lect 6 Interacting electron gas: — Screening, collective modes

*) Screening:

$$H = \sum_{k\sigma} \varepsilon(k)\, C^{+}_{k\sigma} C_{k\sigma} + \frac{1}{2V}\sum_{kk'q} V(q)\, C^{+}_{k+q\sigma} C^{+}_{k'-q\sigma'} C_{k'\sigma'} C_{k\sigma}$$

consider an external perturbation $H_{ex}(t) = \frac{1}{V}\sum_{q} V_{ex}(q,t)\,\rho(-q,t)$

where $\rho(q) = \sum_{k\sigma} C^{+}_{k\sigma} C_{k-q,\sigma}$ ← density operator

From the linear response

$$\delta\rho(q,t) = -\int_{-\infty}^{+\infty} dt'\, \chi_{ret}(q,t-t')\, V_{ex}(t')$$

or $\delta\rho(q,\omega) = -\chi_{ret}(q,\omega)\, V_{ex}(q,\omega)$, where

$$\chi_{ret}(q,\omega) = \frac{i}{\hbar}\int_{-\infty}^{+\infty} dt\; e^{i(\omega+i\eta)t}\, \theta(t)\, \langle \cdot\cdot | \rho(q,t)\, \rho(-q,0) | \cdot\cdot \rangle$$

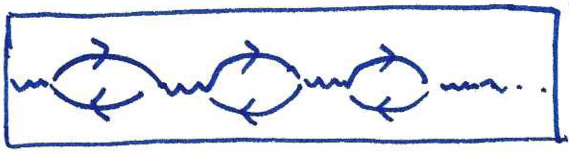
$\langle | \; | \rangle$ means the ground state average at zero temperature or thermal average at finite ~~temperature~~.

$\chi_{ret}(q,\omega)$ is the response for inter-acting systems. We can use the idea of self-consistency to appoximate as

response of the free electron system

$$\delta\rho(q,\omega) = -\chi_{0}(q,\omega)\, V_{tot}(q,\omega)$$

$$\delta\rho(q,\omega) = -\chi_0(q,\omega)\{ V_{ex} + V_{ind}]$$



$$-\nabla^2 V_{ind} = 4\pi e^2\, \delta\rho(q,\omega) \Rightarrow V_{ind} = \frac{4\pi e^2}{q^2}\, \delta\rho(q,\omega)$$

$$= v(q)\, \delta\rho(q,\omega)$$

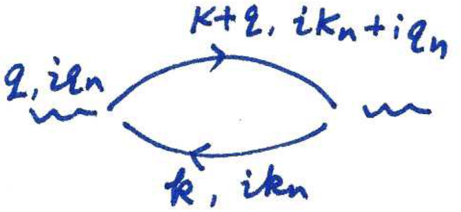
$$\Rightarrow \delta\rho(q,\omega) = \frac{-\chi_0(q,\omega)}{1 + v(q)\chi_0(q,\omega)}\, V_{ext}(q,\omega)$$

$$V_{tot} = V_{ex} + V_{ind} = \frac{1}{1+v(q)\chi_0(q,\omega)}\, V_{ex}(q,\omega)$$

$$\Rightarrow \boxed{\varepsilon(q,\omega) = 1 + \frac{4\pi e^2}{q^2}\chi_0(q,\omega)} \leftarrow \text{dielectric function}$$

$$\chi^0_{ret}(q,\omega) = \frac{i}{\hbar}\int_{-\infty}^{+\infty} dt\ \theta(t)\ \langle | \rho(qt)\, \rho(-q,0) | \cdot\cdot \rangle .$$

→ Matsubara representation



$$\chi^0(q, iq_n) = \frac{1}{V}\int_0^\beta d\tau\ e^{i\omega\tau}\ \langle T_\tau | \rho(q,\tau)\, \rho(-q,0) | \rangle$$

$$= \frac{\overset{\swarrow \text{spin}}{-2}}{V\beta}\sum_{k,\sigma}\ \sum_{ik_n} \mathcal{G}^0(k+q, ik_n+iq_n)\ \mathcal{G}^0(k, ik_n)$$

Ex: frequency summation: define $S = \frac{+1}{\beta}\sum_{ik_n} \frac{1}{ik_n + iq_n - \varepsilon_{k+q}}\ \frac{1}{ik_n - \varepsilon_k}$

$$I = \lim_{R\to\infty} \oint \frac{dz}{2\pi i}\ f(z)\ \frac{1}{e^{\beta z}+1} = 0, \text{ where } f(z)$$

$$= \frac{1}{iq_n + z - \varepsilon_{k+q}}\ \frac{1}{z - \varepsilon_k}$$

$$\Rightarrow \; -\frac{1}{\beta}\sum_n f(i\omega_n) + \sum_i \mathrm{Res}\left(\frac{f(z)}{e^{\beta z}-1}\right)\Big|_{z=z_i} = 0$$

$$\Rightarrow S = \frac{1}{-iq_n + \varepsilon_{k+q} - \varepsilon_k}\,\frac{1}{e^{\beta(\varepsilon_{k+q}+iq_n)}+1} + \frac{1}{iq_n + \varepsilon_k - \varepsilon_{k+q}}\,\frac{1}{e^{\beta\varepsilon_k}+1}$$

$$= \frac{n_f(\varepsilon_k) - n_f(\varepsilon_{k+q})}{iq_n - (\varepsilon_{k+q} - \varepsilon_k)}$$

$$\Rightarrow \chi^0(q, iq_n) = -2\int\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{n_f(\varepsilon_k) - n_f(\varepsilon_{k+q})}{iq_n - (\varepsilon_{k+q} - \varepsilon_k)}$$

Real frequency: $\chi^0(q, \omega+i\eta) = -2\int\frac{d^3\vec{k}}{(2\pi)^3}\,\frac{n_f(\varepsilon_k) - n_f(\varepsilon_{k+q})}{\omega - (\varepsilon_{k+q} - \varepsilon_k) + i\eta}$ ← Lindhardt response

at small q-limit: $q \ll k_f$

$$n_f(\varepsilon_k) - n_f(\varepsilon_{k+q}) = -\frac{\partial n}{\partial \varepsilon}(\varepsilon_{k+q} - \varepsilon_k) = \delta(\varepsilon - \mu)\,\vec{v}_F\cdot\vec{q}$$

$$\chi^0(q, \omega + i\eta) = N_0\int\frac{d\Omega}{4\pi}\,\frac{-\cos\theta\, v_F q}{\omega - v_F q\cos\theta + i\eta}\,, \quad \text{where } N_0 = \frac{2}{(2\pi)^3}\int \underbrace{k^2 dk\int d\Omega}_{\delta(\varepsilon-\mu)}$$

$$= N_0\left[1 - \int\frac{d\Omega}{4\pi}\,\frac{s}{s - \cos\theta + i\eta}\right], \quad \text{where } s = \frac{\omega}{v_F q}.$$

$$\mathrm{Re}\int\frac{d\Omega}{4\pi}\,\frac{s}{s-\cos\theta} = \int_{-1}^{1}\frac{dx}{2}\,\frac{s}{s-x} = -\frac{s}{2}\ln|s-x|\Big|_{-1}^{1} = \frac{s}{2}\ln\left|\frac{s+1}{s-1}\right|$$

$$\mathrm{Im}\int\frac{d\Omega}{4\pi}\,\frac{-s}{s-\cos\theta+i\eta} = \frac{s}{2}\int_{-1}^{1}dx\,(-\pi\,\delta(s-x)) = +\frac{\pi s}{2}\,\theta(|s|<1)$$

$$\Rightarrow \quad \chi_0(q,\omega+i\eta) = N_0\left[1-\frac{s}{2}\ln\left|\frac{1+s}{1-s}\right|\right] + i\frac{\pi}{2}N_0 s\,\theta(|s|<1).$$

RPA response $$\chi_{RPA}(q,\omega+i\eta) = \frac{\chi_0(q,\omega+i\eta)}{1+V(q)\,\chi_0(q,\omega+i\eta)}.$$

☆ Static screening

$$\varepsilon(q,\omega) = 1+v(q)\,\chi_0(q,\omega+i\eta)$$

$$\omega=0 \Rightarrow \varepsilon(q,0) = 1+2\cdot\frac{4\pi e^2}{q^2}\int\frac{d^3k}{(2\pi)^3}\frac{-n(\varepsilon_{k+q})+n(\varepsilon_k)}{\varepsilon_{k+q}-\varepsilon_k}$$

$$\boxed{\begin{array}{l}\vec{k}+\vec{q}\to-\vec{k}\\ \vec{k}\to-\vec{k}-\vec{q}\end{array}}$$

$$= 1+2\cdot\frac{4\pi e^2}{q^2}\int\frac{d^3k}{(2\pi)^3}\frac{n_f(\varepsilon_k)\times 2}{\varepsilon_{k+q}-\varepsilon_k}$$

$$= 1+\frac{4\pi e^2}{q^2}\int\frac{d^3k}{(2\pi)^3}\frac{4}{\frac{\hbar^2k_f^2}{2m}\left[2\frac{\vec{k}}{k_f}\cdot\frac{\vec{q}}{k_f}+\left(\frac{q}{k_f}\right)^2\right]}$$

$$= 1+\frac{4\pi e^2}{q^2}\int\frac{k^2dk}{(2\pi)^3}\int_{-1}^{1}d\cos\theta\,\frac{4\cdot 2\pi}{\varepsilon_f\left[\frac{2kq\cos\theta}{k_f^2}+\left(\frac{q}{k_f}\right)^2\right]} \qquad \text{define } x=\frac{q}{2k_f}$$

$$= 1+\frac{4\pi e^2}{q^2}\frac{k_f^3}{\varepsilon_f}\frac{1}{4\pi^2}\int_0^1 d\left(\frac{k}{k_f}\right)\left(\frac{k}{k_f}\right)^2\int_{-1}^{1}d\cos\theta\,\frac{1}{\left[\frac{k}{k_f}x\cos\theta+x^2\right]}$$

$$= 1+\frac{4\pi e^2}{q^2}N_0\left[\frac{1}{2}+\frac{1-x^2}{4x}\ln\left|\frac{1+x}{1-x}\right|\right]$$

as $q\to 0$ $$\varepsilon(q) = 1+\frac{4\pi e^2}{q^2}N_0 \Rightarrow V(q) = \frac{V_0(q)}{\varepsilon(q)} = \frac{4\pi e^2}{q^2+(1/\lambda)^2}$$

Thomas-Fermi $$V(r) = \frac{1}{r}e^{-\lambda r}, \quad \lambda = \left[4\pi e^2 N_0\right]^{1/2}$$

$$\Rightarrow \lambda\cdot k_F = \frac{1}{\sqrt{4/\pi}}\frac{1}{\left[e^2k_F/\frac{\hbar^2k_F^2}{m}\cdot 2\right]^{1/2}} \sim \sqrt{E_K/E_{int}} \quad \Rightarrow \boxed{1/\lambda \sim k_F}$$

☆ Friedel oscillation,

$$\varepsilon(q,0) = 1 + \frac{\lambda^2}{q^2} S(x), \quad \text{where } S(x) = \frac{1}{2}\left[1 + \frac{1-x^2}{2x} \ln\left|\frac{1-x}{1+x}\right|\right]$$

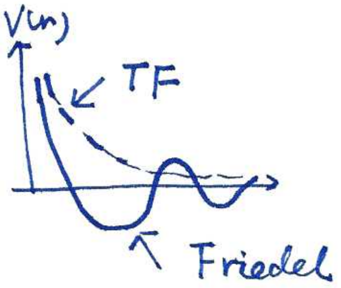
$$x = \frac{q}{2k_f}.$$

at $x = \frac{q}{2k_f} = 1$, $S(x)$ has a sudden drop $\leftarrow$ because $\varepsilon_{k+q} - \varepsilon_k > 0$ for all k if $q > 2k_f$.

$$V(r) = \int d^3\vec{q}\; e^{i\vec{q}\cdot\vec{r}} \frac{4\pi Z e^2}{q^2 + \lambda^2 S(q/2k_f)} \leftarrow$$

singular behavier at $x=1$.

as $r \to +\infty$, $V(r) \sim$ const. $\frac{\cos 2k_f r}{r^3}$



☆ Plasmon frequency at $s \gg 1$

$$1 + \frac{4\pi e^2}{q^2} \chi_0(q,\omega) = 0. \quad \Rightarrow$$

The pole of $\chi(q,\omega)$, or, the zero of $\varepsilon(q,\omega)$, describes the intrinsic excitations.

why? It means even $V_{ex} = 0$, we still have responses.

at $s \gg 1$

$$\chi_0(q,\omega) = N_0\left[-\frac{1}{3s^2} - \frac{1}{5s^4}\right]$$

$$\varepsilon(q,\omega) = 1 + \frac{4\pi e^2 N_0}{q^2}\left[-\frac{1}{3s^2} - \frac{1}{5s^4}\right] = 0$$

$$\Rightarrow \frac{\omega^2}{\omega_p^2} = 1 + \frac{3}{10}\left(\frac{v_F q}{\omega_p}\right)^2 \leftarrow$$

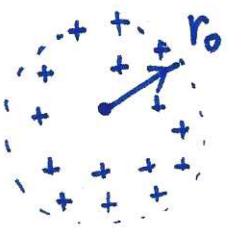
no-damping plasmon

☆ $$\delta \mathcal{E}_{HF}(k) \rightarrow -\sum_{q} n_{k+q} \frac{4\pi e^2}{q^2 + 4\pi e^2 \chi_0(q,0)}$$

☆ $\boxed{\text{Wigner crystal}}$

$$R_s = \frac{E_{int}}{E_k} = \frac{\frac{e^2}{d}}{\frac{\hbar^2}{md^2}} = \frac{d}{\frac{\hbar^2 e^2}{m}} \sim \frac{d}{a_0}$$

at $R_s >> 1$, perturbation picture does not apply. → Crystallization.

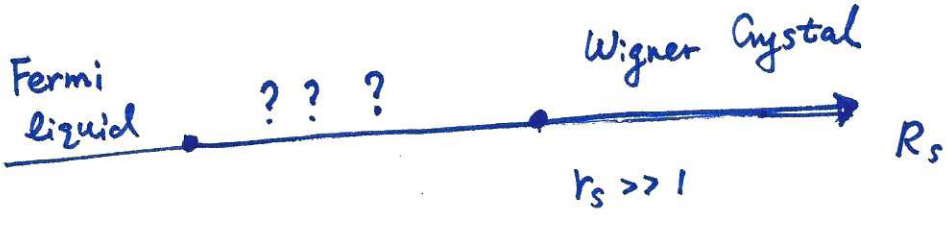


$$E \, 4\pi r^2 = 4\pi \cdot \frac{4\pi}{3} \rho r^3$$

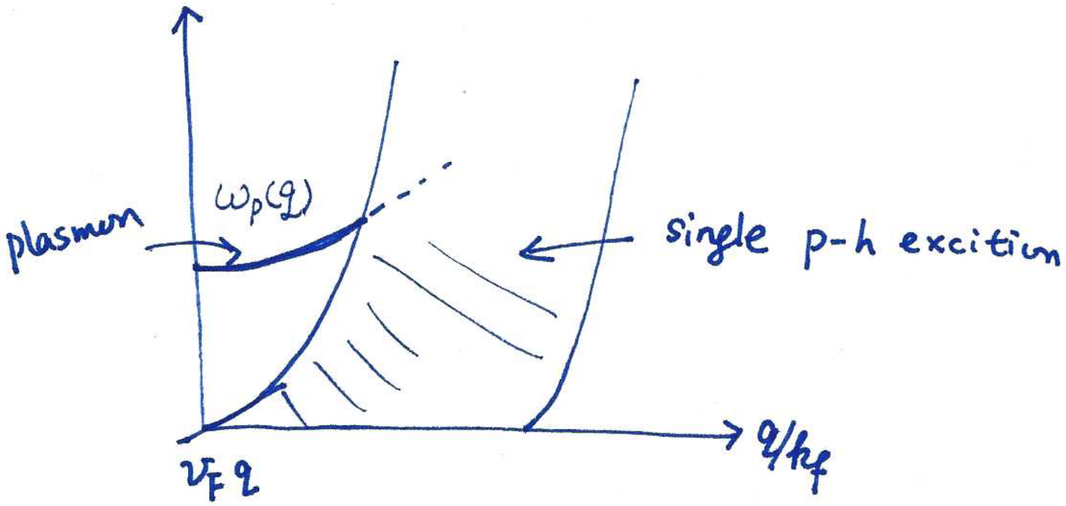
$$E = \frac{4\pi}{3} \rho r = \frac{e}{r_0^3} r$$

$$F = eE = \frac{e^2}{r_0^3} r$$

$$\Rightarrow \boxed{\omega^2 = \frac{e^2}{m r_0^3} = \frac{e^2}{m(R_s a_0)^2} = \frac{1}{3}\omega_p^2}$$



☆



Lect 12: two-body correlation functions & linear responses

§1 density-density correlation

$$i\, P^{00}(t, x) = \langle T\, (\rho(t,x)\, \rho(0,0)\rangle \qquad \rho(x,t) = c^{\dagger}(t,x)\, c(0,0)$$

by Wick theorem ⇒ $i\, P^{00}(t,x) = \langle T\, [\, c^{\dagger}(t+0^{+},x)\, c(t,x)\, c^{\dagger}(0^{+})\, c(0)]\rangle$

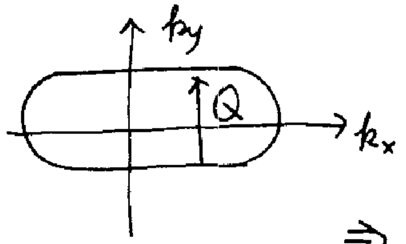
$$= \rho_0^{2} - i\, G(-t,-x)\, i\, G(t,x)$$

as $t \to 0$, for spherical Fermi surface, we have

$$iG(-0^{+}, x)\big|_{x\to+\infty} \sim \cos\left(k_F|x| - \frac{\pi(d+1)}{4}\right) \frac{1}{|x|^{(d+1)/2}}$$

$$\to i\, p^{00}(t,x) - \rho_0^{2} \longrightarrow \left[1 - \sin\left(2k_F|x| - \frac{\pi(d+1)}{2}\right)\right] \frac{1}{|x|^{d+1}} .$$

For nested Fermi surface



the effective dimension is reduced to $d=1$

$$\Rightarrow \quad (1 + \sin(Q|x|))\, \frac{1}{|x|^{2}}$$

↳ algebraic long range crystal.

Retarded density-density response:

$$i\, \Pi^{00}(t, q) = \frac{\theta(t)}{V} \left\langle \left[\, \sum_{k} (c^{\dagger}_{k}\, c_{q+k})(t)\, , \sum_{k'} c^{\dagger}_{k'}\, c_{k'-q}(0) \right] \right\rangle$$

$$= \frac{\Theta(t)}{V} \sum_{k} (1-n_F(\xi_{q+k}))\, n_F(\xi_{k})\, e^{-it(\xi_{q+k}-\xi_{k})}$$

$$- (1-n_F(\xi_{k}))\, n_F(\xi_{q+k})\, e^{+it(\xi_{k}-\xi_{q+k})}$$

$$\rightarrow \Pi^{00}(\omega, \overset{q}{\ }\ ; = V^{-1} \sum_{k} \frac{-(n_F(\xi_{q+k})-n_F(\xi_{k}))}{\omega-(\xi_{q+k}-\xi_{k})+i0^+}$$

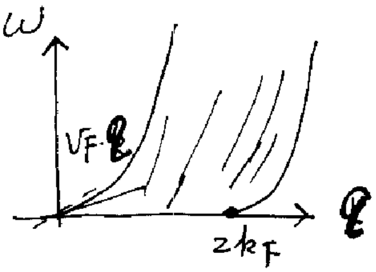
as $|q| \ll k_F$

$$\Pi^{00}(\omega, \overset{q}{\ }\ ) = +\int \frac{d^d q}{(2\pi)^d} \frac{\partial n_F}{\partial \xi} \frac{-\vec{v}\cdot\vec{q}}{\omega-\vec{v}\cdot\vec{\ } + i0^+} \ ; \quad \left(\frac{\partial n_F}{\partial \xi} = -\delta(\xi_{k}) \text{ at } 0K \right).$$

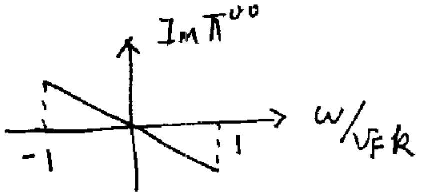
$$\mathrm{Im}\,\Pi^{00}(\omega, \overset{q}{\ }\ ) = -S N_0 \int \frac{d\Omega}{4\pi}\, \delta(S-\cos\theta) \quad (\text{for } 3d)$$ (No is the density of states).

where $S = \frac{\omega}{v_F q}$, N_0 is the density of states.

$$= -\frac{\pi}{2} N_0 S\ \Theta(-|S|+1)\ .$$



Imaginary part means dissipation:



the figure of particle hole continuum.

how about Real part, at 3d, we have

$$\mathrm{Re}\,\Pi^{00}(\omega, q) = N_0 \int \frac{d\Omega}{4\pi} \frac{\cos\theta}{S-\cos\theta}$$ [illegible]

$$= -N_0\left[1 - \frac{S}{2} \ln\left|\frac{1+S}{1-S}\right|\right] .$$

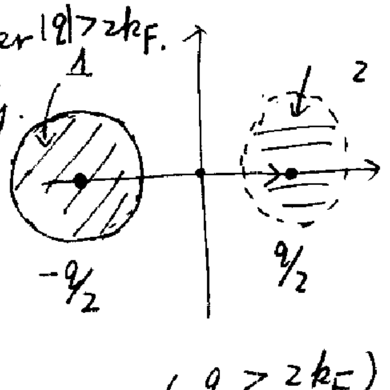
The compressbility $\chi = \underset{\lim_{q\to0}}{-}\Pi^{00}(q,0) = N_0$

~~optical conductivity~~ $\cancel{\sigma(\omega) = -\lim_{k\to0} \mathrm{Im}\frac{\omega}{k^2}\Pi^{00}(\omega,k) = N_0\frac{\omega^2}{k^2}}$

how about $\Pi^{00}(\omega, q)$ as $|q| \to 2k_F$

$$\Pi^{00}(\omega, q) = \int \frac{d^dk}{(2\pi)^d} \frac{n_F(\xi_{k-q/2}) - n_F(\xi_{k+q/2})}{\omega - (\xi_{k+q/2} - \xi_{k-q/2}) + i0^+}$$

Let us translate the Fermi sphere, and consider $|q| > 2k_F$. left and right hand side with $\mp \frac{\vec{q}}{2}$, respectively.



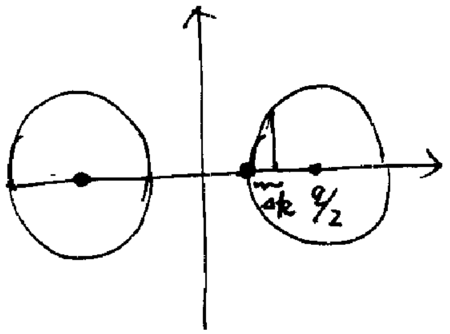
if k is inside the shaded area 1,
then $n_F(\xi_{k+q/2}) = 1$, $n_F(\xi_{k-q/2}) = 0$.

if k is inside the area 2 ⇒
$n_F(\xi_{k-q/2}) = 1$, $n_F(\xi_{k+q/2}) = 0$.

If $\omega > 0$, we need $\xi_{k+q/2} > \xi_{k-q/2}$ to make the denominator vanish, The smallest value $\xi_{k+\frac{q}{2}} - \xi_{k-q/2} =$
$= \hbar^2(|q - k_F|^2 - k_F^2)/2m = \omega_0$

i.e. we take the region 2

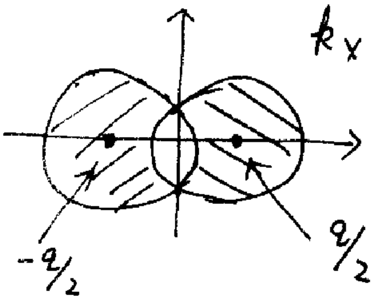
as $\omega > \omega_0$, the area of the intersection is $\mathrm{Im}\,\Pi^{00}(\omega, q > 2k_F) \propto$
$\pi\left(\sqrt{k_F \cdot \Delta k}\right)^2$



$$\Delta\omega = \left\{(q - k_F + \Delta k)^2 - (k_F - \Delta k)^2\right\} - (\omega_0)$$
$$= q \cdot \Delta k$$

⇒ at 3D, $\mathrm{Im}\,\Pi^{00}(\omega, q > k_F) \propto \Theta(\omega - \omega_0)\,|\omega|$, and for general dimensions d,
$\propto \Theta(\omega - \omega_0)\,|\omega|^{\frac{d-1}{2}}$.

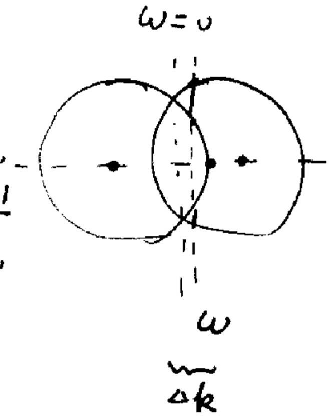
If $|q| < 2k_F$, after we translate the Fermi surface to $-\frac{q}{2}$ and $\frac{q}{2}$, respectively, the two new Fermi surfaces have a overlap region.



Let us calcutate area satisfying $\omega = \xi_{k+q/2} - \xi_{k-q/2} > 0$

~~when $\omega > \xi(k_F) - \xi(q-k_F) = \omega_0$,~~

~~we again have $Im\, \Pi^0(q,\omega) \propto \Theta(\omega-\omega_0)\,(\omega-\omega_0)^{\frac{d-1}{2}}$,~~



as $0 < \omega < \omega_0$, $\Rightarrow \Delta k = \frac{\omega}{q}$,

the area is a ring,

$\omega_0 = \xi(k_F) - \xi(q-k_F)$

the outradius : $\sqrt{k_F^2 - (\frac{q}{2} - \Delta k)^2}$

inner radium $\sqrt{k_F^2 - (\frac{q}{2} + \Delta k)^2}$

$$\Rightarrow Im\, \Pi^{00}(\omega, |q| < 2k_F) \propto \left(\frac{q}{2} + \Delta k\right)^2 - \left(\frac{q}{2} - \Delta k\right)^2 = q \cdot \Delta k$$

$$\propto \omega .$$

as $\omega > \omega_0$, the cross-section is a circle, the area is

$Area = k_F^2 - (\frac{q}{2} - \Delta k)^2$, while $\omega \propto (\frac{q}{2} + \Delta k)^2 - (\frac{q}{2} - \Delta k)^2 \propto q \cdot \Delta k$

$\sim \left[k_F^2 - \left(\frac{q}{2}\right)^2\right] + q \cdot \Delta k$ $\Rightarrow Im\, \Pi^{00}(\omega, |q| < 2k_F) \sim \left[k_F^2 - (\frac{q}{2})^2\right] + \underline{\omega}$

§ Current - current correlation function

$$i\Pi^{ij}(t,x) = \theta(t)\langle [j^i, j^j]\rangle \Rightarrow$$

$$\Pi^{ij}(\omega,q) = \int \frac{d^dk}{(2\pi)^d} \frac{[n_F(\xi_k) - n_F(\xi_{k+q})]\, v^i(k+q/2)\, v^j(k-q/2)}{\omega - (\xi_{k+q}-\xi_k) + i0^+}$$

At zero ~~temperature~~, and $q \to 0 \Rightarrow$ (at 3d)

$$\Pi^{ij}(\omega,q) = N_0 \int \frac{d\Omega}{4\pi} \frac{\hat{v}_F\cdot\vec{q}\;\; v_F^i v_F^j}{\omega - \hat{v}_F\cdot\vec{q} + i0^+}$$

$$= \underset{\widehat{v_F^2}}{N_0} \int \frac{d\Omega}{4\pi} \frac{\cos\theta \cdot \hat{\Omega}^i \hat{\Omega}^j}{s - \cos\theta + i0^+}$$ (we set q along $\hat{z}$)

The longitudinal part can be infered from Π^{00}, or directly

Calculate $\Pi^{zz} = N_0 v_F^2 \int \frac{d\Omega}{4\pi} \frac{\cos^3\theta}{s - \cos\theta + i0^+}$.

The transverse part can be calculated as Π^{xx} or Π^{yy}

$$\Rightarrow \Pi^{xx} = \Pi^{yy} = \frac{N_0 v_F^2}{2} \int \frac{d\Omega}{4\pi} \frac{\cos\theta \sin^2\theta}{s - \cos\theta + i0^+}$$

$$\Rightarrow \text{longitudinal } \Pi^{zz}(\omega,q) = \begin{cases} -2s^2 - i\pi s^3 & (s \ll 1) \\ \frac{2}{3} + \frac{2}{5}(1/s)^2 & (s \gg 1) \end{cases}$$

$$\Pi^{xx}(\omega,q) = \Pi^{yy}(\omega,q) = \begin{cases} 2s^2 - i\frac{\pi}{2} s & (s \ll 1) \\ \frac{2}{3} + \frac{2}{15}(1/s)^2 & (s \gg 1) \end{cases}$$

§ Conductivity

We assume there's impurity scattering, which results in a finite lift time of particle.

$$iG(t,k) = \Theta(t)\Theta(\xi_k)\,e^{-it\xi_k - t\Gamma} - \Theta(-t)\Theta(-\xi_k)\,e^{-it\xi_k-(-t)\Gamma}$$

$$iG^\beta(t,k) = \Theta(t)(1-n_F(\xi_k))\,e^{-it\xi_k - t\Gamma} - \Theta(-t)\,n_F(\xi_k)\,e^{-it\xi_k-(-t)\Gamma}$$

$$\Rightarrow G^\beta(\omega,k) = G^\beta_+ + G^\beta_- = \frac{1-n_F(\xi_k)}{\omega-\xi_k+i\Gamma} + \frac{n_F(\xi_k)}{\omega-\xi_k-i\Gamma}$$

$$\pi A_+(\omega,k) = \Gamma\,\frac{1-n_F(\xi_k)}{(\omega-\xi_k)^2+\Gamma^2}, \quad \pi A_-(\omega,k) = \Gamma\,\frac{n_F(\xi_k)}{(\omega-\xi_k)^2+\Gamma^2}$$

however, this approximation doesn't satisfy

$$A_+^\beta(\omega) = e^{\beta\omega}A_-^\beta(\omega),$$

which means the above approx isn't self-consistent, but we can cure this by define.

$$\pi A_+(\omega,k) = \Gamma\,\frac{(1-n_F(\omega))}{(\omega-\xi_k)^2+\Gamma^2}, \quad \pi A_-(\omega,k) = \Gamma\,\frac{n_F(\omega)}{(\omega-\xi_k)^2+\Gamma^2}$$

Let us calculate $\Theta(t)\langle[\rho(t,q)\ \rho(0,-q)]\rangle$

$$= \langle[C_k^\dagger(t)\,C_{k+q}(t)\ \ C_{k+q}^\dagger(0)\,C_k(0)]\rangle$$

$$\approx \langle C_{k+q}(t)\,C_{k+q}^\dagger(0)\rangle\langle C_k^\dagger(t)\,C_k(0)\rangle - \langle C_{k+q}^\dagger(0)\,C_{k+q}(t)\rangle\langle C_k(0)\,C_k^\dagger(t)\rangle$$

$$= iG(t,k+q)\,(-iG(-t,k)) - (-iG(-t,k+q))^*\,(iG(t,k))^*$$

expand G interms of spectra representation

$$G_+(\omega,k) = \int dt\,\Theta(t)\,G(t,k)\,e^{i\omega t} = \int d\nu\,\frac{A_+(\nu,k)}{\omega-\nu+i0^+}$$

~~3. non-linear sigma m.~~

$$G_-(\omega, k) = \int dt\, \Theta(-t)\, G(t, k)\, e^{i\omega t} = \int d\nu \frac{A_-(\nu, k)}{\omega - \nu + i0^+}$$

$$\Rightarrow \operatorname{Im} \Pi^{00}(\omega, q) = \frac{\pi}{V} \int d\nu \sum_k \left(A_-(\nu, k+q)\, A_+(\nu-\omega, k) - A_+(\nu, k+q)\, A_-(\nu-\omega, k) \right.$$

Similarly

$$\operatorname{Im} \Pi^{ij}(\omega, q) = \frac{\pi}{V} \int d\nu \sum_k v^i(k\quad)\, v^j(k\quad) \left(A_-(\nu, k+q/2)\, A_+(\nu-\omega, k-q/2) - A_+(\nu, k+q/2)\, A_-(\nu-\omega, k-q/2) \right)$$

$$\sigma^{ij}(\omega) = -\lim_{q\to 0} \frac{\Pi^{ij}(\omega, q)}{\omega} = -\int \frac{d\nu}{\pi} \int \frac{dk}{(2\pi)^d} \frac{\Gamma^2 v^i(k)\, v^j(k)\, (n_F(\nu) - n_F(\nu-\omega))}{\omega\, ((\nu-\xi_k)^2 + \Gamma^2)\, ((\nu-\omega-\xi_k)^2 + \Gamma^2)}$$

set $\omega \to 0$

$$\sigma^{ij}(\omega\to 0) = \int \frac{d\nu}{\pi} \int \frac{d^dk}{(2\pi)^d} \frac{\Gamma^2\, v^i(k)\, v^j(k) \left(-\frac{\partial n_F(\nu)}{\partial \nu}\right)}{\left((\nu-\xi_k)^2 + \Gamma^2\right)^2}$$

$$\approx \int \frac{d^dk}{(2\pi)^2}\, \tau\, v^i(k)\, v^j(k)\, \delta(\xi_k) .$$

where $\tau = \frac{1}{2\Gamma}$

§ functional integral formalism

$$Z = \int D\bar{\psi} D\psi \; e^{-S} .$$

$$S = \int_0^\beta d\tau \sum_k \bar{\psi}(k,\tau)(\partial_\tau - \xi_k)\psi(k,\tau) + \frac{1}{2V}\sum_{q\neq 0}\frac{4\pi e^2}{q}\rho(q\tau)\rho(-q,\tau)$$

where $\rho(q) = \sum_{k\sigma} \bar{\psi}_\sigma(k,\tau)\psi_\sigma(k-q,\tau)$

The Hubbard - Stratonovich transformation

$$\exp\left[-\int_0^\beta d\tau \frac{1}{2V}\sum_{q\neq 0}\frac{4\pi e^2}{q^2}\rho(q,\tau)\rho(-q,\tau)\right]$$

$$= \int D\varphi(q,\tau) \exp\left[-\frac{1}{8\pi}\int_0^\beta d\tau \sum_{q\neq 0} q^2 \varphi(q\tau)\varphi(-q\tau)\right]$$

Please check!

$$\exp\left[-\int_0^\beta d\tau \frac{ie}{2\sqrt{V}}\sum_{q\neq 0}\varphi(q,\tau)\rho(-q,\tau)+\rho(q,\tau)\varphi(-q,\tau)\right]$$

then $Z = \int D\bar{\psi} D\psi D\varphi \; e^{-S(\bar{\psi}\psi\varphi)}$

$$S(\bar{\psi},\psi,\varphi) = \frac{1}{8\pi}\int_0^\beta d\tau \sum_{q\neq 0} q^2 \varphi(q,\tau)\varphi(-q,\tau)$$

$$+ \sum_k \bar{\psi}_\sigma(k,\tau)(\partial_\tau - \xi_k)\psi_\sigma(k,\tau) + \frac{ie}{2\sqrt{V}}\sum_{q\neq 0}\left(\varphi(q,\tau)\rho(-q,\tau)+\rho(q\tau)\varphi(-q,\tau)\right)$$

$\swarrow$

$$\frac{ie}{\sqrt{V}}\sum_{k,q}\bar{\psi}_\sigma(k,\tau)\left[\varphi(-q,\tau)\right]\psi_\sigma(k-q,\tau) .$$

transform back to real space $\varphi(r,\tau) = \frac{1}{\sqrt{V}} \sum_q e^{i\vec{q}\cdot\vec{r}} \varphi(q,\tau)$

$$\Rightarrow S(\bar\psi,\psi,\varphi) = -\int_0^\beta d\tau \int dr \, \frac{1}{8\pi}(\nabla\varphi)^2 + \bar\psi_\sigma(r,\tau)\Big[\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu + ie\varphi(r,\tau)\Big]\,{}^{\wedge}\psi_\sigma(r,\tau)$$

Integrate out fermions $\Rightarrow$

$$Z = \int D\varphi \, \exp\Big[-\int_0^\beta d\tau \int dr \, \frac{1}{8\pi}(\nabla\varphi)^2 \det\Big[\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu + ie\varphi(r,\tau)\Big].$$

Remark: The physical meaning of $\varphi(r,\tau)$ is not clear. Naively the saddle point equation $\varphi(r,\tau) \sim \Big\langle \int iV(r-r')\rho(r')dr' \Big\rangle$, but the mean field hamiltonian is no-hermitian: $-\frac{\hbar^2}{2m}\nabla^2 - \mu + ie\varphi$. Such that the average of $\langle i\int V(r-r')\rho(r')dr' \rangle$ can still be real. We may further think what does it really mean.

The determinaint is defined in the basis of $\varphi(r,\tau)$, let's transform to $(k, i\omega_n)$ space, according to the Fourier transform

$$\varphi(r,\tau) = \frac{1}{(\beta V)^{1/2}} \sum_q \sum_l e^{i\vec{q}\cdot\vec{r} - i\omega_l\tau} \varphi(q,\omega_l)$$

then

$$\Big[\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu + ie\varphi(r,\tau)\Big]_{(k\omega_n,\, k'\omega_n')}$$

$$= \int dr\,d\tau \int dr'\,d\tau' \;\; \langle k\omega_n | r\tau\rangle \left[\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu + ie\varphi(r,\tau)\right]_{r\tau,\, r'\tau'} \langle r'\tau' | k'\omega_n'\rangle$$

$$= \int dr\,d\tau \;\; \frac{e^{-i(\vec{k}\cdot\vec{r} - \omega_n\tau)}}{\sqrt{\beta V}} \left[\partial_\tau - \frac{\hbar^2}{2m}\nabla^2 - \mu + ie\varphi(r,\tau)\right] \frac{e^{i(k'\cdot r - \omega_n'\tau)}}{\sqrt{\beta V}}$$

$$= \Big[-i\omega_n + \underbrace{\frac{\hbar^2}{2m}k^2 - \mu}_{\xi_k}\Big]\, \delta_{k,k'}\, \delta_{\omega_n,\omega_n'} + \frac{ie}{\sqrt{\beta V}}\, \varphi(k-k', \omega_n-\omega_n')$$

$\omega_n = \frac{2\pi n}{\beta}$ (bosonic frequency)

write down
$$\boxed{\begin{aligned} M_{k\omega_n,\, k'\omega_n'} &= (M_0)_{k\omega_n,\, k'\omega_n'} + (M_1)_{k\omega_n,\, k'\omega_n'} \\ &= -\mathcal{G}_0^{-1}(k\, i\omega_n)\, \delta_{k\omega_n,\, k'\omega_n'} + \frac{ie}{(\beta V)^{1/2}}\, \varphi(k-k', \omega_n-\omega_n') \end{aligned}}$$

then $Z = \int D\varphi \;\, e^{-S_{eff}(\varphi)}$, with $S_{eff}(\varphi) = \int_0^\beta d\tau \int dr\, \frac{1}{8\pi}\left(\nabla\varphi(r,\tau)\right)^2$

$$- 2 \ln \det M.$$

spin degenerac

$$\boxed{\ln\det M = \mathrm{tr}\ln M = \mathrm{tr}\ln(M_0 + M_1)}$$

$$\ln(M_0 + M_1) = \ln M_0 + \ln(1 + M_0^{-1} M_1) = \ln M_0 + \ln(1 - \mathcal{G}_0 M_1$$

$$= \ln M_0 - \sum_{n=1}^{\infty} \frac{1}{n}\left(\mathcal{G}_0 M_1\right)^n.$$ only M_1 contains φ-field

$$\Rightarrow Z = \int D\varphi \; e^{-\left\{\int_0^\beta d\tau \int dr\, \frac{1}{8\pi}\left(\nabla\varphi(r,\tau)\right)^2 - \sum_{n=1}^{\infty} \frac{1}{n}\, \mathrm{tr}\left[\mathcal{G}_0 M_1\right]^n\right.}.$$

① $n=1$: $$\mathrm{tr}[\mathcal{G}_0 M_1] = \sum_{kk'} (\mathcal{G}_0)_{kk'} (M_1)_{k'k} = \sum_k \mathcal{G}_0(k) M_{1,kk}$$

$$= \sum_k G_0(k) \left(\frac{ie}{(\beta V)^{1/2}} \varphi(0) \right)$$

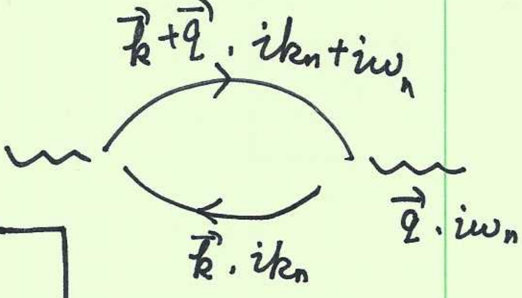
we set $\varphi(0)=0$. $\varphi(0) \sim V(q=0)\,\rho(q=0)$ which is proportional the overall particle density. Set $\underset{=0}{\varphi(0)}$ to neutralize the background.

② Gaussian fluctuation

$$\mathrm{tr}[(\mathcal{G}_0 M_1)^2] = \sum_{kk'} \mathcal{G}_0(k) (M_1)_{kk'} \mathcal{G}_0(k') (M_1)_{k'k}$$

$$= \frac{1}{2} \sum_q \frac{e^2}{\beta V} \left(2 \sum_k \mathcal{G}_0(k) \mathcal{G}_0(k+q) \right) \varphi(q) \varphi(-q)$$

define $$\pi(q) = \frac{2}{\beta V} \sum_k \mathcal{G}_0(k) \mathcal{G}_0(k+q)$$



$$\Rightarrow \boxed{S_{eff} = \frac{1}{2} \sum_q \left[\frac{\vec{q}^2}{4\pi} - e^2 \pi(\vec{q}, iq_n) \right] \varphi(q) \varphi(-q)}$$ vaccum polarization

This $\pi(\vec{q}, iq_n)$ is basically $-\chi^0(\vec{q}, iq_n)$ we calculated before. $\Rightarrow$ $\boxed{\text{Gaussian fluctuation} \equiv \text{RPA approxiation}}$

Itinearant Ferromagnetism (spin-wave

① Define ferromagnetic order parameter $S_\mu(\vec{r}) = \psi^\dagger_\alpha(x)\, \sigma^\mu_{\alpha\beta}\, \psi_\alpha(x)$

and consider the Hamiltonian

$$\boxed{H = \int d^3\vec{r}\ \psi^\dagger_\alpha(\vec{r})\left(-\frac{\hbar^2}{2m}\nabla^2 - \mu\right)\psi_\alpha(\vec{r}) + \frac{1}{2}\int dr dr'\, f(r-r')\, \vec{S}(r)\cdot\vec{S}(r')}$$

Define the H-S field

$$n_\mu(\vec{r}) = -\int dr\ f(r-r')\, S_\mu(r')$$

$$\Rightarrow Z = \int D\psi^\dagger D\psi\ \exp\Big[-\int_0^\beta d\tau \Big\{\int dr\ \psi^\dagger_\alpha(\vec{r})\left(\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu\right)\psi_\alpha(\vec{r})$$

$$+ \frac{1}{2}\int dr dr'\ f(r-r')\ \vec{S}(r)\cdot\vec{S}(r')\Big]$$

$$= \int D\psi^\dagger D\psi\ D\vec{n}\ \exp\Big[\tfrac{1}{2}\int_0^\beta d\tau \int dr dr'\ f^{-1}(r-r')\, n_i(r)\, n_i(r')$$

$$\cdot \exp\Big[-\int_0^\beta d\tau \int dr \Big\{\psi^\dagger_\sigma(r)\Big(\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu\Big]\psi_\sigma(r) - \psi^\dagger_\alpha(r)\, \vec{n}\cdot\vec{\sigma}_{\alpha\beta}\, \psi_\beta(r)\Big\}$$

f^{-1} is defined as

$$\int dr\ f(r_1 - r)\, f^{-1}(r - r_2) = \delta(r_1 - r_2)$$

Now integrate over ψ, we have $Z = \int D\vec{n}\ e^{-S_{eff}(\vec{n})}$

with $$\boxed{S_{eff}(\vec{n}) = -\frac{1}{2}\int_0^\beta d\tau \int dr dr'\ f^{-1}(r-r')\, \vec{n}(r)\cdot\vec{n}(r') \pm T_r\Big[\ln\Big[\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu - \vec{n}\cdot\vec{\sigma}\Big]\Big]}$$

expand $\vec{n} = \bar{n}_\mu + \delta n_\mu(\vec{r},\tau)$

$$\ln\Big[\underset{\nearrow}{}-\Big(\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu - \vec{\bar{n}}\cdot\vec{\sigma}\Big) + \delta n_\mu\cdot\vec{\sigma}\Big] = \ln\Big[G_0^{-1}(\bar{n})\,(1 + G_0(\bar{n})\,\delta n\cdot\vec{\sigma}$$

add a minus sign — a cnst in the Seff because $\ln(-a\ldots] = \ln(-) + \ln a$

$$\Rightarrow \quad S_{eff}(\vec{n}) = \frac{-1}{2}\int_0^\beta d\tau \int dr\,dr'\, f^{-1}(r-r')\,(\bar{n}_i + \delta n_i(r))\,(\bar{n}_i + \delta n_i(r'))$$

$$- \operatorname{Tr}\ln[G_0^{-1}(\bar{n})] \quad - \operatorname{Tr}\ln(1 + G_0(\bar{n})\,\delta\vec{n}\cdot\vec{\sigma})$$

$\swarrow$ Keep to 2nd order

$$- \operatorname{Tr}[G_0(\vec{n})\,\delta\vec{n}\cdot\vec{\sigma}] + \frac{1}{2}\operatorname{Tr}[G_0(\vec{n})\,\delta\vec{n}\cdot\vec{\sigma}\;G_0(\vec{n})\,\delta\vec{n}\cdot\vec{\sigma}]$$

$$\delta S_{eff}(\vec{n}) = -\int_0^\beta d\tau \int dr\,dr'\, f^{-1}(r-r')\,\bar{n}_i\,\delta n_i(r';\tau)$$

$$- \int_0^\beta d\tau \int dr\; \langle \vec{r},\alpha\tau |\, G_0(\bar{n})\,\sigma^i \,|\vec{r}\beta\tau\rangle\, \delta n_i(\vec{r};\tau)$$

$$- \frac{1}{2}\int_0^\beta d\tau \int dr\,dr'\; f^{-1}(r-r')\,\delta n_i(r)\,\delta n_i(r')$$

$$+ \int d\tau\,d\tau' \int dr\,dr'\; \langle r\alpha\tau |\, G_0(\bar{n})\,\delta n_i(r)\,\sigma^i \,| r'\beta\tau'\rangle$$

$$\langle r'\beta\tau' |\, G_0(\bar{n})\,\delta n_j(r)\,\sigma^j \,| r\alpha\tau\rangle$$

Saddle point $\Rightarrow \dfrac{\delta S_{eff}}{\delta n(\vec{r}';\tau')} = 0$ plug in $\dfrac{\delta n(\vec{r},\tau)}{\delta n(\vec{r}',\tau')} = \delta(r-r')\,\delta(\tau-\tau')$

$$- \int dr\; f^{-1}(r-r')\,\bar{n}_i - \langle \vec{r}'\alpha\tau' |\, G_0(\bar{n})\,\sigma^i \,| \underset{\alpha}{\vec{r}'}\,\tau'\rangle = 0$$

$$\int dr\, f^{-1}(r-r') = f^{-1}(q=0)$$

$$n(r) = \frac{1}{V\beta}\sum_{q,i\omega_n} n(q,i\omega_n)\, e^{i(\vec{q}\cdot\vec{r}-\omega_n\tau)}$$

$$f(r-r') = \frac{1}{V}\sum_{\vec{q}} e^{i\vec{q}\cdot(\vec{r}-\vec{r}')}\, f(q)$$

$$\langle \vec{r}'\alpha\tau' \,|\, G_0(\bar{n})\sigma^i \,|\, \vec{r}'\alpha\tau'\rangle$$

$$= \frac{1}{V\beta}\sum_{\vec{k},ik_n} tr\left[G_0(k,\vec{\bar{n}})\,\sigma^i \right] \quad \leftarrow \text{small } tr \text{ refer to spin index}$$

$$\Rightarrow\ -f^{-1}(q=0)\,\bar{n}_i - \frac{1}{V\beta}\sum_{\vec{k},ik_n} tr\left[G_0(k,\vec{\bar{n}})\,\sigma^i\right] = 0$$

无妨设 $\bar{n}_i = \bar{n}\,\delta_{i,z} \ \Rightarrow$

$$\boxed{-f^{-1}(q=0)\,\bar{n} - \frac{1}{V\beta}\sum_{\vec{k},ik_n} tr\left[G_0(\vec{k},ik_n;\bar{n})\,\sigma_z\right] = 0}$$

$$G_0(\vec{k},ik_n;\bar{n}) = \frac{1}{2}\left[\frac{1+\sigma_z}{ik_n-(\varepsilon_k-\mu-\bar{n})} + \frac{1-\sigma_z}{ik_n-(\varepsilon_k-\mu+\bar{n})}\right]$$

$$tr\left[\underset{\wedge\,\sigma_z}{G_0(\vec{k},ik_n;\bar{n})}\right] = \frac{1}{ik_n-(\varepsilon_k-\mu-\bar{n})} - \frac{1}{ik_n-(\varepsilon_k-\mu+\bar{n})}$$

因为 $\displaystyle\frac{1}{\beta}\sum_{ik_n}\frac{1}{ik_n-\varepsilon} = n_f(\varepsilon)$

$$\Rightarrow \frac{1}{\beta}\sum_{ik_n} tr\left[G_0(\vec{k}, ik_n; \bar{n})\sigma_z\right] = n_f(\varepsilon_k - \mu - \bar{n}) - n_f(\varepsilon_k - \mu + \bar{n})$$

$\Rightarrow$ self-consistent Eq

$$-f^{-1}(\vec{q}=0)\,\bar{n} - \frac{1}{V}\sum_{k}\left[n_f(\varepsilon_{k\uparrow}) - n_f(\varepsilon_{k\downarrow})\right] = 0$$

$$\because f^{-1}(\vec{q}=0) < 0 \quad \Rightarrow \quad \boxed{\frac{\bar{n}}{|f(0)|} = \int\frac{d^3\vec{k}}{(2\pi)^3}\left[n_f(\varepsilon_{k\uparrow}) - n_f(\varepsilon_{\downarrow})\right]}$$

self-consistent Eq for FM.

② Gaussian fluctuations

$$\delta S_{eff} = -\frac{1}{2}\int_0^\beta d\tau \int dr\,dr\; f^{-1}(r-r')\,\delta n(r,\tau)\,\delta n(r',\tau)$$

$$+\int d\tau\,d\tau'\int dr\,dr' \underbrace{\langle r\alpha\tau|}_{\delta n_i(r\tau)\,\delta n_j(r',\tau')} G_0(\bar{n})\,\sigma^i\,|r'\beta\tau'\rangle\langle r'\beta'\tau'|\,G_0(\bar{n})\,\sigma^i\,|r\alpha\tau\rangle$$

$$\swarrow \text{ or } \int d\tau\,d\tau'\int dr\,dr'\;\delta n_i(r\tau)\,\delta n_j(r',\tau')\;G_{0,\alpha\alpha'}(\bar{n}, r\,r', \tau\tau')\,\sigma^i_{\alpha\beta}\;G_{0,\beta\beta}\underbrace{(\bar{n}\,r'r,\tau'\tau)}_{\sigma^i_{\beta\alpha}}$$

G_0 is diagonal in spin-index.

$$\boxed{\begin{aligned}&\int d\tau \int dr dr' \; f^{-1}(r-r')\, \delta n(r,\tau)\, \delta n(r',\tau) \\ &= \frac{1}{V\beta} \sum_{q, i\omega_n} \delta n(q\, i\omega_n)\, f^{-1}(q)\, \delta n(-q, -i\omega_n)\end{aligned}} = \int d\tau\, dr dr' \frac{1}{V\beta} \sum_{q i\omega_n} e^{iqr - \omega_n \tau} \delta n(q, \omega_n)$$

$$\frac{1}{V} \sum_{q'} e^{iq'(r-r')} f^{-1}(q)$$

$$\frac{1}{V\beta} \sum_{q'', \omega_n'} e^{iq'' x'' - \omega_n' \tau} \delta n(q'', \omega_n')$$

$$\int d\tau d\tau' \int dr dr' \; \delta n_i(r,\tau)\, \delta n_j(r',\tau')\; G_{0,\alpha\alpha}(\bar{n}, r r'; \tau\tau')\, \sigma^i_{\alpha\beta}\, G_{0\beta\beta}(\bar{n}\, r' r, \tau'\tau)\, \sigma^j_{\beta\alpha}$$

plug in $\delta n_i(r,\tau) = \frac{1}{V\beta} \sum_{q, i\omega_n} e^{i(qr - \omega_n \tau)} \delta n(q, \omega_n)$

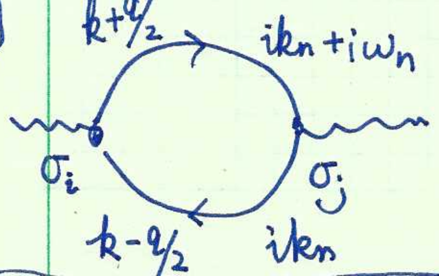
$$G_{0,\alpha\alpha}(\bar{n}, r r'; \tau\tau') = \frac{1}{V\beta}\, G_{0,\alpha\alpha}(\bar{n}, k, \omega_n)\, e^{i[\vec{k}(\vec{r}-\vec{r}\,') - \omega_n(\tau - \tau')]}$$

$$\Rightarrow \frac{1}{(V\beta)^2} \sum_{\substack{k, ik_n \\ k', ik_n'}} \delta n_i(k-k', ik_n - ik_n') \left[G_{0,\alpha\alpha}(\bar{n}, k, k_n)\, \sigma^i_{\alpha\beta}\, G_{0,\beta\beta}(\bar{n}, k', k_n')\, \sigma^j_{\beta\alpha}\right]$$
$$\times\, \delta n_j(k'-k, ik_n' - ik_n)$$

$\Rightarrow$ Gaussian fluctuation

$$\delta S = \frac{1}{2V\beta} \sum_{q, i\omega_n} \delta n_i(q, i\omega_n)\, \mathcal{L}_{ij}(q, i\omega_n)\, \delta n_j(-q, -i\omega_n)$$

where $\mathcal{L}_{ij}(q, i\omega_n) = -f^{-1}(q)\, \delta_{ij} + \frac{1}{V\beta} \sum_{k, i\omega_n} \mathrm{tr}\left[G_0[\bar{n}, k + q/2, ik_n + i\omega_n]\, \sigma^i \right.$
$\left. G_0[\bar{n}, k - q/2, ik_n]\, \sigma^j \right]$



③ Landau damping in the disordered phase

$$\mathcal{L}_{ij}(q,\omega) = -f^{-1}(q)\,\delta_{ij} + \frac{1}{V\beta}\sum_{k,i\omega_n} tr[\sigma^i\sigma^j]\,\frac{1}{ik_n + i\omega_n - \varepsilon_{k+q/2}}\,\frac{1}{ik_n - \varepsilon_k}$$

$$= \delta_{ij}\left[-f^{-1}(q) + \frac{2}{V}\sum_{k}\frac{n_f(\epsilon_{k-q/2}) - n_f(\epsilon_{k+q/2})}{i\omega_n - (\epsilon_{k+q/2} - \epsilon_{k-q/2})}\right]$$

$$= \delta_{ij}\left[-f^{-1}(q) + 2\int\frac{d^3k}{(2\pi)^3}\,\frac{\vec{q}\cdot\hat{k}\,\delta(k-k_f)}{i\omega_n - v_F\,\hat{k}\cdot\vec{q}}\right]$$

$$= \delta_{ij}\left[-f^{-1}(q) + \frac{2k_f^2}{(2\pi)^3}\int d\Omega\,\frac{q\cos\theta}{i\omega_n - v_F q\cos\theta}\right] \qquad \text{define } s = \frac{\omega}{v_F q}$$

$$i\omega_n \rightarrow \omega + i\eta$$

The integral $\rightarrow$ $$N_f\int\frac{d\Omega}{4\pi}\left[-1 + \frac{s}{s + i\eta - \cos\theta}\right] = -N_F + \overset{N_F}{\vee} s\int\frac{d\Omega}{4\pi}\,\frac{1}{s + i\eta - \cos\theta}$$

where $$N_f = 2\int\frac{d^3k}{(2\pi)^3}\,\delta[\varepsilon - v_F(k-k_f)] = \frac{2}{v_F}\frac{k_f^2}{(2\pi)^3}\int d\Omega$$

$$\int\frac{d\Omega}{4\pi}\,\frac{1}{s + i\eta - \cos\theta} = \frac{1}{2}\int_{-1}^{1} d\cos\theta\,\frac{1}{s + i\eta - \cos\theta} = \frac{-1}{2}\ln\left|\frac{1-s}{1+s}\right| - \frac{\pi}{2}i\,\theta(s<1)$$

$$\Rightarrow \mathcal{L}_{ij}(q,\omega) = \delta_{ij}\left\{-f^{-1}(q) - N_F\left[1 - \frac{s}{2}\ln\left|\frac{1+s}{1-s}\right| + i\frac{\pi}{2}s\,\theta(s<1)\right]\right\}$$

at $s \ll 1$,

$$\mathcal{L}_{ij}(q,\omega) = \delta_{ij}\left\{-f^{-1}(q) - N_F + N_F\left[s^2 - i\frac{\pi}{2}s\right]\right\}$$

$$-f^{-1}(q) = |f_0^{-1}| + \chi q^2 ,$$ f_0 need to be negative

a paramter put by hand (phenemilegical)

or $$f(q) = \frac{f_0}{1+\kappa|f_0|q^2}$$

or if we keep - q^2 order for 

define $\delta = |f_0^{-1}| - N_F$, if $\delta > 0 \rightarrow$ paramagnetic phase

$\delta < 0$ — FM phase

$$\delta = 0 \Rightarrow \boxed{N_F f_0 = -1} \leftarrow \text{Pomenranchuk instability.}$$

onside $\delta > 0 \Rightarrow$

$$\boxed{\mathcal{L}_{ij}(q.\omega) = \delta_{ij}(\delta + \chi q^2 - N_F i\frac{\pi}{2}s)}$$

real time

over-damped modes.

$$s \propto \frac{-\kappa}{N_F}\frac{2}{\pi} i\, q^2$$

$$\Rightarrow \boxed{\omega \propto i q^3}$$

We also need it's imaginary expressiun for thermal study later

The key integral is

$$N_F \int \frac{d\Omega}{4\pi} \frac{\frac{i\omega_n}{v_F q}}{\frac{i\omega_n}{v_F q} - \cos\theta} = \frac{N_F}{2}\int_{-1}^{1} dx \frac{\frac{i\omega_n}{v_F q}}{\frac{i\omega_n}{v_F q} - x}$$

$$= \frac{N_F}{2}\cdot\frac{1}{2}\int_{-1}^{1} dx \left[\frac{i\omega_n/v_F q}{i\omega_n/v_F q - x} + \frac{i\omega_n/v_F q}{i\omega_n/v_F q + x} \right]$$

$$\frac{\pi}{2} N_F \left|\frac{\omega_n}{v_F q}\right|$$

as $\omega_n \ll v_F q$

$$= \frac{N_F}{2}\int_{-1}^{1} dx \frac{\left(\frac{\omega_n}{v_F q}\right)^2}{x^2 + \left(\frac{\omega_n}{v_F q}\right)^2} = \frac{N_F}{2}\left|\frac{\omega_n}{v_F q}\right| \arctan\left(\frac{x}{\left|\frac{\omega_n}{v_F q}\right|}\right)\Bigg|_{-\left|\frac{v_F q}{\omega_n}\right|}^{\left|\frac{v_F q}{\omega_n}\right|}$$

$$\Rightarrow \boxed{\mathcal{L}_{ij}(q, i\omega_n) = \delta_{ij}\left(\delta + xq^2 + \frac{\pi}{2} N_F \left|\frac{\omega_n}{v_F q}\right|\right)}$$

③ RPA calculation of spin-wave spectrum

we need to consider the Gaussian fluctuations in the ordered phase.

Set $\bar{n}_i = \bar{n}\,\delta_{iz}$. $\Rightarrow G_0(k, ik_n) = \frac{1}{2}\left[\frac{1+\sigma_z}{ik_n - \mathcal{E}_\uparrow(k)} + \frac{1-\sigma_z}{ik_n - \mathcal{E}_\downarrow(k)}\right].$

$$\mathcal{E}_{\uparrow,\downarrow} = \mathcal{E}_0(k) - \mu \mp \bar{n}$$

Conside transverse modes:

A: $\frac{1}{V\beta}\sum_{k, i\omega_n} tr\left[G_0[\bar{n}, k+q/2]\,\sigma_x\, G_0[\bar{n}, k-\frac{q}{2}]\,\sigma_x\right]$ ← $\delta n_x(q\,\omega)\,\delta n_x(-q-\omega)$

先求 trace $\frac{1}{4} tr\left[\left(\frac{1+\sigma_z}{ik_n + iq_n - \mathcal{E}_\uparrow(k+q/2)} + \frac{1-\sigma_z}{ik_n + iq_n - \mathcal{E}_\downarrow(k+q/2)}\right)\sigma_x\left(\frac{1+\sigma_z}{ik_n - \mathcal{E}_\uparrow(k-q/2)} + \frac{1-\sigma_z}{ik_n - \mathcal{E}_\downarrow(k-q/2)}\right)\right]$

$$tr[(1\pm\sigma_z)\sigma_x(1\pm\sigma_z)\sigma_x] = tr[\sigma_x\sigma_x + \sigma_z\sigma_x\sigma_z\sigma_x] = 0$$

$$tr[(1\pm\sigma_z)\sigma_x(1\mp\sigma_z)\sigma_x] = tr[\sigma_x\sigma_x - \sigma_z\sigma_x\sigma_z\sigma_x] = 2$$

⇒ The above expression [illegible]

$$\rightarrow \frac{1}{2V\beta}\sum_{\vec{k}, ik_n}\left[\frac{1}{ik_n + iq_n - \mathcal{E}_\uparrow(k+q/2)}\,\frac{1}{ik_n - \mathcal{E}_\downarrow(k-q/2)} + \frac{1}{ik_n + iq_n - \mathcal{E}_\downarrow(k+q/2)}\,\frac{1}{ik_n - \mathcal{E}_\uparrow(k-q/2)}\right]$$

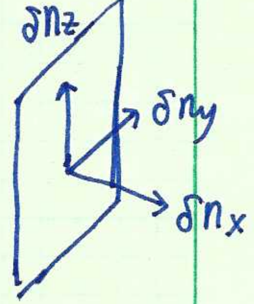
define $A(q, iq_n) = \frac{1}{V\beta}\sum_{k, ik_n}\frac{1}{ik_n + iq_n - \mathcal{E}_\uparrow(k+q/2)}\,\frac{1}{ik_n - \mathcal{E}_\downarrow(k-q/2)}$

$$B(q, iq_n) = \frac{1}{V\beta}\sum_{k, ik_n}\frac{1}{ik_n + iq_n - \mathcal{E}_\downarrow(k+q/2)}\,\frac{1}{ik_n - \mathcal{E}_\uparrow(k-q/2)}$$

$$\Leftrightarrow \boxed{A(-q, -iq_n) = B(q, iq_n)}$$

[redacted]

δn_x, δn_y 未耦合



$$\Rightarrow \boxed{\frac{1}{2}\,\delta n_x(q,\omega)\,\delta n_x(-q,-\omega)\left[A(q\,\omega)+B(q,\omega)\right]}$$

Similarly for y-y [redacted]

$$\frac{1}{2}\,\delta n_y(q\,\omega)\,\delta n_y(-q,-\omega)\left[A(q\omega)+B(q\omega)\right]$$

if $\delta n_z \neq 0$, no verticle reflection plane sym.

consider $\delta n_x(q\,\omega)\,\delta n_y(-q-\omega)$

$$tr\left[G(\bar{n},k+q/2)\,\sigma_x\,G(\bar{n},k-q/2)\,\sigma_y\right]$$

$$=\tfrac{1}{4}tr\left[\left(\frac{1+\sigma_z}{ik_n+iq_n-\varepsilon_\uparrow(k+q/2)}+\frac{1-\sigma_z}{ik_n+iq_n-\varepsilon_\downarrow(k+q/2)}\right)\sigma_x\left(\frac{1+\sigma_z}{ik_n-\varepsilon_\uparrow(k-q/2)}+\frac{1-\sigma_z}{ik_n-\varepsilon_\downarrow(k-q/2)}\right)\sigma_y\right]$$

$$tr\left[(1\pm\sigma_z)\sigma_x(1\pm\sigma_z)\sigma_y\right]=\pm tr\left[\sigma_z\sigma_x\sigma_y\right]\pm tr\left[\sigma_x\sigma_z\sigma_y\right]=0$$

$$tr\left[(1\pm\sigma_z)\sigma_x(1\mp\sigma_z)\sigma_y\right]=\pm tr\left[\sigma_z\sigma_x\sigma_y\right]\mp tr\left[\sigma_x\sigma_z\sigma_y\right]=\pm 2i$$

$$\rightarrow\ \frac{i}{2}\,\delta n_x(q\,\omega)\,\delta n_y(-q,-\omega)\left[A(q,\omega)-B(q,\omega)\right]$$

similarly

$$\frac{-i}{2}\,\delta n_y(q\,\omega)\,\delta n_x(-q,-\omega)\left[A(q\omega)-B(q\omega)\right]$$

$$\Rightarrow \mathcal{L}=\sum_{q\omega}\left\{\frac{1}{\sqrt{2}}\left[\delta n_x(q,\omega)+i\,\delta n_y(q\omega)\right]\right\}\left\{\frac{\delta n_x(-q-\omega)-i\,\delta n_y(-q,-\omega)}{\sqrt{2}}\right\}\ B(q,\omega)$$

$$+\left\{\frac{\delta n_x(q\omega)-i\,\delta n_y(q\omega)}{\sqrt{2}}\right\}\left\{\frac{\delta n_x(-q-\omega)+i\,\delta n_y(-q,-\omega)}{\sqrt{2}}\right\}\ A(q,\omega)$$

不独立，实为一个表达式。 [redacted]

$$=\ [redacted]\ \sum_{q,\omega}\left(\delta n_x(q\,\omega)+i\,\delta n_y(q,\omega)\right)\left(\delta n_x(-q-\omega)-i\,\delta n_y(-q,-\omega)\right)\ B(q,\omega)$$

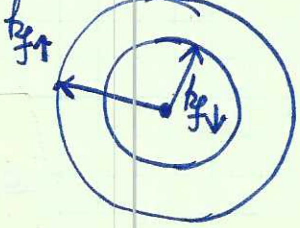
define $\delta n_\pm(q,\omega) = \delta n_x(q\,\omega) \pm i\,\delta n_y(q,\omega)$

$$\mathcal{L} = \sum_{q\omega} \delta n_+(\vec{q}\,\omega)\,\delta n_-(-\vec{q},-\omega)\left[\frac{\chi}{2}q^2 + \frac{1}{2|f_0|} + B(q,\omega)\right]$$

$\delta n_\pm$ are complex field, $\left[n_+(\vec{q},\omega)\right]^* = n_-(-\vec{q},-\omega)$.

$$B(q,\omega) = \frac{1}{V\beta}\sum_{k,ik_n} \frac{1}{ik_n+\omega-\varepsilon_\downarrow(k+q/2)}\;\frac{1}{ik_n-\varepsilon_\uparrow(k-q/2)}$$

$$= \frac{1}{V}\sum_k \frac{n_f(\varepsilon_\uparrow(k-q/2)) - n_f(\varepsilon_\downarrow(k+q/2))}{\omega+\varepsilon_\uparrow(k-q/2)-\varepsilon_\downarrow(k+q/2)}$$



$$\varepsilon_{\uparrow,\downarrow}(k-q/2) = \varepsilon(k-q/2) \mp n$$

$$\Rightarrow B(q\,\omega) = \frac{1}{V}\sum_k\left[\frac{n_f(\varepsilon_\uparrow(k))}{\omega-2\bar{n}+\varepsilon(k)-\varepsilon(k+q)} - \frac{n_f(\varepsilon_\downarrow(k))}{\omega-2\bar{n}+\varepsilon(k-q)-\varepsilon(k)}\right]$$

$$\varepsilon_{\vec{k}\pm\vec{q}} - \varepsilon_{\vec{k}} = \pm\vec{q}\cdot\vec{\nabla}\,\varepsilon_k + \frac{1}{2}(\vec{q}\cdot\vec{\nabla})^2\varepsilon_k$$

$$\frac{1}{\omega-2\bar{n}-(\vec{q}\cdot\vec{\nabla})\varepsilon_k+\frac{1}{2}(\vec{q}\cdot\vec{\nabla})^2\varepsilon_k} = -\frac{1}{2n}\left[\frac{1}{1-\frac{\omega-(\vec{q}\cdot\nabla)\varepsilon_k-\frac{1}{2}(q\cdot\nabla)^2\varepsilon_k}{2n}}\right]$$

$$= -\frac{1}{2n}\left\{1+\frac{\omega-(\vec{q}\cdot\nabla)\varepsilon_k-\frac{1}{2}(q\cdot\nabla)^2\varepsilon_k}{2n}+\frac{\left[\omega-(\vec{q}\cdot\nabla)\varepsilon_k-\frac{1}{2}(\vec{q}\cdot\vec{\nabla})^2\varepsilon_k\right]^2}{4n^2}\right.$$

$$\sim \frac{-1}{2n}\left\{1+\frac{\omega-\frac{1}{2}(q\cdot\nabla)^2\varepsilon_k}{2n}+\frac{\omega^2+\left[(\vec{q}\cdot\nabla)\varepsilon_k\right]^2}{4n^2}\right\}$$ ← 含 $\vec{q}\cdot\nabla$ 奇次项都消去了，在对角度积分后

$\frac{\omega^2}{4n^2}$ 可以略去，因为 $\omega\sim q^2$，这个 term 也是高阶的。
在色散关系处。

$$\Rightarrow \frac{1}{\omega - 2\bar{n} - (\vec{q}\cdot\vec{\nabla})\varepsilon_k - \frac{1}{2}(\vec{q}\cdot\vec{\nabla})^2\varepsilon_k} \sim \frac{-1}{2n}\left[1 + \frac{\omega}{2n} - \frac{(q\cdot\nabla)^2\varepsilon_k}{4n} + \frac{(q\cdot\nabla\varepsilon_k)^2}{4n^2}\right]$$

Similarly

$$\frac{1}{\omega - 2\bar{n} + \varepsilon(k-q) - \varepsilon(k)} = \frac{1}{\omega - 2\bar{n} - (\vec{q}\cdot\vec{\nabla})\varepsilon_k + \frac{1}{2}(\vec{q}\cdot\nabla)^2\varepsilon_k}$$

$$\sim \frac{-1}{2n}\left[1 + \frac{\omega}{2n} + \frac{(q\cdot\nabla)^2\varepsilon_k}{4n} + \frac{(q\cdot\nabla\varepsilon_k)^2}{4n^2}\right]$$

$$\Rightarrow B(q\,\omega) = \int_{k_{f\downarrow}}^{k_{f\uparrow}}\frac{d^3k}{(2\pi)^3}\left(\frac{-1}{2\bar{n}}\right) + \int_{k_{f\downarrow}}^{k_{f\uparrow}}\frac{d^3k}{(2\pi)^3}\frac{-\omega}{(2\bar{n})^2} - \underbrace{\int_{k_{f\downarrow}}^{k_{f\uparrow}}\frac{d^3k}{(2\pi)^3}\frac{(q\cdot\nabla\varepsilon_k)^2}{8n^3}}$$

$$+ \underbrace{\int\frac{d^3k}{(2\pi)^3}\frac{(n_f(\varepsilon_\uparrow)+n_f(\varepsilon_\downarrow))(q\cdot\nabla)^2\varepsilon_k}{8n^2}} \longrightarrow \text{give correction to spin-wave stiffness.}$$

Due to ~~self-consistent~~ Eq.

$$\boxed{\frac{1}{|f_0|} = \int_{k_{f\downarrow}}^{k_{f\uparrow}}\frac{d^3k}{(2\pi)^3}\frac{1}{\bar{n}}}$$

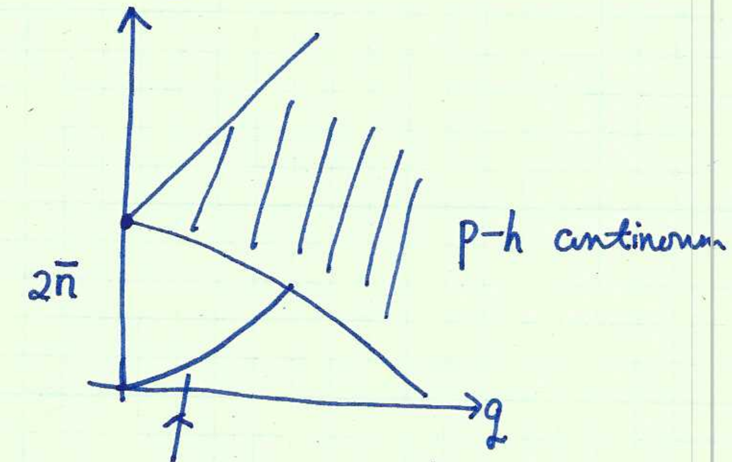
$$\Rightarrow \underset{\frac{\chi q^2}{2}+}{\wedge}\frac{1}{2|f_0|} + B(q,\omega) = \frac{-\omega}{4\bar{n}}\frac{1}{|f_0|} + \frac{\chi'}{2}q^2$$

$$\text{where } \chi' = \chi + \frac{1}{8n^2}\int\frac{d^3k}{(2\pi)^2}\left\{\frac{[n_f(\varepsilon_\uparrow)+n_f(\varepsilon_\downarrow)]}{m} - \frac{\overset{[n_f(\varepsilon_\uparrow)-n_f(\varepsilon_\downarrow)]\ \leftarrow v_F}{}}{n}\right\}$$

$$\Rightarrow \boxed{\delta L = \underset{\text{Spin-wave}}{\underset{\uparrow}{\sum_{q\omega}}} \delta n_+(\vec{q}.\omega)\,\delta n_-(\vec{q}.\omega)\left[-\frac{\omega}{4\bar{n}}\frac{1}{|f_0|} + \frac{\chi'}{2}q^2\right]}$$

linear in $\omega \Rightarrow \delta n_+$ and δn_- are conjugate to each other.

$$\omega_{kq} = \varepsilon_{k+q,\downarrow} - \varepsilon_{k\uparrow} = 2n + \varepsilon_{k+q} - \varepsilon_{k}$$



spin-wave model
outside the p-h continuum.