Interacting electron gas — life time, Fermi surface ①

§ Fermi golden rule — many body version

Consider put an electron in the plane wave state $\vec{k}$ outsider the ground state $|G\rangle$ (interacting systems). This is not the eigenstate, and the $\vec{k}$-electron will decay, say, to $\vec{k}-\vec{q}$, and the Fermi sea is also excited to be $|n\rangle$.

$|ini\rangle = C_k^\dagger |G\rangle$ $\qquad$ $|f\rangle = C_{k-q}^\dagger |n\rangle$

The scattering Hamiltonian $H'_{int} = \frac{1}{V} v(q)\, C_{k-q}^\dagger C_k \rho_q^\dagger$, where $k, k-q > k_f$. The frequency transfer $\omega_q = \frac{1}{\hbar}(\varepsilon_k - \varepsilon_{k-q})$.

The Fermi golden rule (transition rule)

$$W(\vec{q}, \omega_q) = \frac{2\pi}{\hbar} \sum_n |\langle f | H'_{int} | i \rangle|^2 \, \delta(\hbar\omega_q - (E_f - E_i))$$

— Check unit, correct!

Summing over $\vec{q}$, we have

$$\frac{1}{\tau_k} = \sum_q{}' W(\vec{q}, \omega_q) = \frac{2\pi}{\hbar V^2} \sum_q v^2(q) \sum_n \langle n | \rho_q^\dagger | 0 \rangle \, \delta(\hbar\omega_q - (E_n - E_0))$$

where $|n\rangle = |f\rangle$, and we'll use $\hbar\omega_{n0} = E_n - E_0$. $\sum_q{}'$ means the constraint on phase space with $\varepsilon_{k-q} > 0$ and $\varepsilon_k - \varepsilon_{k-q} > 0$.

Remember the sum rule

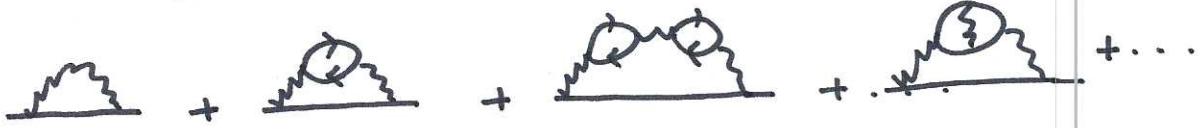
$$\mathrm{Im}\,\frac{1}{\varepsilon(q,\omega)} = \frac{\pi v(q)}{\hbar V}\sum_n |\langle n|\rho_q|0\rangle|^2\left(\delta(\omega+\omega_{n0})-\delta(\omega-\omega_{n0})\right).$$

for isotropic system, $|\langle n|\rho_q|0\rangle|^2 = |\langle n|\rho_{-q}|0\rangle|^2 = |\langle n|\rho_q^+|0\rangle|^2$

$$\Rightarrow W(\vec{q},\omega_q) = \frac{-2v(q)}{V\hbar}\,\mathrm{Im}\,\frac{1}{\varepsilon(q,\omega_q)} \quad \text{for } \omega_q = \varepsilon_k - \varepsilon_{k-q} > 0.$$

$$\boxed{\frac{1}{\tau_k} = \frac{-2}{\hbar V}\sum_q{}' v(q)\,\mathrm{Im}\,\frac{1}{\varepsilon(q,\omega_q)}, \quad \text{with } \omega_q = \varepsilon_k - \varepsilon_{k-q}}$$

§ From Green's function point of view $\frac{1}{\tau_k} = -2\,\mathrm{Im}\,\Sigma_{ret}(k,\varepsilon_k)$. Let's derive it digrammatically.



$$\Sigma(k,ik_n) = -\frac{1}{V}\sum_q v_q \frac{1}{\beta}\sum_{iq_n}\frac{\mathcal{G}^0(k+q,ik_n+iq_n)}{\varepsilon(q,iq_n)}$$

use spectra representation

$$\mathcal{G}^0(k+q,ik_n+iq_n) = \int_{-\infty}^{+\infty}\frac{d\varepsilon'}{2\pi}\,\frac{A(k+q,\varepsilon')}{ik_n+iq_n-\varepsilon'}$$

$$\frac{v_q}{\varepsilon(q,iq_n)} = \int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi}\,\frac{B(q,\omega')}{iq_n-\omega'}$$

and $$-\frac{1}{\beta}\sum_{iq_n}\frac{1}{ik_n+iq_n-\varepsilon'}\,\frac{1}{iq_n-\omega'} = \frac{n_B(\omega')+n_F(\varepsilon')}{ik_n+\omega'-\varepsilon}$$ please check!

$$\Rightarrow \Sigma(k, ik_n) = \frac{1}{V}\sum_q \int_{-\infty}^{+\infty}\frac{d\varepsilon'}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega'}{2\pi} A(k+q,\varepsilon')\, B(q,\omega')\, \frac{n_B(\omega') + n_F(\varepsilon')}{ik_n + \omega' - \varepsilon'}$$

at $T=0$, $n_B(\omega') = -\Theta(-\omega')$, & $n_F(\epsilon') = \Theta(-\epsilon')$

$$\Sigma_{ret}(k,\varepsilon) = \int\frac{d\vec{q}}{(2\pi)^3}\left[\int_{-\infty}^{0}\frac{d\epsilon'}{2\pi}\int_{0}^{+\infty}\frac{d\omega'}{2\pi} - \int_{0}^{+\infty}\frac{d\epsilon'}{2\pi}\int_{-\infty}^{0}\frac{d\omega'}{2\pi}\right]\frac{A(k+q,\epsilon')\, B(q,\omega')}{\varepsilon+\omega'-\varepsilon'+i\eta}$$

The spectra functions: $A(k+q,\varepsilon') = 2\pi\delta(\varepsilon' - \varepsilon_{k+q})$

$$B(q,\omega') = -2\, \mathrm{Im}\frac{v_q}{\varepsilon_{ret}(q,\omega'+i\eta)}$$

$$\Rightarrow \Sigma_{ret}(k,\varepsilon) = \int\frac{d\vec{q}}{(2\pi)^3}\left[\int_{-\infty}^{0}\frac{d\epsilon'}{2\pi}\int_{0}^{+\infty}\frac{d\omega'}{2\pi} - \int_{0}^{+\infty}\frac{d\epsilon'}{2\pi}\int_{-\infty}^{0}\frac{d\omega'}{2\pi}\right] 2\pi\delta(\epsilon'-\epsilon_{k+q})(-2)\, \mathrm{Im}\frac{v_q}{\varepsilon_{ret}(q,\omega'+i\eta)}$$

$$\times \frac{1}{\varepsilon+i\eta+\omega'-\varepsilon'}$$

For bosonic fregnency spectra function, it's odd respect to ω'

$$\Rightarrow \Sigma_{ret}(k,\varepsilon) = -\int\frac{d^3\vec{q}}{(2\pi)^3}\int_0^{+\infty}\frac{d\omega'}{\pi}\left\{\Theta(-\epsilon_{k+q})\frac{1}{\varepsilon+i\eta+\omega'-\varepsilon_{k+q}} + \Theta(\varepsilon_{k+q})\frac{1}{\varepsilon+i\eta-\omega'-\varepsilon(k+q)}\right\}$$

$$v_q\, \mathrm{Im}\frac{1}{\varepsilon_{ret}(q,\omega')}$$

$$\mathrm{Im}\,\Sigma_{ret}(k,\varepsilon) = \int\frac{d^3\vec{q}}{(2\pi)^3}\int_0^{+\infty}d\omega'\, \Theta(\varepsilon_{k+q})\,\delta(\varepsilon-\omega'-\varepsilon(k+q))\, v_q\, \mathrm{Im}\frac{1}{\varepsilon_{ret}(q\omega')}$$

$$\begin{cases} \quad \text{for } (\varepsilon>0). \\ \displaystyle\int\frac{d^3\vec{q}}{(2\pi)^3}\int_0^{+\infty}d\omega'\, \Theta(-\varepsilon_{k+q})\,\delta(\varepsilon+\omega'-\varepsilon_{k+q})\, v_q\, \mathrm{Im}\frac{1}{\varepsilon_{ret}(q\omega')} \quad (\text{for } \varepsilon<0) \end{cases}$$

$$\operatorname{Im}\Sigma_{ret}(k,\varepsilon) = \begin{cases} \displaystyle\int \frac{d^3\vec{q}}{(2\pi)^3}\, \Theta(\varepsilon_{k+q})\,\Theta(\varepsilon-\varepsilon_{k+q})\, v_q \operatorname{Im}\left(\frac{1}{\varepsilon_{ret}(q\;\;)}\right) & \varepsilon > 0 \\ \displaystyle\int \frac{d^3\vec{q}}{(2\pi)^3}\, \Theta(-\varepsilon_{k+q})\,\Theta(\varepsilon_{k+q}-\varepsilon)\, v_q \operatorname{Im}\frac{1}{\varepsilon_{ret}(q,\varepsilon-\varepsilon_{k-q})} & \varepsilon < 0 \end{cases}$$

$\varepsilon-\varepsilon_{k-q}$

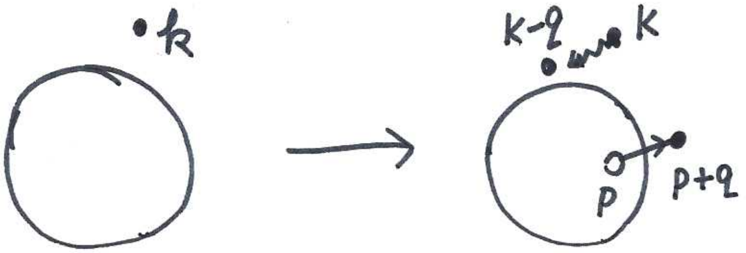
take $k > k_f$, $\varepsilon = \varepsilon_k > 0$, $\Rightarrow$ (also change $q \to -q$)

$$\operatorname{Im}\Sigma_{ret}(k,\varepsilon_k) = \underbrace{\int \frac{d^3\vec{q}}{(2\pi)^3}\, \Theta(\varepsilon_k-\varepsilon_{k-q})\,\Theta(\varepsilon_{k-q})}_{\frac{1}{V}\sum_q'} v_q \operatorname{Im}\frac{1}{\varepsilon_{ret}(q,\varepsilon_k-\varepsilon_{k-q})}$$

$$\Rightarrow \boxed{\frac{1}{\tau_k} = -2\operatorname{Im}\Sigma_{ret}(k,\varepsilon_k)}$$

3: Change a view —— screened Coulumb potential.

In this picture, we view the Fermi sea as free electron plane wave Slater determinaint state. The states after scattering are also plane-waves, but the interaction we will use is the screened Coulumb.



This is equivalent to change a representation, but the matrix elements are the same.

Fermi golden rule $\displaystyle\frac{1}{\tau_k} = \sum_q{}' W(\vec{q},\varepsilon_k-\varepsilon_{k-q})$

$$= \frac{2\pi}{\hbar^2 V}\sum_q{}'\sum_{p\sigma} \frac{v^2(q)}{|\varepsilon(q,\varepsilon_k-\varepsilon_{k-q})|^2}\, n_p(1-n_{p+q})\,\delta(\omega-(\varepsilon_k-\varepsilon_{k-q}))$$

Remember in the Lindhardt function:

$$\chi^{0}_{ret}(q,\omega) = -\frac{2}{V}\sum_{k}\frac{n_k - n_{k+q}}{\hbar(\omega-\omega_{kq})+i\eta} \qquad \leftarrow n_k - n_{k+q} = n_k(1-n_{k+q}) - n_{k+q}(1-n_k)$$

The positive ω part $\rightarrow$ $\mathrm{Im}\,\chi^{0}_{ret}(q,\omega) = \frac{2}{\hbar V}\sum_{k}\pi n_k(1-n_{k+q})\,\delta(\omega-\omega_{kq})$

For the RPA form $\varepsilon(q,\omega) = 1 + \frac{4\pi e^2}{q^2}\chi^{0}(q,\omega+i\eta)$

$$\Rightarrow \mathrm{Im}\,\varepsilon(q,\omega) = \frac{\pi}{\hbar V}v(q)\sum_{k\sigma}n_k(1-n_{k+q})\,\delta(\omega-\omega_{kq}) \quad \text{for } \omega>0.$$

$$\Rightarrow \frac{1}{\tau_k} = \frac{2}{\hbar V}\sum_{q}{}' v(q)\frac{\mathrm{Im}\,\varepsilon(q,\omega_q)}{|\varepsilon(q,\omega_q)|^2} = \boxed{-\frac{2}{\hbar V}\sum_{q}{}' v(q)\,\mathrm{Im}\frac{1}{\varepsilon_{RPA}(q,\omega_q)} = \frac{1}{\tau_k} \qquad \omega_q = \omega_k - \omega_{k+q}}$$

§4: Now we ready to do real calculation

$$\mathrm{Im}\,\varepsilon(q,\omega\to 0) = \frac{\pi}{2}\left(\frac{k_{TF}}{q}\right)^2\frac{\omega}{v_F q} \rightarrow \frac{\pi}{2}\left(\frac{k_{TF}}{q}\right)^2\frac{(\varepsilon_k - \varepsilon_{k-q})}{\hbar v_F q}$$

for small frequency, we can have $\mathrm{Re}\,\varepsilon(q,\omega) \gg \mathrm{Im}\,\varepsilon(q,\omega)$, and we can set $\mathrm{Re}\,\varepsilon(q,\omega) \simeq \varepsilon(q,0)$. $\Rightarrow$

$$\frac{1}{\tau_k} = \frac{\pi}{V\hbar}\sum_{q}{}' v(q)\left(\frac{k_{TF}}{q}\right)^2\frac{(\varepsilon_k - \varepsilon_{k-q})}{\hbar q v_F|\varepsilon(q,0)|^2}$$

in the long wave length limit, $\varepsilon(q,0) = 1 + \frac{k_{TF}^2}{q^2}$

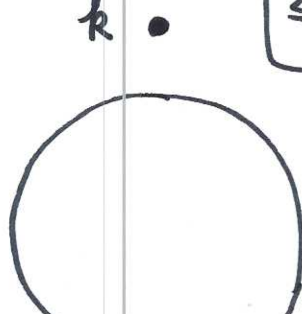
and $\omega_k - \omega_{k-q} \simeq \frac{k_F q}{m}\cos\theta$, where θ is the angle between $\hat{k}$ and $\hat{q}$.

we also need the constraint

$$\varepsilon_k > \varepsilon_{k-q} > \varepsilon_F$$

$\leftarrow k^2+q^2-2kq\cos\theta > k_f^2 \Rightarrow \boxed{\cos\theta \le \frac{k^2-k_f^2}{2kq}}$

k



$$\Rightarrow \quad 0 \le \cos\theta \le \frac{k^2-k_F^2}{2k_f q} \equiv z_m$$

$$\Rightarrow \frac{1}{\tau_k} = e^2 k_{TF}^2 \int_0^{2k_f} dq \int_0^{z_m} d\cos\theta \, \frac{\cos\theta}{q^2\left(1+\frac{k_{TF}^2}{q^2}\right)^2}$$

$$= \frac{e^2 k_{TF}^2}{8k_f^2}(k^2-k_f^2)^2 \int_0^{2k_f} dq \, \frac{1}{(q^2+k_{TF}^2)^2}$$

$\leftarrow$ the upper limit of $q = 2k_f$ should not be considered seriously. The TF approximation for $\varepsilon(q,0)$ only valid at $q \ll k_f$.

$$= \frac{e^2 k_{TF}^2}{8k_f^2}(k^2-k_f^2)^2 \, k_{TF}^{-3} \int_0^{+\infty} dx \, \frac{1}{(x^2+1)^2}$$

$$\approx \frac{e^2}{8k_{TF}} \frac{4k_f^2}{k_f^2}(k-k_f)^2 \cdot \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{1}{\tau_k} = \frac{\pi}{8}\frac{e^2}{k_{TF}}\frac{\varepsilon_f^2}{v_F^2}\left(\frac{\varepsilon-\varepsilon_f}{\varepsilon_f}\right)^2 = \frac{\pi}{32}\frac{e^2 k_f^2}{k_{TF}}\left(\frac{\varepsilon-\varepsilon_f}{\varepsilon_f}\right)^2$$

$$\sim \frac{\pi^2\sqrt{3}}{128}\,\omega_p\left(\frac{\varepsilon-\varepsilon_f}{\varepsilon_f}\right)^2,$$

where $\omega_p^2 = \frac{4\pi \rho e^2}{m}$ is the plasma frequency.

$$\Rightarrow \operatorname{Im}\Sigma_{ret}(k,\varepsilon) = -\frac{\pi^2\sqrt{3}}{256}\,\omega_p\left(\frac{\varepsilon-\varepsilon_f}{\varepsilon_f}\right)^2$$

— long life time well-defined quasi-particle.

Landau Fermi liquid theory (I)

§ Concept of quasi-particles

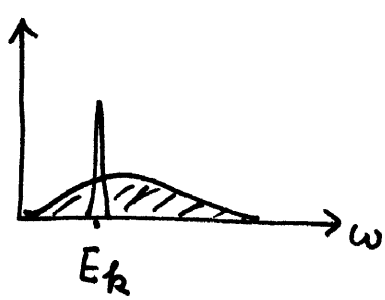
Suppose we have a many-body fermion ground state $|G\rangle$. At time $t=0$, an extra particle is added at the plane wave state $C_k^\dagger$. After a time interval of T, we check what's the amplitude remaining in this state.

$$G_k(t) = \langle G| e^{iHT} C_k \, e^{-iHT} C_k^\dagger | G \rangle = \langle G | C_k(T) \, C_k^\dagger(0) | G \rangle$$

in terms of Lehmann's representation

$$G_k(t) = \sum_m \langle G | C_k(T) | m \rangle \langle m | C_k^\dagger(0) | G \rangle = \sum_m |\langle G | C_k | m \rangle|^2 \, e^{-i(E_m - E_g)T}$$

In the Fermi liquid state, the distribution of the spetral weight has a special value of δ-peak (may be brodened), and a continuum (incoherent part)



$$Z = |\langle \Psi_k | C_k | G \rangle|^2 \leftarrow$$ wavefunction renormalization factor

$$\Rightarrow G_k(t) = Z \, e^{-iE_k' T} \leftarrow$$ quasi-particle

$$+ \int \frac{d\omega}{2\pi} A(\omega) \, e^{-i\omega t}$$

"$0 < Z < 1$" justifies the validity of Fermi liquid state. Even through in an interacting system, we can still look it as if a free system. Quasi-particle is like "a running horse running in a dusty road, dressed by a cloud of dust

§ Physical content of the self-energy P_{249} Negele & Orland.

no-interacting Green function $G_0(k,\omega) = \dfrac{1}{\omega - \varepsilon_k + i\,\mathrm{sgn}(\omega)\eta} \rightarrow$ single pole with residue "1".

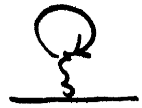
For the full green's function

$$G(k,\omega) = \frac{1}{\omega - \varepsilon_k - \Sigma(k,\omega)},$$ we need to exam the structures of poles and the residues.

Let us only exam the first two order pertubation theory

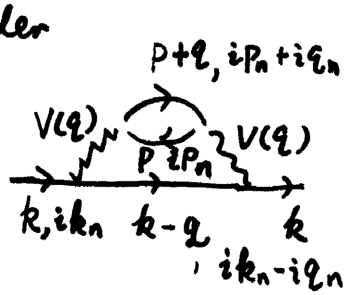
$$G(k,\omega) = \frac{1}{\omega - \varepsilon_k - \Sigma_1(k) - \Sigma_2(k,\omega)}$$

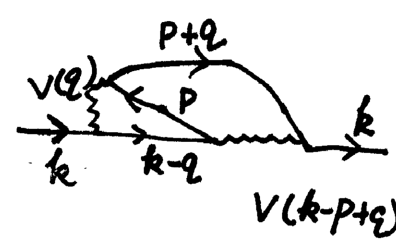
The first order is HF which is frequency independent

$\Sigma_1(k) =$  $+$  $= \sum_{k'} v(q=0)\, n_{k'} - V(k-k')\, n_{k'}$.

(v has no fregency dependence $\rightarrow$ $\Sigma_1(k)$ has no frequency dependence.

The second order

$\Sigma_2(k,\omega) =$ 

$+$ 

frequency summation

$$\frac{1}{\beta}\sum_{iq_n}\left[\frac{1}{\beta}\sum_{ip_n}\frac{1}{ip_n+iq_n-\varepsilon_{p+q}}\,\frac{1}{ip_n-\varepsilon_p}\right]\frac{1}{ik_n-iq_n-\varepsilon_{k-q}}$$

$$= \frac{1}{\beta}\sum_{iq_n}\frac{n_f(\varepsilon_p)-n_f(\varepsilon_{p+q})}{iq_n-(\varepsilon_{p+q}-\varepsilon_p)}\,\frac{1}{ik_n-iq_n-\varepsilon_{k-q}}$$

define $$S = \frac{1}{\beta}\sum_{iq_n} \frac{1}{iq_n - (\varepsilon_{p+q} - \varepsilon_p)} \, \frac{1}{ik_n - iq_n - \varepsilon_{k-q}}$$

& $$f(z) = \frac{1}{z - (\varepsilon_{p+q} - \varepsilon_p)} \, \frac{1}{ik_n - z - \varepsilon_{k-q}}$$

$$I = \lim_{R\to\infty} \int \frac{dz}{2\pi i} f(z) \frac{1}{e^{\beta z}+1} = 0. \Rightarrow$$

$$\frac{1}{\beta}\sum_n f(iq_n) + \frac{1}{e^{\beta(\varepsilon_{p+q}-\varepsilon_p)}-1} \, \frac{1}{ik_n - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{k-q})}$$

$$+ \frac{1}{e^{\beta(ik_n - \varepsilon_{k-q})}-1} \, \frac{-1}{ik_n - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{k-q})} = 0$$

$$\Rightarrow \frac{1}{\beta}\sum_n f(iq_n) = - \frac{[n_B(\varepsilon_{p+q}-\varepsilon_p) + 1 - n_f(\varepsilon_{k-q})]}{ik_n - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{k-q})}$$

$$\Rightarrow \Sigma_2(k,\omega) = \sum_p \sum_q \frac{[n_f(\varepsilon_p) - n_f(\varepsilon_{p+q})][1 - n_f(\varepsilon_{k-q}) + n_B(\varepsilon_{p+q} - \varepsilon_p)]}{\omega + i\eta - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{k-q})} [V^2(q) - V(q)V(k-p+q)]$$

$\uparrow$ real frequency

$\omega > 0$ for $k > k_f$.

$\Sigma_2(k,\omega)$ explicitly depends on ω. Σ_2 has an infinite number of poles at $\omega = \varepsilon_{p+q} - \varepsilon_p + \varepsilon_{k-q}$ $\longrightarrow$ finite imaginary part.

☆ Let us consider a simple example. If $\Sigma_2(k,\omega)$ has two poles

$$\Sigma_2(\omega) = \frac{A_1}{\omega - E_1 + i\eta} + \frac{A_2}{\omega - E_2 + i\eta},$$

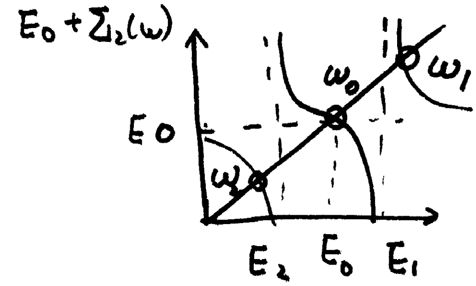
then what's the poles and residues of

$$G(\omega) = \frac{1}{\omega - E_0 - \Sigma_2(\omega)}$$

we assume E_1 is above E_0, and E_2 is below E_0, and the residues in Σ_2 are very small, satisfying $\frac{A_1}{|E_1-E_0||E_2-E_0|}, \frac{A_2}{|E_1-E_0||E_2-E_0|} \ll 1$.

Since Σ_2 is small at $\omega \neq E_1, E_2$, we first consider Σ_1, which gives the order of the pole of $\omega = E_0$. Including Σ_2, we solve $\omega = E_0 + \Sigma_2(\omega)$

ω_0 is close to E_0; $\omega_{1,2}$ close to $E_{1,2}$.



We expand $G(\omega)$ around each poles

$$\omega_i = E_0 + \Sigma_2(\omega_i) \qquad i = 0, 1, 2$$

$$\omega - E_0 - \Sigma_2(\omega) = \omega - E_0 - \Sigma_2(\omega_i) + (\omega - \omega_i)\Sigma_2'(\omega_i)$$

$$= (\omega - \omega_i)(1 - \Sigma_2'(\omega_i)) \Rightarrow G(\omega) \simeq \frac{1}{1-\Sigma_2'(\omega)} \frac{1}{\omega - \omega_i + i\eta}$$

$$\frac{1}{1-\Sigma_2'(\omega_0)} = \frac{1}{1 + \sum_i \frac{A_i}{(\omega_0 - E_i)^2}} \approx 1 - \sum_{i=1}^{2} \frac{A_i}{(E_0 - E_i)^2}$$

$$\frac{1}{1-\Sigma_2'(\omega_1)} = \frac{1}{1 + \frac{A_1}{(\omega_1 - E_1)^2} + \frac{A_2}{(\omega_1 - E_2)^2}} \approx \frac{(\omega_1 - E_1)^2}{A_1}$$

consider $\omega_1 = E_0 + \frac{A_1}{\omega_1 - E_1} + \frac{A_2}{\omega_1 - E_2}$

$$\Rightarrow \omega_1 - E_0 \approx \frac{A_1}{\omega_1 - E_1} \quad \text{or} \quad \omega_1 - E_1 \approx \frac{A_1}{\omega_1 - E_0}$$

$$\Rightarrow \frac{1}{1-\Sigma_2'(\omega_1)} \approx \frac{A_1}{(E_1 - E_0)^2}, \quad \text{similarly} \quad \frac{1}{1-\Sigma_2'(\omega_2)} \approx \frac{A_2}{(E_2 - E_0)^2}.$$

After switching on interaction, the single pole is fragmented into three poles. The major one is still close to E_0, but with a smaller residue $1 - \sum_i \frac{A_i}{(E_0 - E_i)^2}$. This pole is called the quasi-particle pole.

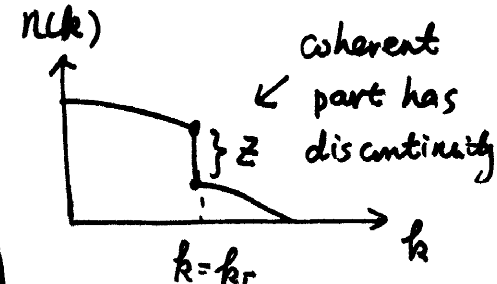
The ~~strength~~ removed from the quasi-particle pole has been redistributed to poles at $\omega_{1,2}$. These poles represent complicated many-body medium excitations of

Now consider the real case.

$$\boxed{\left.\frac{1}{1 - \frac{\partial}{\partial\omega}\Sigma_2(\omega)}\right|_{\omega \simeq E_0} \approx 1 - \sum_{p,q} \frac{A_{p,q}^2}{[E_0 - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{p-q})]^2} = Z}$$

depleted to the incoherent background.

The Landau fermi-liquid is based on the assuption that Z remains finite. $\rightarrow$



Now let's consider damping

$$\frac{1}{\tau} = -2\,\mathrm{Im}\,\Sigma_2 = 2\pi \sum_{p,q} \overbrace{|A_{p,q}|^2}\, \delta[\omega - (\varepsilon_{p+q} - \varepsilon_p + \varepsilon_{k-q})]$$

$$C_k^\dagger C_k = \underbrace{Z}_{\text{WF renormalization}} \underbrace{\tilde{C}_k^\dagger \tilde{C}_k}_{\text{quasi-particle}} + \cdots$$

$$\frac{1}{\tau} \approx |V|^2 \int_0^{\varepsilon_k} \underbrace{d\epsilon_{k-q}}_{\text{particle}} \int_0^{\epsilon_k - \epsilon_{k-q}} \underbrace{d\epsilon_{p+q}}_{\text{particle}} \; \rho(\epsilon_{k-q})\,\rho(\epsilon_{p+q})\,\rho(\underbrace{\varepsilon_p}_{\text{hole}} = -\varepsilon_k + (\varepsilon_{p+q} + \varepsilon_{k-q}))$$

$$\le |V|^2 \rho_{max}^3 \, \varepsilon_k^2 \quad \Longrightarrow \quad \boxed{\tau(\epsilon) \propto |\epsilon - \epsilon_F|^{-2}}$$

☆ effective mass.

$$\mathcal{E} = \frac{k^2}{2m} - \mu + \Sigma(k, \mathcal{E})$$

$$\frac{d\epsilon}{dk} = \frac{k}{m} + \frac{\partial \Sigma}{\partial k} + \frac{\partial \Sigma}{\partial \mathcal{E}} \frac{d\mathcal{E}}{dk} \Rightarrow \frac{d\mathcal{E}}{dk} = \left(1 - \frac{\partial \Sigma}{\partial \epsilon}\right)^{-1} \left(\frac{k}{m} + \frac{\partial \Sigma}{\partial k}\right)$$

$$= \left(1 - \frac{\partial \Sigma}{\partial \epsilon}\right)^{-1} \frac{k}{m} \left(1 + \frac{m}{k} \frac{\partial \Sigma}{\partial k}\right)$$

define $\frac{d\epsilon}{dk} = \frac{k}{m^*}$ $\Rightarrow$ $$\boxed{m^* = m \left(1 + \frac{m}{k} \frac{\partial \Sigma}{\partial k}\right)^{-1} \times \left(1 - \frac{\partial \Sigma}{\partial \omega}\right)\Big|_{\omega = \mathcal{E}}}$$

Interacting electron gas — ground state energy

§1: Hartree - Fock approximation energy

$$H = H_0 + H_{int} = \sum_{k\sigma} \frac{\hbar^2 k^2}{2m} C^+_{k\sigma} C_{k\sigma} + \frac{1}{2V} \sum_q{}' v(q) \left(\rho^+_q \rho_q - N \right)$$

$$\rho_q = \sum_{k\sigma} C^+_{k-q\sigma} C_{k\sigma} , \quad \rho_{-q} = \rho^+_q = \sum_{k\sigma} C^+_{k\sigma} C_{k-q,\sigma}$$

$$v(q) = \frac{4\pi e^2}{q^2} , \quad V(r) = \frac{1}{V} \sum_q e^{i\vec{q}\cdot\vec{r}} v(q)$$

HF approximation assumes a determinant wavefunction with filled Fermi surfaces. We have derived before that

$$\frac{1}{N} E_0(HFA) = \frac{3}{5} \frac{\hbar^2 k_F^2}{2m} - \frac{3e^2}{4\pi} k_F \quad \text{where } k_F^3 = 3\pi^2 N/V = 3\pi^2 \rho$$

if use the dimensionless parameter r_s defined through $\rho \frac{4\pi}{3} (r_s a_0)^3 = 1$

with $a_0 = \frac{\hbar^2}{me^2}$, we have $k_F = \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_s a_0}$, we express

$$\frac{1}{N} E_0(HFA) = \left[\frac{3}{10} \left(\frac{9\pi}{4}\right)^{2/3} \frac{1}{r_s^2} - \frac{3}{2\pi} \left(\frac{9\pi}{4}\right)^{1/3} \frac{1}{r_s} \right] \frac{e^2}{2a_0}$$

$$= \left(\frac{2.21}{r_s^2} - \frac{0.916}{r_s} \right) R_y , \quad \text{where } R_y = \frac{e^2}{2a_0} = 13.6\,\text{eV}.$$

§2 Dielectric-function

Response function: consider a many-body system with an external perturbation $H_e(t)$. The Schrödinger Eq reads (at $t \to -\infty$, $H_e(t) \to 0$).

$$i\hbar \frac{\partial}{\partial t} \psi = H\psi + H_e(t)\psi, \quad \longrightarrow \text{change to interaction picture}$$

$$\psi(t) = e^{-\frac{i}{\hbar}Ht} \varphi(t) \quad \to \quad i\hbar \frac{\partial}{\partial t} \varphi(t) = H_e'(t)\,\varphi$$

$$H_e'(t) = e^{\frac{i}{\hbar}Ht} \, H_e(t) \, e^{-\frac{i}{\hbar}Ht}$$

The time evolution:

$$\varphi(t) = \Phi_0 + \frac{1}{i\hbar}\int_{-\infty}^{t} H_e'(t')\,\varphi(t')\,dt' \,,$$

$\boxed{|\Phi_0\rangle \text{ is ground state of } H}$

$\xrightarrow{\text{linear order}}$

$$\varphi(t) = \left[1 + \frac{1}{i\hbar}\int_{-\infty}^{t} H_e'(t')\,dt'\right] \Phi_0$$

operator evolution $\quad A(t) = e^{i/\hbar\, Ht} \, A \, e^{-i/\hbar\, Ht}$

$$\Rightarrow \quad \bar{A} = \langle \varphi(t) | A(t) | \varphi(t) \rangle = \langle \Phi_0 | A(t) | \Phi_0 \rangle + \frac{1}{i\hbar}\int_{-\infty}^{t} \langle \Phi_0 | [A(t), H_e'(t')] dt' | \Phi_0 \rangle$$

$|\Phi_0\rangle$ is an eigenstate of $H \Rightarrow \langle \Phi_0 | A(t) | \Phi_0 \rangle = \langle \Phi_0 | A | \Phi_0 \rangle$

$\Rightarrow$ the response

$$\Delta A = \bar{A}(t) - \bar{A}(t \to -\infty) = \frac{1}{i\hbar}\int_{-\infty}^{+\infty} dt'\, \Theta(t-t') \langle \Phi_0 | [A(t), H_e'(t')] | \Phi_0 \rangle$$

Consider the perturbation

$$H_e(t) = \frac{1}{V} \sum_q \rho(-q, t)\, V_{ex}(q, t)$$

$$\delta\rho(q,t) = -\frac{i}{\hbar}\int_{-\infty}^{+\infty} dt'\, \Theta(t-t')\, \frac{1}{V}\, \langle \Phi_0 | [\rho(q t)\ \rho(-q t')] | \Phi_0 \rangle\, V_{ex}(q t')$$

$$= -\int_{-\infty}^{+\infty} dt'\ \chi_{ret}(q, t-t')\ V_{ex}(q, t')$$

$\rightarrow$ Fourier ~~trans~~form $\Rightarrow$ $\boxed{\delta\rho(q,\omega) = -\chi_{ret}(q,\omega)\ V_{ex}(q,\omega)}$

where $\displaystyle \chi_{ret}(q,\omega) = \frac{i}{\hbar}\frac{1}{V}\int_{-\infty}^{+\infty} dt\ e^{i(\omega+i\eta)t}\ \langle \Phi_0 | [\rho(q t)\ \rho(-q, 0)] | \Phi_0 \rangle$

From poissin equation: $-\nabla^2 V_{ind} = 4\pi e^2\, \delta\rho(q,\omega) \Rightarrow V_{ind}(q,\omega) = \dfrac{4\pi e^2}{q^2}\, \delta\rho(q,\omega)$

$$V_{tot} = V_{ex} + V_{ind} = V_{ex} + \frac{4\pi e^2}{q^2}\delta\rho(q,\omega) = \frac{1}{\epsilon} V_{ex}$$

$$\Rightarrow \boxed{\frac{1}{\epsilon(q,\omega)} = 1 - \chi_{ret}(q,\omega)\cdot\frac{4\pi e^2}{q^2}}$$

Now we use Lehman Representation

$$\chi_{ret}(q,\omega) = \frac{i}{V\hbar}\int_{-\infty}^{+\infty} dt\ \underset{\Theta(t)}{e^{i(\omega+i\eta)t}} \Big\{ \langle \Phi_0 | e^{iHt} \rho(q) e^{-iHt} | m \rangle \langle m | \rho(-q) | \Phi_0 \rangle - \langle \Phi_0 | \rho(-q) | m \rangle \langle m | e^{iHt} \rho(q) e^{-iHt} | \Phi_0 \rangle \Big\}$$

$$= \frac{1}{V}\sum_m \frac{i}{\hbar}\int_{-\infty}^{+\infty} dt\ \Theta(t) \Big[e^{i(\omega+i\eta)t + \frac{(E_0-E_m)t}{\hbar}} \langle \Phi_0 | \rho(q) | m \rangle \langle m | \rho(-q) | \Phi_0 \rangle - e^{i(\omega+i\eta + \frac{(E_m-E_0)}{\hbar})t} \langle \Phi_0 | \rho(-q) | m \rangle \langle m | \rho(q) | \Phi_0 \rangle \Big]$$

$$= \frac{1}{V}\Big[\sum_m \frac{-1}{\hbar\omega + E_0 - E_m + i\eta} \langle \Phi_0 | \rho(q) | m \rangle \langle m | \rho(-q) | \Phi_0 \rangle$$

$$+ \sum_m \frac{1}{\hbar\omega + E_m - E_0 + i\eta} \langle \Phi_0 | \rho(-q) | m \rangle \langle m | \rho(q) | \Phi_0 \rangle \Big]$$

$$\boxed{\chi_{ret}(q,\omega) = \frac{1}{V}\sum_m |\langle m|\rho(q)|0\rangle|^2 \left[\frac{1}{\hbar\omega - \hbar\omega_{m,0} + i\eta} - \frac{1}{\hbar\omega + \hbar\omega_{m,0} + i\eta}\right]}$$

we used $|\langle m|\rho_{-q}|0\rangle| = |\langle m|\rho_q|0\rangle|$ for isotropic systems.

and $|0\rangle$ for ground state

$$\Rightarrow \boxed{\frac{1}{\epsilon(q,\omega)} = 1 - \frac{4\pi e^2}{q^2 V}\sum_n |\langle n|\rho_q|0\rangle|^2 \left(\frac{1}{\hbar\omega + \hbar\omega_{n,0} + i\eta} - \frac{1}{\hbar\omega - \hbar\omega_{n,0} + i\eta}\right)}$$

Check dimension: $|\langle n|\rho_q|0\rangle|$ is dimensionless

Take imaginary part

$$\mathrm{Im}\frac{1}{\varepsilon(q,\omega)} = \pi v(q)\frac{1}{\hbar V}\sum_n |\langle n|\rho_q|0\rangle|^2\left[\delta(\omega + \omega_{n,0}) - \delta(\omega - \omega_{n,0})\right]$$

↙ sum rule

$$\int_0^\infty d\omega\, \mathrm{Im}\frac{1}{\varepsilon(q,\omega)} = -\pi v(q)\frac{1}{\hbar V}\sum_n \langle n|\rho_q|0\rangle\langle 0|\rho_{-q}|n\rangle$$

$$= -\frac{\pi v(q)}{\hbar V}\langle 0|\rho_q^\dagger \rho_q|0\rangle$$

$$\Rightarrow \boxed{\langle 0|H_{int}|0\rangle = \frac{1}{2V}\sum_q{}' \left[\langle 0|v(q)\rho_q^\dagger\rho_q|0\rangle - N v(q)\right] = -\left[\sum_q{}' \frac{\hbar}{2\pi}\int_0^\infty d\omega\, \mathrm{Im}\left[\frac{1}{\varepsilon(q,\omega)}\right] + \frac{1}{2}v(q)N\right]}$$

3. Feynman-Hellman theorem

Consider Hamiltonian containing parameter λ, denoted $H(\lambda)$, then the eigenvalue $E_n(\lambda)$ for the n-th eigenstate satisfies

$$\frac{\partial E_n(\lambda)}{\partial \lambda} = \langle \psi_n(\lambda) | \frac{\partial H}{\partial \lambda} | \psi_n(\lambda) \rangle,$$ where the wavefunction $|\psi_n(\lambda)\rangle$

is normalized, and $E_n(\lambda) = \langle \psi_n(\lambda) | H | \psi_n(\lambda) \rangle$. — Please prove it.

Consider $H(\lambda) = H_0 + \lambda H_{int}$, then $H(0) = H_0$ and $H(1) = H_0 + H_{int}$.

Then $E_G = E_0(\lambda=0) + \int_0^1 d\lambda \langle \psi_0(\lambda) | \frac{\partial H(\lambda)}{\partial \lambda} | \psi_0(\lambda) \rangle$

$$\boxed{E_G = E_0(\lambda=0) + \int_0^1 \frac{d\lambda}{\lambda} \langle \psi_0(\lambda) | \lambda H_{int} | \psi_0(\lambda) \rangle .}$$

The first term is the kinetic energy of the ground state of the free system. We use this trick because the kinetic energy on the true ground state is difficult to direct calculate. We define ε_λ as the dielectric function for the ground state for $H(\lambda)$. Then

$$\langle \psi_0(\lambda) | \lambda H_{int} | \psi_0(\lambda) \rangle = - \sum_q{}' \left[\frac{\hbar}{2\pi} \int_0^\infty d\omega \, \text{Im}\left[\frac{1}{\varepsilon_\lambda(q\omega)} \right] + \frac{\lambda}{2} v(q) N \right]$$

$$\Rightarrow \boxed{E_G = \frac{3}{5} N E_F - \sum_q{}' \left\{ \frac{\hbar}{2\pi} \int_0^\infty d\omega \int_0^1 \frac{d\lambda}{\lambda} \, \text{Im}\left[\frac{1}{\varepsilon_\lambda(q\omega)} \right] + \frac{1}{2} v(q) N \right\}}$$

where $\frac{\hbar}{\pi} \int_0^\infty d\omega \, \text{Im}\left[\frac{1}{\varepsilon_\lambda(q,\omega)} \right] = -\lambda v(q) \langle \psi_0(\lambda) | \rho_q^\dagger \rho_q | \psi_0(\lambda) \rangle$

§4: Hartree - Fock Approximation

The HFA does not change the ground state wavefunction, $|\psi_0(\lambda=0)\rangle$, but directly does 1st order perturbation theory based on $|\psi_0(\lambda=0)\rangle$, i.e

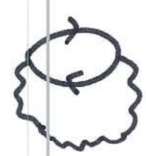
$$\frac{1}{\epsilon_{HFA}(q\omega)} = 1 - v(q)\,\chi^0_{ret}(q,\omega),$$ where $\chi^0_{ret}(q,\omega)$ is the Lindhardt response function. $\chi^0_{ret}(q,\omega)$ is the retarded response function for the free system: it's Lehman reepresentation

$$\chi^0_{ret}(q\omega) = \frac{1}{V}\sum |\langle \psi_m(\lambda=0)|\rho(q)|\psi_0(\lambda=0)\rangle|^2 \left[\frac{1}{\hbar\omega - \hbar\omega^0_{m,0} + i\eta} - \frac{1}{\hbar\omega + \hbar\omega^0_{m0} + i\eta}\right]$$

$\boxed{\text{These states are for non-interacting systems.}}$

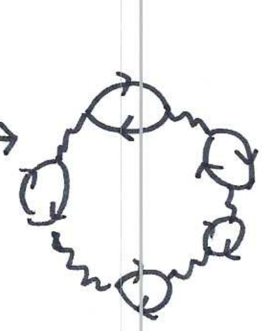
repeat the same process, we arrive at

$$\int_0^\infty d\omega \, I_m \frac{1}{\epsilon_{HFA}(q\omega)} = -\frac{\pi v(q)}{\hbar V}\langle \psi_0(\lambda=0)|\rho^+(q)\,\rho(q)|\psi_0(\lambda=0)\rangle,$$

— this is precisely the spirit of HFA. $\rightarrow$ 

§5 RPA

$$\frac{1}{\varepsilon^{RPA}_\lambda(k,\omega)} = 1 + \sum_{n=1}^{\infty}\left[-\lambda v(q)\,\chi^0_{ret}(q,\omega)\right] = \frac{1}{1+\lambda v(q)\,\chi^0_{ret}(q,\omega)}$$

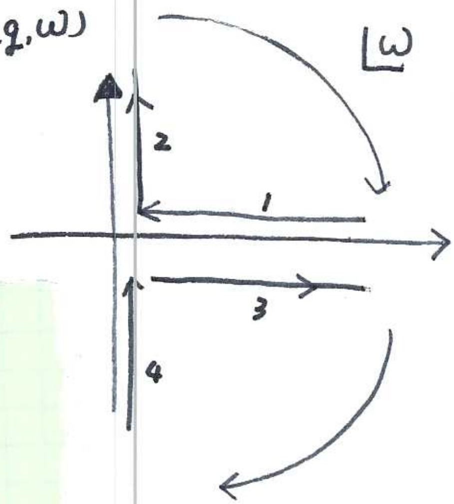
where $$\chi^0_{ret}(q,\omega) = \frac{-2}{\hbar V}\sum_k \frac{n_k - n_{k+q}}{\omega - \omega_{kq} + i\eta}.$$ $\rightarrow$ 

The RPA result

$$-\frac{\hbar}{2\pi}\sum_q{}' \int_0^1 \frac{d\lambda}{\lambda}\int_0^{+\infty} d\omega\; \mathrm{Im}\left[\frac{1}{\varepsilon_\lambda^{RPA}(k,\omega)}\right] = -\frac{\hbar}{2\pi}\sum_q{}'\int_0^1 d\lambda \int_0^\infty \mathrm{Im}\,\frac{-v(q)\,\chi^0_{ret}(q\,\omega)}{1+\lambda v(q)\chi^0_{ret}}$$

we use $\mathrm{Im}\left[\frac{1}{\varepsilon_\lambda^{RPA}}\right] = \mathrm{Im}\left[\frac{1}{\varepsilon_\lambda^{RPA}}-1\right] = \frac{-\lambda v(q)\chi^0}{1+\lambda v(q)\chi^0}$.

Define function $B(q,\omega) \equiv \frac{-v(q)\,\chi^0_{ret}(q,\omega)}{1+\lambda v(q)\,\chi^0_{ret}(q,\omega)}$



$B(q,\omega+i\eta)$ is analytic on the upper-half plane

$B(q,\omega-i\eta)$ is analytic on the lower-half plane

In the upper-half plane, we have

$$\int_{+\infty}^{0} d\omega\; B(q,\omega+i\eta) + i\int_0^{+\infty} d\nu\; B(q,i\nu) = 0$$

In the lower-half plane, we have

$$\int_0^{+\infty} d\omega\; B(q,\omega-i\eta) + i\int_{-\infty}^{0} d\nu\; B(q,i\nu) = 0$$

$$\Rightarrow i\int_{-\infty}^{+\infty} d\nu\; B(q,i\nu) = -\int_0^{+\infty} d\omega\; B(q,\omega-i\eta) - \int_{+\infty}^{0} d\omega\; B(q,\omega+i\eta)$$

$$= \int_0^{+\infty} d\omega\,\big(B(q,\omega+i\eta) - B(q,\omega-i\eta)\big) = 2i\int_0^{+\infty} d\omega\; \mathrm{Im}\,B(q,\omega+i\eta)$$

$$\Rightarrow \boxed{\int_0^{+\infty} d\omega\; \mathrm{Im}\,B(q,\omega+i\eta) = \frac{1}{2}\int_{-\infty}^{+\infty} d\nu\; B(q,i\nu)}$$

RPA:

$$E_G = E_G^0 - \sum_q' \left\{ \frac{\hbar}{4\pi} \int_0^1 d\lambda \int_{-\infty}^{+\infty} d\nu \frac{-v(q)\chi^0(q\,i\nu)}{1+\lambda v(q)\chi^0(q\,i\nu)} + \frac{1}{2}v(q)N \right\}$$

$$= E_G^0 + \sum_q' \left\{ \frac{\hbar}{4\pi} \int_{+\infty}^{+\infty} d\nu \ \ln(1+v(q)\chi^0(q,i\nu)) - \frac{1}{2}v(q)N \right\}$$

The difference between RPA and HFA is called the correlation energy at the RPA level

$$\boxed{\varepsilon_c^{RPA} = \sum_q \frac{\hbar}{4\pi} \int_{+\infty}^{+\infty} d\nu \left\{ \ln\big(1+v(q)\chi^0(q,i\nu)\big) - v(q)\chi^0(q,i\nu) \right\}}$$

$$\chi^0(q\,i\nu) = -2\,\frac{1}{V}\sum_k \frac{n_f(\varepsilon_k)-n_f(\varepsilon_{k+q})}{i\hbar\nu - (\varepsilon_{k+q}-\varepsilon_k)} = 2\int \frac{d^3k}{(2\pi)^3} \frac{-\delta(\varepsilon_k-\mu)\hbar\vec{v}_F\cdot\vec{q}}{i\hbar\nu - \hbar\vec{v}_F\cdot\vec{q}}$$

$$= N_0 \int \frac{d\Omega}{4\pi} \frac{-\cos\theta}{\frac{i\nu}{v_f q} - \cos\theta} = \frac{N_0}{2}\int_{-1}^{1} d\cos\theta \frac{-\cos\theta}{\frac{i\nu}{v_f q} - \cos\theta}$$

where $N_0 = \frac{2}{(2\pi)^3}\int k^2 dk \int d\Omega\, \delta\left(\frac{\hbar^2k^2}{2m} - \frac{\hbar^2 k_f^2}{2m}\right) = \frac{2\cdot 4\pi}{8\pi^3} k_f^2 \Big/ \frac{\hbar^2 k_f}{m} = \frac{m k_f}{\pi^2\hbar^2}$

define $x = \cos\theta$, $\quad S = \frac{\nu}{v_f q}$ $\Rightarrow$ $\int_{-1}^{1} dx \frac{x}{x-iS} = \int_0^1 dx \left[\frac{x}{x-iS} + \frac{x}{x+iS}\right]$

$$= \int_0^1 dx \frac{2x^2}{x^2+S^2}$$

$$\Rightarrow \boxed{\chi^0(q,i\nu) = N_0 \int_0^1 dx \frac{x^2}{x^2+S^2} = N_0\left[1 - S\tan^{-1}(1/S)\right]} \leftarrow \text{as } q\to 0.$$

→ dimensinless

$$\chi^0(q, i\nu) = N_0 R(s), \text{ where } s = \frac{\nu}{v_F q} \text{ and } N_0 = \frac{m k_f}{\pi^2 \hbar^2}$$

$$1 + v_q \chi^0(q, i\nu) = 1 + \frac{k_{TF}^2}{q^2} R(s) = 1 + \frac{\lambda_1^2}{x^2} R(s) \text{ where } x = q/k_F,$$

$$k_{TF}^2 = 4\pi e^2 N_0 = \frac{4\pi e^2 m k_f}{\pi^2 \hbar^2} = \frac{4 e^2}{\pi \hbar^2} m k_f,$$

$$\lambda_1^2 = k_{TF}^2 / k_F^2 = \frac{4 e^2 m}{\pi \hbar^2 k_f} = \frac{4 m e^2}{\pi \hbar^2} \left(\frac{4}{9\pi}\right)^{1/3} r_s a_0 = \frac{4}{\pi} \left(\frac{4}{9\pi}\right)^{1/3} r_s.$$

$$\mathcal{E}_c = \frac{\hbar V}{4\pi N} \int_0^{+\infty} \frac{d^3\vec{q}}{(2\pi)^3} \int_{-\infty}^{+\infty} d\nu \left\{ \ln\left[1 + v(q)\chi^0(q, i\nu)\right] - v(q)\chi^0(q, i\nu) \right\}$$

define $x = q/k_F \Rightarrow v(q)\chi^0(q, i\nu) = \frac{\lambda_1^2}{x^2} R(s)$.

$$\int d^3\vec{q} = \int q^2 dq \cdot 4\pi = 4\pi k_f^3 \int x^2 dx \qquad \int_{-\infty}^{+\infty} d\nu = v_F q \int ds = v_F k_F x \int ds$$

$$\Rightarrow \mathcal{E}_c = \frac{\hbar}{8\pi^3} \frac{v_F k_f^4}{\rho} \lambda_1^4 \int_0^{+\infty} dx \int_{-\infty}^{+\infty} ds \left\{ \frac{x^3}{\lambda_1^4} \ln\left[\left(1 + \frac{\lambda_1^2}{x^2} R(s)\right] - \frac{x}{\lambda_1^2} R(s) \right\}$$

$$\text{The prefactor} = \frac{\hbar}{8\pi^3} \frac{\hbar k_f}{m} \frac{3\pi^2}{k_f^3} \left(\frac{4e^2}{\pi \hbar^2}\right)^2 m^2 k_f^2 = \frac{6}{\pi^3} \frac{m e^4}{\hbar^2} = \frac{12}{\pi^3} R_y$$

The expression $\chi^0(q, i\nu) = N_0 [1 - s \tan^{-1} 1/s]$ is only valid at $q \ll k_F$.

At $q > k_F$, it decays as q^{-2}, or x^{-2}. The ultra-voilet part is convergent, and we only worry the infrared.

at $q > k_F$, $\chi^0(q, i\nu) = N_0 R_x(s) \sim N_0 x^{-2}$,

it cannot be a function only of s.

then $\frac{x^3}{\lambda_1^4} \ln\left(1 + \frac{\lambda_1^2}{x^2} R_x(s)\right) - \frac{x}{\lambda_1^2} R_x(s) \approx \frac{1}{x} R_x(s) \sim \frac{1}{x^3} \rightarrow$ Converge quickly

we can set $+\infty \rightarrow 1$, we have

$$\varepsilon_c \simeq \frac{12}{\pi^3} \int_0^1 dx \int_{-\infty}^{+\infty} ds \left\{ \frac{x^3}{\lambda_1^4} \ln\left[1 + \frac{\lambda_1^2}{x^2} R(s)\right] - \frac{x}{\lambda_1^2} R(s) \right\}$$

$$= \frac{-12}{\pi^3} \int_{-\infty}^{+\infty} ds \; F[s, \lambda_1^2], \text{ where } F[s, \lambda_1^2] = -\frac{R^2[s]}{4} \left\{ \frac{\ln\left(1 + \lambda_1^2 R(u)\right) - \lambda_1^2 R(u)}{\lambda_1^4 R^2(s)} - \ln\left(1 + \frac{1}{\lambda_1^2 R(s)}\right) \right\}$$

this term $\rightarrow \infty$.

$\lambda_1^2 = \frac{4}{\pi}\left(\frac{4}{9\pi}\right)^{1/3} r_s \ll 1$ in the high density limit

as $r_s \rightarrow 0$

$$F(s, \lambda_1^2) \simeq -\frac{1}{4} R^2(s) \ln r_s + \cdots \text{ (terms not dependent on } r_s)$$

$$\Rightarrow \varepsilon_c \simeq \frac{3}{\pi^3}\left[\int_{-\infty}^{+\infty} ds \; R^2(s)\right] \ln r_s + \text{const not dependent on } r_s$$

$$\int_{-\infty}^{+\infty} ds \; R^2(s) = \int_{-\infty}^{+\infty} ds \int_0^1 dy\, dz \; \frac{y^2 z^2}{(y^2+s^2)(z^2+s^2)} = \pi \int_0^1 dy \int_0^1 dz \; \frac{yz}{y+z} = \frac{2\pi}{3}(1 - \ln 2)$$

$$\Rightarrow \boxed{\varepsilon_c = \frac{2}{\pi^2}(1 - \ln 2) \ln r_s + \text{const} \cdots [Ry]}$$

at $r_s \ll 1$, $\ln r_s$ negative

Correlations save more energy!

More precisely complicate calituations

$$\boxed{\left.\frac{E_G}{N}\right|_{RPA} = \frac{2.21}{r_s^2} - \frac{0.916}{r_s} - 0.094 + 0.0622 \ln r_s + O(r_s \ln r_s) \; [Ry]}$$