_2\; U(t,t')\; O_1\; U(t',-\infty)\,|0\rangle$$

- assume adibatic process, the system return to $|0\rangle$ at $t\to+\infty$,

$\langle 0|U(+\infty,-\infty)|0\rangle$ is a phase.

$$\Rightarrow \boxed{\langle 0|O_2(t)\,O_1(t')|0\rangle = \frac{\langle 0|U(+\infty,t)\,O_2\,U(t,t')\,O_1\,U(t',-\infty)|0\rangle}{\langle 0|U(+\infty,-\infty)|0\rangle}}$$

we generalize the time-evolution to complex plane



$$\mathcal{U}^{\theta}(t_b, t_a) = e^{-i\{H(t_b - t_a)\, e^{-i\theta}\}}, \quad t_b > t_a \text{ and } \theta > 0.$$

we can choose arbitary state $|\psi\rangle$, after we apply $\mathcal{U}^{\theta}(t_b, t_a)|\psi\rangle$ as $t_b - t_a \to +\infty$, only the ground state is projected out. Then we have

$$\langle 0|\, O_2(t)\, O_1(t')\,|0\rangle = \frac{\langle\psi|\,\mathcal{U}^{\theta}(\infty, t)\, O_2\, \mathcal{U}^{\theta}(t, t')\, O_1\, \mathcal{U}^{\theta}(t', -\infty)\,|\psi\rangle}{\langle\psi|\,\mathcal{U}^{\theta}(\infty, -\infty)\,|\psi\rangle}$$

let's choose $|\psi\rangle = \delta(x - x_0) = |x_0\rangle$ as position eigenstate

$$\Rightarrow \langle x_0|\, \mathcal{U}^{\theta}(\infty, -\infty\, |x_0\rangle = \int \mathcal{D}x(t)\; e^{i\int_{-\infty}^{+\infty} dt\, \mathcal{L}(x, \dot{x})}$$

path initial & final position at x_0

$$\langle x_0|\, \mathcal{U}^{\theta}(\infty\, t)\, \underbrace{O_2}_{(x(t))}\, \mathcal{U}^{\theta}(t\, t')\, O_1(x(t'))\, \mathcal{U}^{\theta}(t', -\infty)\,|x_0\rangle$$

$$= \int \mathcal{D}x(t)\;\; e^{i\int_{t}^{+\infty} dt\, \mathcal{L}(x, \dot{x})}\; O_2(x(t))\; e^{i\int_{t'}^{t} dt\, \mathcal{L}(x x')}\; O_1(x(t'))\; e^{i\int_{-\infty}^{t'} dt\, \mathcal{L}(x x')}$$

$$= \int \mathcal{D}x(t)\;\; O_2(x(t))\, O_1(x(t'))\;\; e^{i\int_{-\infty}^{+\infty} dt\, \mathcal{L}(x, x')}$$

where O_1, and O_2 are functions of coordinates. ~~the~~ The integral argument $t \to t e^{-i\theta}$

The key point of the expression of $\int Dx(t)\, O_2(x(t))\, O_1(x(t'))\, e^{i\int_{-\infty}^{+\infty} dt\, \mathcal{L}}$ is that: when we map it to operator formalism, the operator should be time-ordered! Thus if $t' > t$, we have

$$\int Dx(t)\, O_2(x(t)) O_1(x(t'))\, e^{i\int_{-\infty}^{+\infty} dt\, \mathcal{L}} = \int Dx(t)\, e^{i\int_{t'}^{+\infty} dt\, \mathcal{L}}\, O_1(x(t'))\, e^{i\int_{t}^{t'} dt\, \mathcal{L}}\, \underbrace{O_2(x(t))\, e^{i\int_{-\infty}^{t} dt\, \mathcal{L}}}$$

$$= \langle x_0 | U^0(\infty, t')\, O_1(x(t'))\, U^0(t', t)\, O_2(x(t))\, U^0(t, -\infty) | x_0 \rangle$$

$\Rightarrow$ Combine both cases of $t > t'$ and $t' > t$, we have

$$\boxed{\frac{\int Dx(t)\, O_2(x(t))\, O_1(x(t'))\, e^{i\int_{-\infty}^{+\infty} dt\, \mathcal{L}(x, x')}}{\int Dx(t)\, e^{i\int_{-\infty}^{+\infty} dt\, \mathcal{L}(x, x')}} = T\langle 0 | O_2(t)\, O_1(t') | 0 \rangle}$$

we define the time-ordered Green's function as

$$G(t_2, t_1) = -i\langle 0 | T(O_2(t_2)\, O_1(t_1)) | 0 \rangle = \begin{cases} -i\langle 0 | O_2(t_2)\, O_1(t_1) | 0 \rangle & t_2 > t_1 \\ -i\langle 0 | O_1(t_1)\, O_2(t_2) | 0 \rangle & t_1 > t_2 \end{cases}$$

Example: For harmonic oscillator $H = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 x^2$, we define

$$G(t_2 t_1) = -i \langle 0 | T\, x(t_2)\, x(t_1) | 0 \rangle ,$$ then

$$i G(t_2 t_1) = \frac{\int Dx(t)\; x(t_2) x(t_1)\; e^{i \int_{-\infty}^{+\infty} dt\; -\frac{x}{2}\left(m \frac{d^2}{dt^2} x\right) - \frac{m}{2}\omega_0^2 x^2}}{\int Dx(t)\; e^{i \int_{-\infty}^{+\infty} dt\; -\frac{x}{2}\left(m \frac{d^2}{dt^2} x\right) - \frac{m}{2}\omega_0^2 x^2}}$$

"t" is understood as $t \to t e^{-i\theta}$ with $\theta = 0^+$,

and $\frac{d^2}{dt^2} \to \frac{d^2}{dt^2} e^{2i\theta}$.

According to the formula of Gaussian integral

$$\int \prod_i dx_i\; x_{i_1} x_{i_2}\, e^{-\frac{1}{2} x_i A_{ij} x_j} \Big/ \int \prod_i dx_i\, e^{-\frac{1}{2} x_i A_{ij} x_j} = (A^{-1})_{i_1 i_2}$$

the, A operator is just

$$\left\{ i \left[m \left(e^{i\theta} \frac{d}{dt}\right)^2 + \omega_0^2 e^{-i\theta} \right] \right\}^{-1}$$

$\Rightarrow$ $iG(t_2, t_1)$ satisfies the different equation

$$iG(t_2, t_1) = \left\{ i \left[m \left(e^{i\theta} \frac{d}{dt}\right)^2 + \omega_0^2 e^{-i\theta} \right] \right\}^{-1}_{t_2, t_1}$$

$$\Rightarrow \left[m e^{2i\theta} \frac{d^2}{dt_2^2} + \underset{\widehat{e^{-i\theta}}}{m\omega_0^2} \right] G(t_2, t_1) = -\delta(t_2, t_1)$$

change to frequency space

$$G(\omega) = \int dt\, e^{i\omega t}\, G(t,0) \Rightarrow$$

$$\left(e^{i2\theta}\, m\,\omega^2 - m\omega_0^2\, e^{-i\theta}\right) G(\omega) = 1$$

only $\omega = \omega_0$, we need to worry, the imaginary part, and we only need to know the imaginary part $\rightarrow i0^+$, or $i0^-$.

Apparently, the above Eq. $\rightarrow$

$$\left\{m(\omega^2 - \omega_0^2) + i0^+\right\} G(\omega) = 1$$

or $$\boxed{G(\omega) = \frac{1/m}{\omega^2 - \omega_0^2 + i0^+}}$$

$G(\omega)$ is just the inverse of the operator $-m\frac{d^2}{dt^2} - m\omega_0^2$.

$G(\omega)$ canbe expressed as (time-ordered)

$$G(\omega) = \frac{1}{2\omega_0 m}\left[\frac{1}{\omega - \omega_0 + i\,\mathrm{sgn}\,\omega_0\,\eta} - \frac{1}{\omega + \omega_0 + i\,\mathrm{sgn}\,\omega_0\,\eta}\right]$$

Compare the retarded one, we derived before. Plug in $\langle 0|x^2|0\rangle = \frac{1}{2m\omega_0}$,

$$\Rightarrow D(\omega) = \frac{1}{2\omega_0 m}\left[\frac{1}{\omega - \omega_0 + i\eta} - \frac{1}{\omega + \omega_0 + i\eta}\right]$$

§ Relation between different Green functions

The response function (retaroteel) is directly related to physical observables, but time-ordered one is easy to calculate through path integrals. Both of them can be connected by the imaginary Green's function, or, Matsubara Green's function, defined as

$$\boxed{\mathcal{G}(\tau) = -\langle T_\tau [O_2(\tau) O_1(0)] \rangle, \text{ where } O(\tau) = e^{H\tau} O e^{-H\tau} \quad (-\beta \le \tau \le \beta)}$$

if O_2, O_1 are bosonic operators, they satisfy

$$\mathcal{G}(\tau) = \mathcal{G}(\tau+\beta),$$

if O_2, O_1 are fermionic operaters, they satisfy

$$\mathcal{G}(\tau) = -\mathcal{G}(\tau+\beta). \qquad \longleftarrow \text{Please prove.}$$

$\Rightarrow$ For bosonic operators

$$\mathcal{G}(i\omega_n) = \frac{1}{2}\left[\int_{-\beta}^{0} d\tau\, \mathcal{G}(\tau) e^{i\tau\omega_n} + \int_0^\beta d\tau\, \mathcal{G}(\tau) e^{i\tau\omega_n}\right]$$

$$= \int_0^\beta d\tau\, \mathcal{G}(\tau) e^{i\tau\omega_n} \frac{1+e^{i\omega_n\beta}}{2}, \quad \text{only for } \omega_n = \frac{2n\pi}{\beta}$$

Fourier componets • suries

$$\boxed{\mathcal{G}(i\omega_n) = \int_0^\beta d\tau\, \mathcal{G}(\tau) e^{i\tau\omega_n} \quad \text{for } \omega_n = \frac{2n\pi}{\beta}}$$

Similarly for fermionic operators

$$\boxed{\mathcal{G}(i\omega_n) = \int_0^\beta d\tau \; \mathcal{G}(\tau)\, e^{i\tau\omega_n} \quad \text{for } \omega_n = \frac{(2n+1)\pi}{\beta}}$$

At $\tau > 0$,

$$\mathcal{G}(\tau) = -\, e^{\beta\Omega} \sum_{nm} \langle n| e^{-\beta H} e^{H\tau} O_2 e^{-H\tau} |m\rangle\langle m| O_1 |n\rangle$$

(where $Z = e^{\beta\Omega}$)

$$= -e^{\beta\Omega} \sum_{nm} e^{-\beta E_n} e^{(E_n - E_m)\tau} \langle n| O_2 |m\rangle\langle m| O_1 |n\rangle$$

$$\Rightarrow \mathcal{G}(i\omega_n) = -e^{\beta\Omega} \int_0^\beta d\tau \; e^{i\omega_n\tau} e^{(E_n - E_m)\tau} \langle n| O_2 |m\rangle\langle m| O_1 |n\rangle e^{-\beta E_n}$$

$$\boxed{\mathcal{G}(i\omega_n) = e^{\beta\Omega} \sum_{nm} \frac{\langle n| O_2 |m\rangle\langle m| O_1 |n\rangle}{i\omega_n + E_n - E_m} \left(e^{-\beta E_n} \mp e^{-\beta E_m} \right)}$$

$\mp$ for bosonic / fermionic operaters, respectively.

This expression is often called Lehman - representation.

Now $D_{ret}(t-t') = -i\,\Theta(t-t') \left\langle \left| [O_2(t),\, O_1(t')]_\mp \right| \right\rangle$

$$= -i\,\Theta(t-t')\, e^{\beta\Omega} \sum_{nm} \langle n| O_2 |m\rangle\langle m| O_1 |n\rangle e^{i(E_n - E_m)(t-t')}$$

$$\times \left(e^{-\beta E_n} \mp e^{-\beta E_m} \right)$$

$$\boxed{\begin{aligned} D_{ret}(\omega) &= \int_{-\infty}^{+\infty} d(t-t')\, e^{i\omega(t-t')}\, D_{ret}(t-t') \\ &= \sum_{nm} e^{\beta\Omega} \frac{\langle n|O_2|m\rangle\langle m|O_1|n\rangle \left(e^{-\beta E_n} \mp e^{-\beta E_m}\right)}{\omega + (E_n - E_m) + i\eta} \end{aligned}}$$

Next check the time-ordered Green's function

$$G(t-t') = -i\Theta(t-t')\langle|O_2(t)\,O_1(t')|0\rangle \mp (-i\Theta(t'-t)\langle|O_1(t')\,O_2(t)|\rangle$$

$$= -i\Theta(t-t')\, e^{\beta\Omega} \sum_{nm} \langle n|O_2|m\rangle\langle m|O_1|n\rangle\, e^{-\beta E_n}\, e^{i(E_n-E_m)(t-t')}$$

$$\pm (-i\Theta(t'-t))\, e^{\beta\Omega} \sum_{nm} \langle n|O_2|m\rangle\langle m|O_1|n\rangle\, e^{-\beta E_m}\, e^{i(E_n-E_m)(t-t')}$$

$$\overset{\Theta}{=} e^{\beta\Omega} \sum_{nm} \langle n|O_2|m\rangle\langle m|O_1|n\rangle\, e^{i(E_n-E_m)(t-t')} \left(-i\Theta(t-t')\,e^{-\beta E_n} \pm (-i\Theta(t'-t)\, e^{-\beta E_m}\right)$$

$$\Rightarrow \boxed{G(\omega) = \sum_{nm} e^{\beta\Omega} \langle n|O_2|m\rangle\langle m|O_1|n\rangle \left\{ \frac{e^{-\beta E_n}}{\omega + E_n - E_m + i\eta} \mp \frac{e^{-\beta E_m}}{\omega + E_n - E_m - i\eta} \right\}.}$$

$$\cancel{G(\omega) = \sum_{nm} e^{\beta\Omega} \langle n|O_2|m\rangle\langle m|O_1|n\rangle \left\{ \frac{e^{-\beta E_n}}{\omega + E_n - E_m + i\eta} \mp \frac{e^{-\beta E_m}}{\omega + E_n - E_m - i\eta} \right\}}$$

thus:

$D_{ret}(\omega)$ can be arrived by analytical continuition

$$\mathcal{G}(i\omega_n)\big|_{i\omega_n \to \omega + i\eta}.$$

at zero temperature: $\beta \to +\infty$

$$G(\omega) = \sum \langle 0|O_2|m\rangle\langle m|O_1|0\rangle \frac{1}{\omega + (E_0 - E_m) + i\eta} \mp \langle n|O_2|0\rangle\langle 0|O_1|n\rangle \frac{1}{\omega + (E_n - E_0) - i\eta}$$

$$= \mathcal{G}(i\omega_n)\big|_{i\omega_n \to \omega + i\,\mathrm{sgn}(\omega)\eta}.$$

Define the spectral function as

$$A_+(\omega) = 2\pi \sum_{nm} e^{\beta\Omega} \langle n|O_2|m\rangle\langle m|O_1|n\rangle\, e^{-\beta E_n}\, \delta(\omega + E_n - E_m)$$

$$A_-(\omega) = 2\pi \sum_{nm} e^{\beta\Omega} \langle n|O_2|m\rangle\langle m|O_1|n\rangle\, e^{-\beta E_m}\, \delta(\omega + E_n - E_m)$$

$$= 2\pi \sum_{nm} e^{\beta\Omega} \langle m|O_2|n\rangle\langle n|O_1|m\rangle\, e^{-\beta E_n}\, \delta(\omega - (E_n - E_m))$$

$$A(\omega) = -2\,\mathrm{Im}\, D_{ret}(\omega) = A_+(\omega) \mp A_-(\omega)$$

$$A_+(\omega) = e^{\beta\omega} A_-(\omega)$$

we have $\quad iG(t)\big|_{t>0} = \int \frac{d\omega}{2\pi} A_+(\omega)\, e^{-i\omega t}$

$$iG(t)\big|_{t<0} = \pm \int \frac{d\omega}{2\pi} A_-(\omega)\, e^{-i\omega t}$$

if $O_1 = O_2^\dagger$, we have $A_+(\omega) = \frac{-1}{1 \pm e^{-\beta\omega}} \left(2\,\mathrm{Im}\, G(\omega) \right)$

$$= -[1 \pm n_{B/F}(\omega)]\left(2\,\mathrm{Im}\, D(\omega)\right)$$

Check: $-2\,\mathrm{Im}\, G(\omega) = A_+(\omega) \mp (-) A_-(\omega) = A_+(\omega)\left(1 \pm e^{-\beta\omega}\right)$

$\Rightarrow A_+(\omega) = -2\,\mathrm{Im}\, G(\omega) / (1 \pm e^{-\beta\omega})$

Lect 5 Perturbation theory of fermions

§ Free fermions

We will use operator formalism and path integral in parallel for comparison. For the imaginary time Green's function

$$\mathcal{G}_k = - T_\tau \langle | \hat{\psi}_k(\tau) \psi^\dagger_k(\tau') | \rangle ,$$

and in the path integral: $\mathcal{G}_k = - \langle \psi_k(\tau) \bar{\psi}_k(\tau') \rangle$ based on the partition function:

$$Z_0 = \int D\bar{\psi}_k D\psi_k \exp\left[- \int_0^\beta d\tau \, \bar{\psi}_k(\tau) \left[\partial_\tau + \xi_k \right] \psi_k(\tau) \right]$$

Then $\mathcal{G}_k(\tau, \tau') = - (\partial_\tau + \xi_k)^{-1}_{\tau, \tau'}$ ← continous matrix

The continous matrix should be understood as

$$\boxed{\begin{array}{l} \text{identity } \delta(\tau - \tau'), \text{ and matrix product:} \\ \displaystyle\int_0^\beta d\tau' \, (\partial_\tau + \xi_k)_{\tau, \tau'} \mathcal{G}_k(\tau', \tau'') = - \sum_n \delta(\tau - \tau'' - n\beta) (-)^n \end{array}}$$

due to the anti-periodic boundary condition

what's $(\partial_\tau)_{\tau\tau'}$?

$$(\partial_\tau)_{\tau\tau'} = \lim_{\epsilon \to 0} \frac{\delta(\tau - \tau') - \delta(\tau - \epsilon - \tau')}{\epsilon} \quad \text{plug in}$$

$$(\xi_k)_{\tau\tau'} = \xi_k \delta(\tau - \tau')$$

$$\Rightarrow \boxed{(\partial_\tau + \xi_k) \mathcal{G}_k(\tau - \tau') = - \sum_n \delta(\tau - \tau' - n\beta) (-)^n}$$

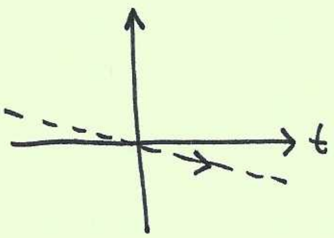
Fourier transform $\mathcal{G}_k(\tau-\tau') = \frac{1}{\beta}\sum_{i\omega_n} \mathcal{G}_k(i\omega_n)\, e^{-i\omega_n(\tau-\tau')}$

$$\Rightarrow \mathcal{G}_k(i\omega_n) = \frac{1}{i\omega_n - \xi_k}$$

$\longrightarrow$ real time / time ordered $\quad Z_0 = \int D\bar{\psi}_k(t) D\psi_k(t)\, e^{i\int dt\, \mathcal{L}_0}$

where $\mathcal{L}_0 = i\bar{\psi}_k(t)\partial_t\psi_k(t) - \bar{\psi}_k \xi_k \psi_k$

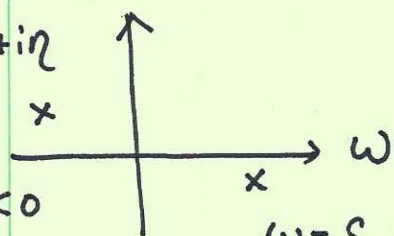
$$i G_k(t\,t') = T\langle|\hat{\psi}_k(t)\hat{\psi}^{\dagger}_k(t')|\rangle = \langle \psi_k(t)\bar{\psi}_k(t')\rangle = -\left(i(i\partial_t - \xi_k)\right)^{-1}_{t,t'}$$



$$t \to t(1-i\eta) \to \omega \to \omega(1+i\eta)$$

$$\to G_k(\omega) = \frac{1}{i(-i\omega(1+i\eta)) - \xi_k} = \frac{1}{\omega - \xi_k + i\,\mathrm{sgn}(\omega)\cdot\eta}$$

$\omega = \xi_k + i\eta$



for $\omega < 0$

$\omega = \xi_k - i\eta$ for $\omega > 0$

- Retarded $\quad i\omega_n \to \omega + i\eta$

$$i G^R(t,t') = \Theta(t-t')\langle|\{\psi_k(t)\, \psi^{\dagger}(t')\}|\rangle \Rightarrow G^R(\omega) = \frac{1}{\omega - \xi_k + i\eta}$$

§ Perturbation theory

Consider a system with Hamitonian H, we have defined time-ordered Green's function in the last lecture

$G(t_2, t_1) = -i\langle\Omega| T\hat{O}_2(t_2)\hat{O}_1(t_1)|\Omega\rangle$ where $|\Omega\rangle$ is the ground state of H. we have also showed that it can be expressed as path integrals as

$$i\,G(t_2,t_1) = \frac{\int D\bar{\psi}\, D\psi\ O_2(\bar{\psi}(t_2),\psi(t_2))\, O_1(\bar{\psi}(t_1), \psi(t_1))\, e^{i\int \mathcal{L}\, dt}}{\int D\bar{\psi} D\psi\ e^{i\int \mathcal{L}\, dt}}$$

[We extend to fermion and many-body system].

Now, we decompose $H = H_0 + H_I$, and assume H_I is small.

then $\mathcal{L} = \bar{\psi}(i\partial_t\psi - H = \bar{\psi}(i\partial_t)\psi - H_0 - H_I = \mathcal{L}_0 + \mathcal{L}_I$

where $\mathcal{L}_I = -H_I$. But remember that H_0 and H_I originally are operators, but when we write Lagrangian, operators become Complex field or grassman field. To simplify notation, we often do not distinguish carefully.

Now, we will develope a perturbation theory using $\mathcal{L}_I$ as small quantity.

we define $Z = \int D\bar{\psi} D\psi \; e^{i\int dt \, \mathcal{L}} = \int D\bar{\psi} D\psi \; e^{i\int dt \, \mathcal{L}_0 + \mathcal{L}_I}$

and $Z_0 = \int D\bar{\psi} D\psi \; e^{i\int dt \, \mathcal{L}_0}$. Then

$$\frac{Z}{Z_0} = \frac{\int D\bar{\psi} D\psi \; e^{i\int dt \, \mathcal{L}_0} \sum_{n=0}^{\infty} \frac{i^n}{n!} \left[\int dt \, \mathcal{L}_I\right]^n}{\int D\bar{\psi} D\psi \; e^{i\int dt \, \mathcal{L}_0}} = \sum_{n=0}^{\infty} \frac{i^n}{n!} \left\langle \left(\int dt \, \mathcal{L}_I\right)^n \right\rangle_0$$

where $\langle \cdots \rangle_0$ means averging with respect to Z_0.

we can re-exponetialized

$$\frac{Z}{Z_0} = \left\langle e^{i\int dt \, \mathcal{L}_I} \right\rangle_0 = \exp\left[1 + \left\langle i\int dt \, \mathcal{L}_I \right\rangle_0 + \frac{1}{2!}\left\{\left\langle \left(i\int dt \, \mathcal{L}_I\right)^2 \right\rangle - \left\langle i\int dt \, \mathcal{L}_I \right\rangle^2\right\} + \cdots\right.$$

$$\Rightarrow \ln \frac{Z}{Z_0} = 1 + \left\langle i\int dt \, \mathcal{L}_I \right\rangle_0 + \frac{1}{2!}\left\{ \left\langle \left(i\int dt \, \mathcal{L}_I\right)^2 \right\rangle_0 - \left(\left\langle i\int dt \, \mathcal{L}_I \right\rangle_0\right)^2 \right\} + \frac{1}{3!} \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle \left(i\int dt \, \mathcal{L}_I\right)^n \right\rangle_{0,\,\text{connected}}$$

$$= \left\langle e^{i\int dt \, \mathcal{L}_I} \right\rangle_{0,\,\text{connected}} \quad \longleftarrow \text{Linked cluster expansion.}$$

Let us assume the interaction takes the form of

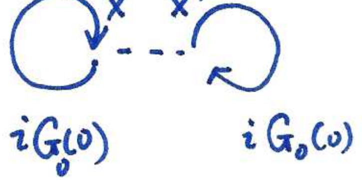
$$\int dt \ \mathcal{L}_I = \int dx\, dx' \ \bar{\psi}(x)\, \bar{\psi}(x')\, V(x-x')\, \psi(x')\, \psi(x)$$ here $x = (\vec{x}, t)$

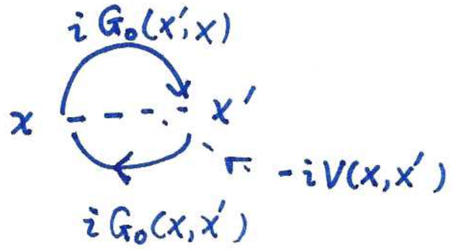
The first order contribution

$$\frac{Z}{Z_0} = 1 + \left\langle \int dx\, dx' \ \bar{\psi}(x)\, \bar{\psi}(x')\, (-i\, V(x-x'))\, \psi(x')\, \psi(x) \right\rangle_0 + \cdots$$

$$= 1 + \int dx\, dx' \ (-i\, G_0(0))\, (-i V(x,x'))\, (-i\, G_0(0))$$

$$+ \int dx\, dx' \ (-i\, G_0(x-x'))\, (-i V(x,x'))\, (i\, G_0(x'-x)) + \cdots$$





§ Self-energy and interaction:

$$i\, G_{\alpha\beta}(x,t) = \frac{\int D\bar{\psi} D\psi \ \psi_\alpha(x\,t)\, \bar{\psi}_\beta(0\,0)\, e^{i\int dt\, \mathcal{L}}}{\int D\bar{\psi} D\psi \ e^{i\int dt\, \mathcal{L}}},$$

restore electron spin index

where $\mathcal{L} = i\bar{\psi}_\alpha \partial_t \psi_\alpha - H$.

$$= \frac{\left[\int D\bar{\psi} D\psi \ \psi_\alpha(x,t)\, \bar{\psi}_\beta(0,0)\, e^{i\int dt\, \mathcal{L}_{int}}\, e^{i\int dt\, \mathcal{L}_0}\right] / Z_0}{\left[\int D\bar{\psi} D\psi \ e^{i\int dt\, \mathcal{L}_{int}}\, e^{i\int dt\, \mathcal{L}_0}\right] / Z_0}$$

$$= \langle \psi_\alpha(xt)\bar{\psi}_\beta(00)\, e^{i\int dt\, L_{int}} \rangle_0 \Big/ \langle e^{i\int dt\, L_{int}} \rangle_0$$

$$= \langle \psi_\alpha(x,t)\bar{\psi}_\beta(00)\, e^{i\int dt\, L_{int}} \rangle_{0,\ \text{connected}}$$

First order

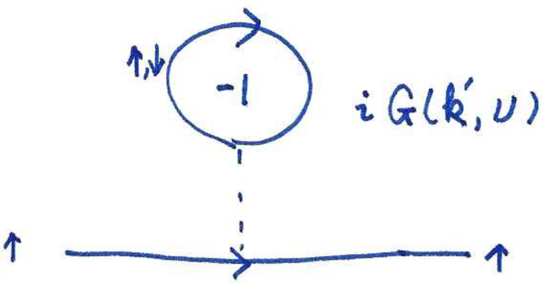
if spin is conserved

$$iG_{\alpha\beta}(xt)\,\delta_{\alpha\beta} = \langle \psi_\alpha(xt)\bar{\psi}_\beta(00)\rangle + \langle \psi_\alpha(xt)\bar{\psi}_\beta(00)\,(-i)\int dt'\, H_I(t')\rangle_{0,\ \text{connect}}$$

$\Rightarrow$

$$iG_{\alpha\beta}(k\omega) = iG_0(k,\omega) + (-)\,(iG_0(k,\omega))^2\,(-iV(0))\,\frac{1}{V}\sum_{k'\alpha'}\int\frac{d\nu}{2\pi}\, iG_0(k',\nu)\, e^{i0^+\nu}$$

$$+ (iG_0(k,\omega))\,\frac{1}{V}\sum_q\int\frac{d\nu}{2\pi}\, iG_0(k-q,\omega-\nu)\,(-iV(q))\, e^{-i0^+\nu}$$



$iG(k',\nu)$

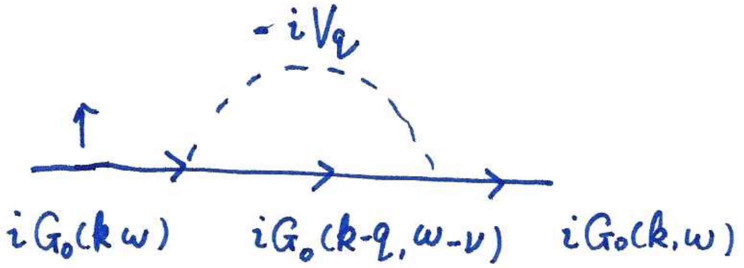
$iG_0(k,\omega)$ $\quad iG_0(k,\omega$

$$(-i)\quad \psi(k,t')\,\bar{\psi}(k,t)\,\frac{1}{2V}\sum \bar{\psi}(k_1+q,t'')\,\bar{\psi}(k_2-q,t'')\,V(q)\,\psi(k_2,t'')\,\psi(k,t'')$$

Four points: ① $q=0$, ② $\frac{1}{2}\times 2$, ③ spin summation, ④ overall $(-)$ sign

④ $\langle \bar{\psi}(k_2,t'')\,\psi(k_2 t'')\rangle \longrightarrow -T\langle \psi(k_2,t''-0^+)\,\psi(k_2,t'')\rangle$

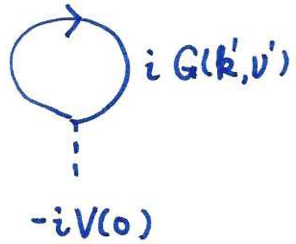
$$= -\int\frac{d\nu}{2\pi}\,\psi(k_2,\nu)\, e^{-i\nu(-0^+)}$$

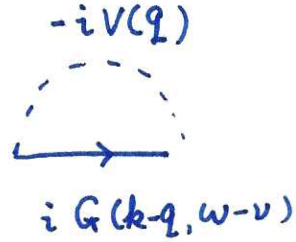


$$i\,G(k,\omega) = i\,G_0(k,\omega) + \left[i\,G_0(k,\omega)\right]^2 \left(-i\,\Sigma(k,\omega)\right)$$

where $$-i\,\Sigma(k,\omega) = (-)\,\frac{-iV(q=0)}{V} \sum_{k'\alpha'} \int \frac{d\nu}{2\pi}\, i\,G_0(k',\nu)\, e^{i0^+\nu}$$

$$+ \frac{1}{V} \sum_{q} \int \frac{d\nu}{2\pi}\, i\,G_0(k-q,\omega-\nu)\,(-iV_q)\, e^{-i0^+\nu}$$

$$\Rightarrow -i\,\Sigma^{(1)}(k,\omega) =$$ 

$+$ 

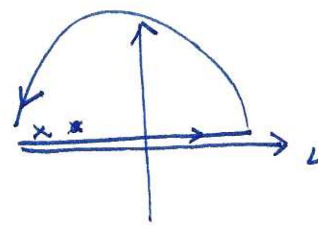
$$iG(k,\omega) = i\,G_0(k,\omega) \sum_{n=0}^{\infty} \left[(i\,G_0(k,\omega))(-i\,\Sigma(k,\omega)) \right]^n$$

$$= \frac{1}{\left[i\,G_0(k,\omega)\right]^{-1} + (-i)\,\Sigma(k,\omega)}$$

$$G(k,\omega) = \frac{1}{G_0(k,\omega) - \Sigma(k,\omega)} = \frac{1}{\omega - \xi_k - \Sigma(k,\omega) + i\,\mathrm{sgn}(\omega)\,\eta}$$

HF level

$$\int \frac{d\nu}{2\pi}\, i\, G_0(k',\nu)\, e^{i0^+\nu} = i\int \frac{d\nu}{2\pi} \frac{1}{\nu - \xi_{k'} + i0^+ \mathrm{sgn}\,\nu}\, e^{i0^+\nu} = -\Theta(-\xi_{k'}) = -n_{k'}$$



$$\int \frac{d\nu}{2\pi}\, i\, G_0(k-q,\omega-\nu)\, e^{-i0^+\nu} = -\Theta(-\xi(k-q)) = -n_{k-q}$$

$$\boxed{\Sigma(k,\omega) = V_0\rho_0 + \frac{1}{V}\sum_q (-n_{k-q})\, V(q)} \leftarrow \text{H-F self-energy}$$

cancel the background charge

For Coulumb potential $V(q) = \dfrac{4\pi e^2}{q^2}$

$$\Sigma_{\text{Fock}} = -\frac{1}{(2\pi)^3}\int_0^{\theta(k'<k_F)} d\vec{k}'\, \frac{4\pi e^2}{|\vec{k}-\vec{k}'|^2},$$

define $\vec{q} = \vec{k}' - \vec{k}$

$k'^2 = k^2 + q^2 + 2kq\cos\theta$

$$= -\frac{4\pi e^2}{(2\pi)^3}\cdot 2\pi \int_0^{+\infty} dq \int_{-1}^{1} d\cos\theta\;\; \theta\left(k_F^2 - (k^2+q^2+2kq\cos\theta)\right)$$

$$= -\frac{2e^2}{\pi} k_f\; \frac{1}{2}\int_0^{\infty} dz \int_{-1}^{1} d\cos\theta\;\; \theta\left(1 - (x^2+z^2+2xz\cos\theta)\right)$$

$$= -\frac{2e^2}{\pi} k_f\, F(x),$$ where $F(x) = \frac{1}{2}\int_0^{\infty} f(z)\,dz$, and

$$f(z) = \begin{cases} 2 & |x+z|<1 \\ \dfrac{1-(x-z)^2}{2x} & \text{otherwise} \\ 0 & |x-z|>1 \end{cases}$$

$$F(x) = \frac{1}{2} + \frac{1-x^2}{4x}\ln\left|\frac{1+x}{1-x}\right|$$

where $z = \dfrac{q}{k_f}$, $x = \dfrac{k}{k_f}$.

Excerise: we can also use Matsubara representatin

$$\mathcal{G}(k, ik_n) = \mathcal{G}^0(k, ik_n) + \mathcal{G}^0(k, ik_n)\,\Sigma(k, ik_n)\,\mathcal{G}(k, ik_n)$$

and then $\mathcal{G}(k, ik_n) = \dfrac{1}{ik_n - \xi_k - \Sigma(k, ik_n)}$

at HF level

$$\Sigma_{HF}(k, ik_n) = \frac{1}{V}\sum_{k', ik_n'} \frac{2V(0)}{ik_n' - \xi_{k'}} + \frac{1}{V}\sum_{q, q_n} \frac{1}{i(k_n - q_n) - \xi_{k-q}} V(q)$$

$$= \int \frac{d\vec{k}'}{(2\pi)^3} V(q=0)\, n_f(\xi_{k'}) - \int \frac{d\vec{q}}{(2\pi)^3} V(q)\, n_f(k-q)$$

which the same as befre.

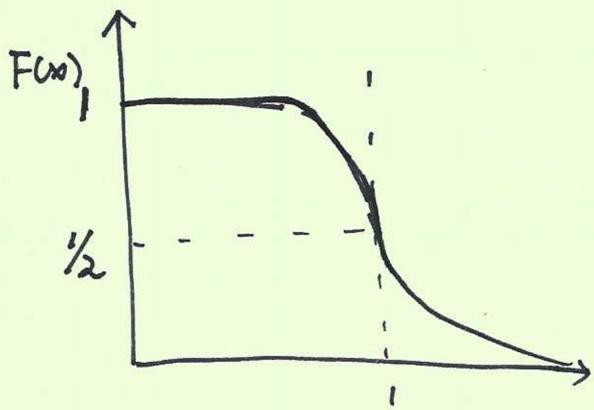
However, HF results have significant problem — artificial divergence of Fermi velocity

$$\delta \varepsilon_{HF}(k) \sim e^2 (k - k_f) \ln\left|\frac{k - k_f}{k_f}\right| ,$$

velocity shift $\delta v(k) = \hbar^{-1} \dfrac{\partial \delta \varepsilon}{\partial k} \sim \ln\left|\dfrac{k_f}{k - k_f}\right| \to \infty .$

→ suppressiun of density of states (DOS)

and $C \sim \dfrac{T}{\ln\left(\frac{T_F}{T}\right)}$ ← factor of DOS suppression.



More about HF: HF-method is to use <u>an optimized free fermion wave function</u> to approximate an interacting system!

$\underbrace{\text{wave function}}_{\text{(slater determinant-like WF)}}$

use $\Psi_{HF} = c^+_{i_1\sigma_1} c^+_{i_2\sigma_2} \cdots c^+_{i_\ell \sigma_\ell} |0\rangle$ to minimize

$$\langle \Psi_{HF} | H | \Psi_{HF} \rangle - \sum_{i\sigma} \lambda_{i\sigma} \int dr\, \phi^*_{i\sigma} \phi_{i\sigma}$$ ← basis of the operater $c^+_{i\sigma}$.

HF self-ansistent Eq

$$\left\{ -\frac{\hbar^2}{2m} \nabla_i^2 + \int d\vec{r}' \frac{\left(\sum_{j\sigma'} n_{j\sigma'} |\phi_j(r')|^2 \right)}{|\vec{r} - \vec{r}'|} \right\} \phi_{i\sigma}(r)$$

$$- \sum_j n_{j\sigma} \int d\vec{r}' \frac{\phi^*_{j\sigma}(\vec{r}') \phi_{j\sigma}(\vec{r})}{|\vec{r} - \vec{r}'|} \phi_{i\sigma}(\vec{r}') = \underset{\substack{\uparrow \\ \text{HF- self energy}}}{\lambda_{i\sigma}} \phi_{i\sigma}(r)$$

$$\lambda_{i\sigma} = \frac{\delta E}{\delta n_{i\sigma}}$$

For homogenous system, the HF wave function is the same as the free Fermi surface. In other words, HF wavefunction is a free fermion WF, <u>no correlation</u>. Any wave function which is not a slater determinat has <u>correlation effect!</u> (Say, Laughlin WF)

HF wavefunction correctly describes that electrons with the same spin avoid each other, but does not describle electron with opposite spin also have correlation!

Exchange hole

$$\langle HF|\rho_\sigma(r)\rho_{\sigma'}(r')|HF\rangle = \sum_{ij} n_{i\sigma} n_{j\sigma'} \{ |\varphi_{k_i}(r)|^2 |\varphi_{k_j}(r)|^2 - \delta_{\sigma\sigma'} \varphi^*_{k_i}(r)\varphi^*_{k_j}(r') \varphi_{k_j}(r)\varphi_{k_i}(r') \}$$

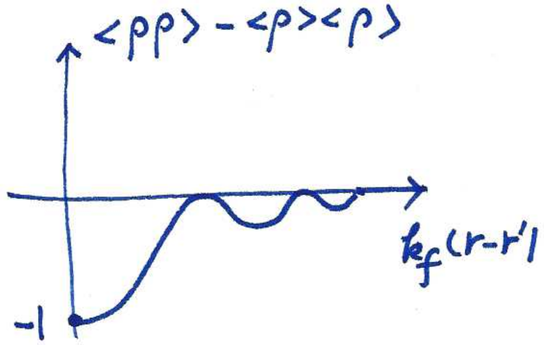
$$\langle HF|\rho_\sigma(r)\rho_{\sigma'}(r')|H_F\rangle - \langle HF|\rho_\sigma(r)|HF\rangle\langle HF|\rho_{\sigma'}(r')|H_F\rangle$$

$$= -\delta_{\sigma\sigma'} \sum_{ij} \delta_{\sigma\sigma'} \varphi^*_{k_i}(r)\varphi^*_{k_j}(r')\varphi_{k_j}(r')\varphi_{k_i}(r) = -\frac{1}{V^2}\sum_{kk'} e^{i(k-k')(r-r')} n_k n_{k'}$$

$$= -\delta_{\sigma\sigma'} \left| \frac{1}{(2\pi)^3}\int d\vec{k}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\theta(k_F-k)\right|^2 = \underbrace{-}_{\delta_{\sigma\sigma'}}\left[\frac{+1}{2\pi^2|r-r'|}\frac{d}{d|r-r'|}\left(\frac{\sin k_f|r-r'|}{|r-r'|}\right)\right]^2$$

$$= -\delta_{\sigma\sigma'}\left(\frac{n}{2}\right)^2 9 \left(\frac{x\cos x - \sin x}{x^3}\right)^2 \quad \text{where } x = k_f|r-r'|$$

Near an electron, it has less possibility to find another electron with the same spin!



lect 11. Free fermion Green's function

§ time-ordered, imaginary time Green's function

$$C_k(t) = e^{iHt} C_k e^{-iHt}$$

$$i G(t_1 - t_2, k_1) \delta_{k_1,k_2} = \langle 0 | T (C_{k_1}(t_1) C^\dagger_{k_2}(t_2) | 0 \rangle = \begin{cases} \langle 0 | C_{k_1}(t_1) C^\dagger_{k_2}(t_2) | 0 \rangle & \text{for } t_1 > t_2 \\ - \langle 0 | C^\dagger_{k_2}(t_2) C_{k_1}(t_1) | 0 \rangle & \text{for } t_1 < t_2 \end{cases}$$

if at finite temperature $\langle 0 | \cdots | 0 \rangle \to \frac{1}{Z} tr [e^{-\beta H} \ldots]$.

$$i G(t, k) = \begin{cases} \theta(t)\, \theta(\xi_k)\, e^{-it\xi_k} - \theta(-t)\, \theta(-\xi_k)\, e^{-i(-t)(-\xi_k)} \Big|_{T = 0\,K} \\ \theta(t)\, (1 - n_F(\xi_k)) e^{-it\xi_k} - \theta(-t)\, n_F(\xi_k)\, e^{-it\xi_k} \Big|_{T > 0} \end{cases}$$

$$\Rightarrow G(\omega, k) = \begin{cases} \dfrac{1}{\omega - \xi_k + i0^+ \,\mathrm{sgn}(\omega)} & \text{for } T = 0\,K \\[2ex] \dfrac{1 - n_F(\xi_k)}{\omega - \xi_k + i0^+} + \dfrac{n_F(\xi_k)}{\omega - \xi_k - i0^+} & \text{for } T > 0 \end{cases} \qquad \boxed{\xi_k = \varepsilon_k - \mu}.$$

$$C_k(\tau) = e^{H\tau} C_k e^{-H\tau}$$

$$\mathcal{G}^\beta(\tau_1, \tau_2; k_1) \delta_{k_1,k_2} = -\frac{1}{Z} tr [e^{-\beta H} C_{k_1}(\tau_1) C^\dagger_{k_2}(\tau_2)] \quad (\text{if } \tau_1 > \tau_2)$$
$$+ \frac{1}{Z} tr [e^{-\beta H} C^\dagger_{k_2}(\tau_2) C_{k_1}(\tau_1)] \quad \text{if } (\tau_2 > \tau_1)$$

for fermions $\mathcal{G}^\beta$ satisfies $\mathcal{G}^\beta(\tau, k) = -\mathcal{G}^\beta(\tau + \beta, k)$

$$\mathcal{G}^\beta(\tau, k) = -\Theta(\tau)\,(1-n_F(\xi_k))\,e^{-\tau\xi_k} + \Theta(-\tau)\,n_F(\xi_k)\,e^{-\tau\xi_k}$$

$$\Rightarrow \mathcal{G}^\beta(\omega_n, k) = \int_0^\beta d\tau\ \mathcal{G}^\beta(\tau, k)\,e^{i\omega_n\tau} = \frac{1}{i\omega_n - \xi_k} \qquad \omega_n = \frac{(2n+1)\pi}{\beta}$$

§ Equal-space Green's function and tunneling :

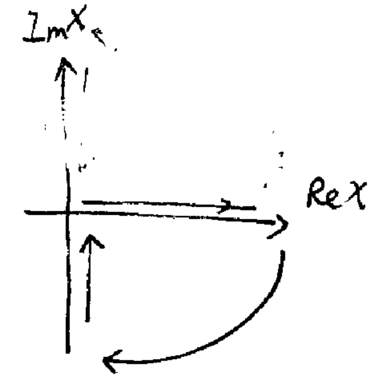
Let's evaluate $\langle T\ c(x_1 t_1) c^\dagger(x_2 t_2)\rangle$ at $x_1 = x_2$

$$i\,G(t,0) = \int \frac{d^d k}{(2\pi)} \left[\Theta(t)\,\Theta(\xi_k)\,e^{-it\xi_k} - \Theta(-t)\,\theta(-\xi_k)\,e^{-it\xi_k}\right]$$

$$\int \frac{d^d k}{(2\pi)^d} = \int_{-\infty}^{+\infty} d\xi\ N(\xi + E_F)\,, \qquad \text{if } N(\xi + E_F) \propto |\xi|^g$$

$$\Rightarrow \Theta(t) \int_0^{+\infty} d\xi\ N(\xi+E_F)\ e^{-it\,(\xi - i0^+)} =$$

$$\propto \Theta(t) \int_0^{+\infty} d\xi\ \ \xi^g\, e^{-it(\xi - i0^+)} = \Theta(t)\ t^{-(1+g)} \int_0^{+\infty} dx\ x^g\ e^{-i(x-i0^+)}$$



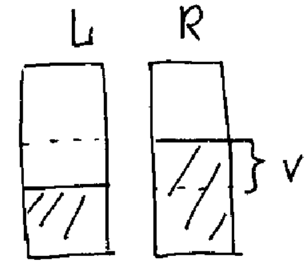
$$\int_0^{+\infty} dx\ \ x^g\ e^{-ix} + \int_{arc} dz\ z^g\ e^{-i(Re\,z + i\,Im\,z)}$$

$$+ \int_{-\infty\cdot i}^{0} dz\ \ z^g\,e^{-iz} = 0$$

$$\Rightarrow \int_0^{+\infty} dx\ x^g\,e^{-ix} = \int_0^{+\infty} d(-ix)\ (-ix)^g\ e^{-x} = (-i)^{g+1}\,\Gamma(g-1)$$

$$\Rightarrow G(t,0) \propto -i\,\mathrm{sgn}(t)\, e^{-i\frac{\pi}{2}(1+g)} \frac{1}{|t|^{1+g}}$$

how to measure this? we can use tunneling spectroscopy.



$$N_{R,L}(\xi) \sim |\xi|^{g_{R,L}}$$

$$H_T = \Gamma\left(e^{iVt} C_R^\dagger C_L + h.c.\right)$$

$$\langle j_T \rangle(t) = -i\int_{-\infty}^{t} dt' \, \langle |[j_T(t),\ H_T(t')]| \rangle$$

$$= -\Gamma^2 \int^t dt' \, e^{iV(t-t')} \langle |[I(t),\ I^\dagger(t')]| \rangle \quad + h.c., \quad \text{where } I(t) = C_R^\dagger(t,0)\, C_l(t,0)$$

$$= -i\Gamma^2 D_R(V) + h.c. = 2\Gamma^2 \,\mathrm{Im}\, D_R(V)$$

$$\langle I(t) I^\dagger(0) \rangle = \langle C_L(t,0) C_L^\dagger(00) \rangle \langle C_R^\dagger(t0)\, C_R(00) \rangle = -G_L(t,0)\,(-)\,G_R(t,0) \quad (\text{for } t>0)$$

$$\propto e^{-i\frac{\pi}{2}(2+g_R+g_L)} \frac{1}{t^{2+g_R+g_L}}$$

$$\cancel{\langle j_T \rangle = -\Gamma^2 \int_{-\infty}^{t} dt' \, e^{iV(t-t')} (t-t')^{-(2+g_R+g_L)} e^{-i\frac{\pi}{2}}}$$

$$\langle I^\dagger(0) I(t) \rangle = \langle C_L^\dagger(00) C_L(t,0) \rangle \langle C_R(00) C_R^\dagger(t,0) \rangle = G_L^*(t,0)\, G_R^*(-t,0) \quad (\text{for } t>0)$$

$$\Rightarrow \langle |[I(t),\ I^\dagger(t')]| \rangle = \propto \left(e^{-i\frac{\pi}{2}(2+g_R+g_L)} - h.c.\right) \frac{1}{|t-t'|^{2+g_R+g_L}}$$

$$= -2i \sin\frac{(2+g_R+g_L)\pi}{2} \quad \frac{1}{|t-t'|^{2+g_R+g_L}}$$

$$\Rightarrow \langle j_T\rangle(t) = +P^2 \underbrace{\int_{-\infty}^{t} dt' \left(e^{iV(t-t')} \frac{i}{|t-t'|^{2+g_R+g_L}} + h.c. \right)}_{2\sin\frac{(2+g_R+g_L)\pi}{2}}$$

$$\propto V^{+(2+g_R+g_L)-1} = V^{+1+(g_R+g_L)}$$

If both side is metal, we arrive at ohm's law, otherwise the tunneling current is suppressed by a factor $V^{+g_R} \cdot V^{g_L}$.

§ Fermion spectral function

we introduce the spectral function defined as

$$i\,G(t,0) = \int d\omega \; A_+^0(\omega)\, e^{-i\omega t} \quad \text{for } t>0$$

$$\pm \int d\omega \; A_-^0(\omega)\, e^{-i\omega t} \quad \text{for } t<0$$

where $A_+^0(\omega) = \sum_{m,n} \delta(\nu - (\epsilon_m - \epsilon_n)) \langle\psi_n| c(x)|\psi_m\rangle\langle\psi_m| c^\dagger(x)|\psi_n\rangle \frac{e^{-\beta\epsilon_n}}{Z}$

$$A_-^0(\omega) = \sum_{mn} \delta(\nu + (\epsilon_m - \epsilon_n)) \langle\psi_n| c^\dagger(x)|\psi_m\rangle\langle\psi_m| c(x)|\psi_n\rangle \frac{e^{-\beta\epsilon_n}}{Z}$$

Similarly, we can introduce the spectral function ρ in momentum space.

$$i\,G(t,k) = \int d\omega \; A_+^0(\omega,k)\, e^{-i\omega t} \quad \text{for } t>0$$

$$\int d\omega \; A_-^0(\omega,k)\, e^{-i\omega t} \quad \text{for } t<0$$

and $G_+(\omega,k) = \int dt\, \theta(t)\, G(t,k)\, e^{-i\omega t} = \int d\nu \dfrac{A_+(\nu,k)}{\omega - \nu + i0^+}$

$G_-(\omega,k) = \int dt\, \theta(-t)\, G(t,k)\, e^{-i\omega t} = \int d\nu \dfrac{A_-(\nu,k)}{\omega - \nu - i0^+}$

where $A_+(\nu,k) = \sum_{mn} \delta[\nu - (\epsilon_m - \epsilon_n)] \langle \psi_n | C_k | \psi_m \rangle \langle \psi_m | C_k^\dagger | \psi_n \rangle \dfrac{e^{-\beta\epsilon_n}}{Z}$

$A_-(\nu,k) = \sum_{mn} \delta[\nu + (\epsilon_m - \epsilon_n)] \langle \psi_n | C_k^\dagger | \psi_m \rangle \langle \psi_m | C_k | \psi_n \rangle \dfrac{e^{-\beta\epsilon_n}}{Z}$.

Let us rederive the tunneling current using the spectral function.

$$\langle [I(t), I^\dagger(0)] \rangle = \langle C_L(t) C_L^\dagger(0) \rangle \langle C_R^\dagger(t,0) C_R(00) \rangle - \langle C_L^\dagger(00) C_L(t,0) \rangle \langle C_R(00) C_R^\dagger(t0) \rangle$$

$$= (iG_L(t))(-iG_R(-t)) - (-iG_L(-t))^* (iG_R(t))^*$$

$$= -\int d\omega_L\, d\omega_R \left[A^0_{L+}(\omega_L) A^0_{R-}(\omega_R) - A^0_{L-}(\omega_L) A^0_{R+}(\omega_R) \right] e^{i(\omega_R - \omega_L)t}$$

The retarded Green function : $iD(t) = \theta(t) \langle | [I(t), I^\dagger(0)] | \rangle$

$$D(\omega) = \int dt\, D^{\phi}(t)\, e^{+i\omega t}$$

$$= \int d\omega_L\, d\omega_R \frac{A^0_{L+}(\omega_L) A^0_{R-}(\omega_R) - A^0_{L-}(\omega_L) A^0_{R+}(\omega_R)}{\omega - (\omega_L - \omega_R) + i0^+}$$

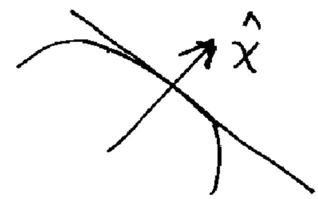
$$\Rightarrow j_T(t) = 2 P^2\, \mathrm{Im}\, D(V)$$

$$= 2\pi P^2 \int d\nu \left[(A^0_{L-}(\nu) A^0_{R+}(\nu - V) - A^0_{L+}(\nu) A^0_{R-}(\nu - V) \right]$$

For spherical Fermi surface, $\Rightarrow \quad C_1 = C_2$

$$i\,G(-0^+, x)\Big|_{x\to\infty} \sim \cos\left(k_F|x| - \frac{\pi(d+1)}{4}\right)\frac{1}{|x|^{d+1/2}} .$$

If Fermi surface has flat piece,



then along that direction

$$N(k, \hat{x}) \simeq \text{const} ,$$

we effective has 1d system along $\hat{x}$-direction

$$\rightarrow i\,G(-0^+, x)\Big|_{x\to+\infty} \sim -i\, e^{i\vec{k}_F(\hat{x})\cdot\vec{x}} \frac{1}{|x|}$$

For free fermions, we have $A_-(\omega,k) = n_F(\xi_k)\,\delta(\omega - \epsilon_k)$

$A_+(\omega,k) = (1-n_F(\xi_k))\,\delta(\omega-\epsilon_k)$

$\Rightarrow A_-^0(\omega) = n_F(\omega)\,N(\omega+E_F) \quad A_+^0(\omega) = (1-n_F(\omega))\,N(\omega+E_F)$

$$\to \langle \dot{J}_T\rangle(t) = 2\pi\Gamma^2 \int d\nu\, \Big((1-n_F(\nu))\, n_F(\nu - V)\, N_L(\nu+E_F) N_R(\nu - V + E_F) - n_F(\nu)(1-n_F(\nu-V))\Big)\, N_L(\nu+E_F)\, N_R(\nu-V+E_F)$$

$$\boxed{\approx 2\pi\Gamma^2 N_L(E_F) N_R(E_F) \int d\nu\, \left(n_F(\nu-V) - n_F(\nu)\right)}$$

§ Equal time Green's function and the shape of Fermi surface

$$G(0^+,x) - G(-0^+,x) = \langle\{c(x), c^\dagger(0)\}\rangle = \delta(x).$$

Let's pick up a direction $\hat{x}$, and study the large-x behavior of

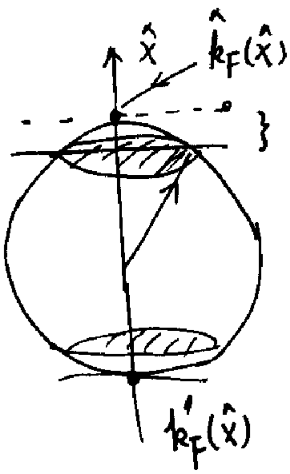
$G(\pm 0^+, x\hat{x})$.

At $T=0$, $G(-0^+,x) = -\int \frac{d^dk}{(2\pi)^d}\, n_F(\xi_k)\, e^{i\vec{k}\vec{x}}$

$$= -\underbrace{\int_{-\infty}^{+\infty} dk\, \tilde{N}(k,\hat{x})\, e^{ik|x|}}_{\text{integral along the } \hat{x}\text{-direction}}, \quad \text{where } \underbrace{\tilde{N}(k,\hat{x}) = \int \frac{d^dk}{(2\pi)^d}\, \Theta(-\xi_k)\,\delta(k - \vec{k}\cdot\hat{x})}_{\text{integral over the other "}d-1\text{" dimension}}$$

$\vec{k}\cdot\hat{x}$ = const defines a $\underbrace{\text{hyper-plane}}_{d-1 \text{ dim}}$, which is tangential to Fermi surface at two points $k_F(\hat{x})$ and $k_F'(\hat{x})$

$\hat{N}(k,\hat{x})$ basically is cross-section area, intersected by the plane $\vec{k}\cdot\hat{x} = k$ and the Fermi sea. Without loss of generality, let us set $\hat{x}$ along the $\hat{z}$-direction



$\hat{k}_F(\hat{x})\cdot\hat{x} - k$ $\rightarrow$ this a second order small number compare to the radius of the cross section

i.e. area $\propto \left(\sqrt{|\hat{k}_F(\hat{x})\cdot\hat{x} - k|}\right)^{d-1}$

Similar reason also applies to $k_F'(\hat{x})$ $\Rightarrow$ $\hat{N}(k,\hat{x})$ has two singularity at two ends.

$$\Rightarrow \hat{N}(k,\hat{x}) = C_1\, \Theta(k_F(\hat{x})\cdot\hat{x} - k)\, |\hat{k}_F(\hat{x})\cdot\hat{x} - k|^{\frac{d-1}{2}} \quad \text{(singluar parts)}$$
$$+ C_2\, \Theta(k_F'(\hat{x})\cdot\hat{x} - k)\, |-\hat{k}_F'(\hat{x})\cdot\hat{x} + k|^{\frac{d-1}{2}}$$

$$\Rightarrow iG(-0^+, x)\Big|_{x\to+\infty} = \int_{-\infty}^{+\infty} dk\; C_1\, \Theta(k_F(\hat{x})\cdot\hat{x} - k)\, |\hat{k}_F(\hat{x})\cdot\hat{x} - k|^{\frac{d-1}{2}}\, e^{ik|\vec{x}|} + \cdots$$

$$\propto \frac{C_1 e^{ik_F(\hat{x})\cdot\vec{x}}}{|x|^{\frac{d+1}{2}}}\, e^{-i\frac{\pi(d+1)}{4}} + \frac{C_2\, e^{-ik_F(\hat{x})\cdot\hat{x}}}{|x|^{\frac{d+1}{2}}}\, e^{i\frac{\pi(d+1)}{4}}$$

(by introducing convegence factor $e^{ik|x|} \rightarrow e^{i(k+i0^+)|x|}$