Lect 7 Boltzmann transport (I)

§ Wigner distribution

We need to consider a spatially inhomogenous (slowly varing) system. We will talk about the particle with momentum $\vec{p}$ at point $\vec{r}$, and the distribution function $n_{\alpha\beta}(p;r,t)$ in the phase space. A more rigours definition is as follows

define $f(p;k;t) = \langle C^\dagger_{p+k/2}\, C_{p-k/2}\rangle(t)$, and perform the Fourier transform over small variable k, and arrive at

$$n_{\alpha\beta}(p;r,t) = \sum_k f(p;k;t)\, e^{ikr}$$

another way to express $n_{\alpha\beta}(p;r,t) = \int dr'\, e^{ip\cdot r'} \langle \psi^\dagger_\alpha(r+\frac{r'}{2})\, \psi_\beta(r-\frac{r'}{2})\rangle$

where "r" is the center of mass coordinate, r' is the ralative coordinate.

$n_{\alpha\beta}(p;r,t)$ is a semiclassical distribution. The resolution of Δp and Δr need to satisfy $\Delta p \cdot \Delta r \geq \hbar/2$.

§ Boltzmann equation:

Let us study the equation of motion of $n(p;r;t)$. It has three contributions: ① Flow in the momentum space

② Flow in the real space

③ Collisions

$$\frac{\partial n(p;r;t)}{\partial t} + \nabla_r [\underset{\uparrow\, dr/dt}{v_p(r,t)}\, n(p;r,t)] + \nabla_p [\underset{\uparrow\, dp/dt}{f_p(r,t)}\, n(p;r,t)] = I_{\text{collision}}$$

where $v_p(r,t) = \nabla_p \mathcal{E}_p(r,t)$, $f_p(r,t) = -\nabla_r \mathcal{E}_p(r,t)$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} n(p;r;t) + \nabla_p \mathcal{E}(p,r,t)\, \nabla_r n(prt) - \nabla_r \mathcal{E}(prt)\, \nabla_p n(prt) = I_{\text{collision}}}$$

linearing the equation:

$$\mathcal{E}(p,r,t) = \mathcal{E}_0(p,r,t) + \frac{1}{V}\sum_{p'} f(p,p')\, \delta n(p'rt)$$

$$n(p;r,t) = n_0(p,r,t) + \delta n(prt)$$

$n_0(p,r,t)$ is a slow varying function of r. but has a sharp discontinuity in momentum space as a function of P.

$$\Rightarrow \frac{\partial}{\partial t}\delta n(prt) + \vec{v}_p \cdot \nabla_r \delta n(prt) - \nabla_p n^0(prt) \Big(\sum_{p'} f_{pp'}\, \nabla_r \delta n(p'rt)\Big) = I_{coll}$$

$$\frac{\partial}{\partial t}\delta n(prt) + \vec{v}_p \cdot \nabla_r \Big[\delta n(prt) - \frac{\partial n^0(p,r,t)}{\partial \mathcal{E}_p} \sum_{p'} f_{pp'}\, \delta n(p'rt)\Big] = I_{coll}$$

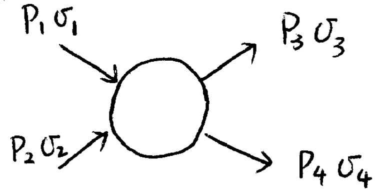
do Fourier transform over vairable $rt \to q, \omega$

$$\delta n(p\, r\, t) = \sum_q n(p;q,\omega)\, e^{i(qr-\omega t)}$$

$$\Rightarrow (-i\omega + i\vec{v}_p\cdot\vec{q})\, \delta n(pq\omega) - \frac{\partial n^0}{\partial \mathcal{E}_p} \vec{v}_p \cdot i\vec{q} \cdot \Big(\sum_{p'} f_{pp'}\, \delta n(p'q\omega)\Big) = I_{coll}$$

$$(\omega - vq\cos\theta)\, \delta n(pq\omega) + \frac{\partial n^0}{\partial \mathcal{E}_p} vq\cos\theta \Big(\sum_{p'} f_{pp'}\, \delta n(p'q\omega)\Big) = i I_{coll}$$

Collision integral



$$\frac{2\pi}{\hbar}|\langle 34|t|12\rangle|^2\,\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)\,n_1n_2(1-n_3)(1-n_4)$$

$$\frac{2\pi}{\hbar}|\langle 12|t|34\rangle|^2\,\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)\,n_3n_4(1-n_1)(1-n_2)$$

define $\frac{2\pi}{\hbar}|\langle 34|t|12\rangle|^2 = \frac{1}{V^2}W(12;34)\,\delta_{p_1+p_2,\,p_3+p_4}\,\delta_{\sigma_1+\sigma_2=\sigma_3+\sigma_4}$

$$\Rightarrow I_{coll}[n_p] = \frac{1}{V^2}\sum_{p_2\sigma_2}\sum_{p_3\sigma_3}\sum_{p_4\sigma_4} W(12;34)\,\delta_{p_1+p_2,\,p_3+p_4}\,\delta_{\sigma_1+\sigma_2,\,\sigma_3+\sigma_4}$$

$$\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4)\,[n_3n_4(1-n_1)(1-n_2)-n_1n_2(1-n_3)(1-n_4)]$$

In most situations, we will use a relaxation time approximation

$$[I(n_p)]_{RT} = -\frac{\delta n}{\tau}$$

In the case of $\omega\tau\gg1$, we can neglect the collision integral.

$$\Rightarrow (s-\cos\theta)\,\delta n(p;q\omega)+\frac{\partial n^0}{\partial \varepsilon_p}\cos\theta\left(\sum_{p'} f_{pp'}\,\delta n(p';q\omega)\right)=0$$

we write $\delta n_p = -\frac{\partial n_p^0}{\partial \varepsilon_p}\nu_p$, $\leftarrow$ the distortion of k_F in the direction of p

$$\Rightarrow (s-\cos\theta)\,\nu_p-\cos\theta\sum_{p'} f_{pp'}\left(-\frac{\partial n_{p'}^0}{\partial \varepsilon_{p'}}\right)\nu_{p'}=0$$

- expand $\nu_p=\sum_l Y_{l0}(\hat p)\,u_{l0}$

$$\Rightarrow \nu_p=\frac{\cos\theta}{s-\cos\theta}\,,\quad \sum_l \frac{1}{2l+1}F_l^s\,Y_{l0}(\hat p)\cdot u_{l0}=0$$

$$\sum_l Y_{l0}(\hat p)\,u_{l0} = F_l^s\sum_l\frac{1}{2l+1}\frac{\cos\theta}{s-\cos\theta}\,Y_{l0}(\hat p)\,u_{l0}$$

$$\Rightarrow \boxed{F_0^S \int \frac{d\Omega}{4\pi} \frac{\cos\theta}{S-\cos\theta} = 1}$$

only keep the zero-th order, we get the zero-sound

$\rightarrow$ the Solution :

$$S_0 = \begin{cases} 1 + 2e^{-2/1+F_0^S} & F_0^S \ll 1 \\ \sqrt{F_0^S/3} & F_0^S \gg 1 \end{cases}$$

§ spin-related transport

define $n_p(r,t) = \frac{1}{2} tr[n_{p,\alpha\beta}]$, $\vec{\sigma}_p(r,t) = \frac{1}{2} tr[n_{p,\alpha\beta} \vec{\tau}_{\beta\alpha}]$

$$\Rightarrow \boxed{n(p;rt) = n_p(r,t)\,\delta_{\alpha\alpha'} + \vec{\sigma}_p(r,t)\cdot\vec{\tau}_{\alpha\alpha'}}$$

(decompose into density and spin)

the quasi-particle energy can be written as

$$\boxed{\varepsilon(prt) = \varepsilon_p(r,t)\,\delta_{\alpha\alpha'} + \vec{h}_p(r,t)\cdot\vec{\tau}_{\alpha\alpha'}.}$$

Plug in the Boltzman Eq

$$\frac{\partial n(rpt)}{\partial t} + \frac{\partial}{\partial r}\left[\frac{\partial\varepsilon}{\partial p}\cdot n(rpt)\right] + \frac{\partial}{\partial p}\left[\frac{\partial\varepsilon}{\partial r}\cdot n(r,pt)\right]$$

$$\underbrace{-\frac{1}{i\hbar}[n(r,pt),\ \varepsilon(r,p,t)]}_{\text{Larmor procession}} = I_{\text{collision}}$$

Seperate variables

$$\boxed{\begin{aligned} &\frac{\partial n_p(rt)}{\partial t} + \frac{\partial}{\partial r_i}\left[\frac{\partial\varepsilon_p}{\partial p_i} n_p + \frac{\partial\vec{h}_p}{\partial p_i}\cdot\vec{\sigma}_p\right] + \frac{\partial}{\partial p_i}\left[-\frac{\partial\varepsilon}{\partial r_i} n_p - \frac{\partial\vec{h}_p}{\partial r_i}\cdot\vec{\sigma}_p\right] = I_{coll} \\ &\frac{\partial\vec{\sigma}_p(rt)}{\partial t} + \frac{\partial}{\partial r_i}\left[\frac{\partial\varepsilon_p}{\partial p_i}\vec{\sigma}_p + \frac{\partial\vec{h}_p}{\partial p_i} n_p\right] + \frac{\partial}{\partial p_i}\left[-\frac{\partial\varepsilon}{\partial r_i}\vec{\sigma}_p - \frac{\partial\vec{h}_p}{\partial r_i}\cdot n_p\right] = \frac{2}{\hbar}\vec{h}_p\times\vec{\sigma}_p + I_{coll} \end{aligned}}$$

Lamor precession: in an external magnet field $\mathcal{H}$, it couples to electron spin as $-\frac{\gamma}{2}\mathcal{H}\cdot\sigma \Rightarrow \frac{\partial \vec{\sigma}_p}{\partial t} = \vec{\sigma}_p \times \gamma \mathcal{H} \Rightarrow \omega_0 = \gamma\mathcal{H}$

In the Fermi liquid, $h_p = -\gamma\frac{\hbar}{2}\mathcal{H} + 2\int\frac{d^3p'}{(2\pi)^3} f^a(p,p')\,\sigma_{p'}$

↳ contribution from interaltion

in the uniform system, we have

$$\frac{\partial \vec{\sigma}_p}{\partial t} = \gamma\,\vec{\sigma}_p \times \vec{\mathcal{H}} - \frac{4}{\hbar}\int\frac{d^3p'}{(2\pi)^3}\,(\vec{\sigma}_p \times \vec{\sigma}_{p'})$$

define $\delta\,\sigma(\hat{n}) = 2\int\frac{d p}{(2\pi)^3}\cdot p^2\sigma(p,\hat{n})$, (we integrate out radius direction).

$$\Rightarrow 2\int\frac{p^2 dp}{(2\pi)^3}\frac{\partial\vec{\sigma}(p,\hat{n})}{\partial t} = 2\int\frac{p^2 dp}{(2\pi)^3}\,\vec{\sigma}(p,\hat{n})\times(\gamma\vec{\mathcal{H}}) - 8\int\frac{d\Omega'}{4\pi}\int\frac{p'^2dp'}{(2\pi)^3}\int\frac{p^2dp}{(2\pi)^3}\,\overset{f^a(pp')}{\sigma_p\times\sigma_{p'}}$$

$$\frac{\partial}{\partial t}\sigma(\hat{\Omega}_p) = \gamma\,\sigma(\hat{\Omega}_p)\times\vec{\mathcal{H}} - \frac{2}{\hbar}\int\frac{d\Omega_p'}{4\pi}\,f^a(p,p')\,\hat{\sigma}(\Omega_p)\,\hat{\sigma}(\Omega_{p'})$$

In the external field $\sigma(\hat{\Omega}_p) = \sigma^0 + \delta\sigma(\hat{p}) \Rightarrow$

$$\frac{\partial}{\partial t}\delta\vec{\sigma}(\Omega_p) = \gamma\,\delta\vec{\sigma}(\Omega_p)\times\vec{\mathcal{H}} - \frac{2}{\hbar}\int\frac{d\Omega_p'}{4\pi}\,f^a(pp')\left[\delta\sigma(\hat{p})\times\sigma^0 + \sigma^0\times\delta\sigma(\hat{p}')\right]$$

define $\delta\sigma_+(\hat{\Omega}_p) = \delta\sigma_x(\hat{\Omega}_p) + i\,\delta\sigma_y(\hat{\Omega}_p)$

$$\frac{\partial}{\partial t}\delta\sigma_+(\Omega_p) = -i\Big[\omega_0\,\delta\sigma_+(\Omega_p) - \frac{2\sigma^0}{\hbar}\int\frac{d\Omega'}{4\pi}\,f^a(p,p')\big(\delta\sigma_+(\Omega_p) - \delta\sigma_+(\Omega_p')\big)$$

$$= -i\Big[\big(\omega_0 - \frac{2}{\hbar}N^{-1}(0)\,F_0^a\,\sigma^0\big)\,\delta\sigma_+(\Omega_p) + \frac{2}{\hbar}\sigma^0\int\frac{d\Omega'}{4\pi}\,f^a(p,p')\,\delta\sigma_+(\Omega_p')\Big]$$

expand $\delta\sigma_+(\Omega_p) = \sum_{lm}\delta\sigma_+(lm)\,Y_{lm}(\Omega_p)$

$$\int \frac{d\Omega'}{4\pi} f^a(p,p')\, \delta\sigma_+(\Omega_{p'}) = N^{-1}(0) \int \frac{d\Omega_{p'}}{4\pi} F^a_l \frac{4\pi}{2l+1} Y_{lm}(\Omega_p) Y^*_{lm}(\Omega_{p'}) \sum_{l'm'} Y_{l'm'}(\Omega_{p'})\, \delta\sigma_+(l'm')$$

$$= N^{-1}(0) \sum_{l,m} \frac{F^a_l}{2l+1} Y_{lm}(\Omega_p)\, \delta\sigma_+(lm)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\delta\sigma_+(lm) = -i\Big\{\omega_0 - \frac{2}{\hbar} N^{-1}(0) F^a_0 \sigma^0 , \quad + N^{-1}(0) \frac{F^a_l}{2l+1}\Big]\delta\sigma_+(lm)$$

$$\Rightarrow \quad \omega_{l+} = \Big\{\omega_0 - \frac{2}{\hbar}\sigma_0 N^{-1}(0)\Big[F^a_0 - \frac{F^a_l}{2l+1}\Big]\Big\}$$

$$\sigma^0 = \frac{\gamma\hbar}{2} \frac{N(0)}{1+F^a_0} \mathcal{H}, \quad \omega_0 = \gamma\mathcal{H}$$

$$\Rightarrow \boxed{\frac{\omega_{l+}}{\omega_0} = \frac{1+F^a_l/2l+1}{1+F^a_0}}$$

the $l=0$ channel Larmor frequency is not modified by interaction because Spin is conserved by interaction!

~~§ spin hydrodynamic equations~~

Integrate over momentum for the Boltzman transport equation $\Rightarrow$

$$\frac{\partial}{\partial t}\vec{\sigma}(r,t) + \frac{\partial}{\partial r_i}\vec{j}_i(\vec{r},t) = -\frac{2}{\hbar}\int\frac{d\vec{p}}{(2\pi)^3}\,\vec{\sigma}_p(r,t) \times \Big[-\frac{\gamma}{2}\mathcal{H} + \int\frac{d\vec{p}'}{(2\pi)^3} f^a(p,p')\,\sigma_{p'}\Big)$$

$$\boxed{\frac{\partial}{\partial t}\vec{\sigma}(r,t) + \frac{\partial}{\partial r_i}\vec{j}_i(\vec{r},t) = \gamma\,\vec{\sigma}(r,t)\times\mathcal{H}(r,t)}$$ (inteaction part cancels).

where $\vec{\sigma}(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\,\vec{\sigma}(r,p,t)$

$$\vec{j}_i(r,t) = 2\int\frac{d^3p}{(2\pi)^3}\Big(\frac{\partial\mathcal{E}_p}{\partial p_i}\vec{\sigma}(r,p,t) + \frac{\partial \vec{h}_p}{\partial p_i} n_p(r,p,t)\Big]$$

$$\vec{j}_i(r,t) = 2\int \frac{d^3p}{(2\pi)^3}\left[\frac{\partial \varepsilon_p^0}{\partial p_i}\,\vec{\sigma}(r,p,t) - \frac{\partial n_p^0}{\partial p}(r,p,t)\,\vec{h}_p\right] \quad \text{(linearize)}.$$

$$= 2\int \frac{d^3p}{(2\pi)^3}\, v_{p_i}\left(\vec{\sigma}_p - \frac{\partial n_p^0}{\partial \varepsilon_p}\,\vec{h}_p\right)$$

$$= 2\int \frac{d^3p}{(2\pi)^3}\, v_{p_i}\left(\vec{\sigma}_p - \frac{\partial n_p^0}{\partial \varepsilon_p}\left(-\frac{\gamma}{2}\frac{\hbar}{2}\mathcal{H} + 2\int \frac{d^3p'}{(2\pi)^3} f^a(p,p')\,\vec{\sigma}_{p'}\right)\right]$$

$$= 2\int \frac{d^3p}{(2\pi)^3}\, v_{p_i}\,\vec{\sigma}_p - 4\int \frac{d^3p}{(2\pi)^3}\, v_{p_i}\frac{\partial n_p^0}{\partial \varepsilon_p}\int \frac{d^3p'}{(2\pi)^3} f^a(p,p')\,\vec{\sigma}_{p'}$$

$$2\int \frac{d^3p}{(2\pi)^3}\, v_{p_i}\frac{\partial n_p^0}{\partial \varepsilon_p} f^a(p,p') = -N(0)\int \frac{d\Omega}{4\pi}\sum_l f_l^a \quad P_l(\cos\theta)\; v_F \cdot \cos\theta \quad \text{(set } p' \text{ along z axis)}.$$

$$= -\frac{F_1^a}{3}\, v_F$$

$$\Rightarrow \vec{j}_i(r,t) = 2\int \frac{d^3p}{(2\pi)^3}\, v_{p_i}\,\vec{\sigma}_p + 2\int \frac{d^3p'}{(2\pi)^3}\,\frac{F_1^a}{3}\, v_{p_i'}\,\vec{\sigma}_{p'}$$

$$\boxed{\vec{j}_i(r,t) = 2\int \frac{d^3p}{(2\pi)^3}\, v_{p_i}\,\vec{\sigma}_p\left(1 + \frac{F_1^a}{3}\right)}$$

Lect 10: Landau Fermi liquid – microscopic theory

As we learned before, the single fermion Green's function

$$G(x_1,x_2) = -i\langle G_0 | T(\psi(x_1)\,\psi^\dagger(x_2)) | G_0\rangle, \quad x=(\vec{x},t).$$

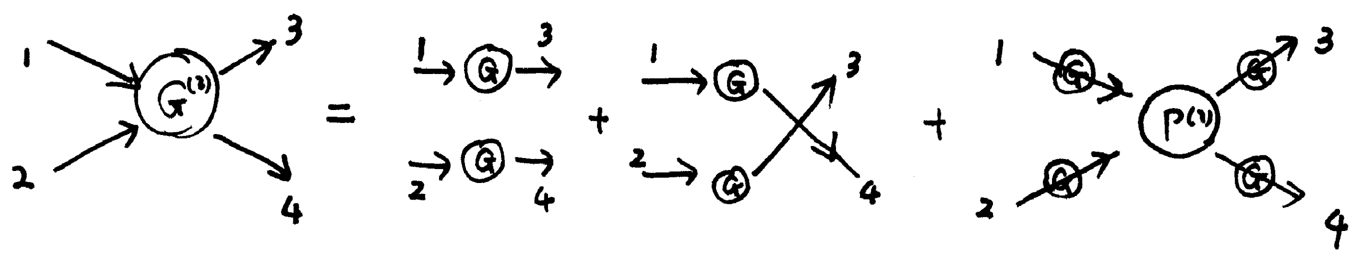
$$\rightarrow \quad G(\vec{p},\omega) = \frac{1}{\omega - \left[\frac{p^2}{2m}-\mu\right] - \Sigma(\vec{p},\omega)} = \frac{\left[1-\frac{\partial\Sigma}{\partial\varepsilon}\right]^{-1}}{\omega - v_F(p-p_F) + i\,\mathrm{sgn}\,\omega} + \cdots$$

the residue at the quasi-particle pole $Z = \left[1-\frac{\partial\Sigma}{\partial\varepsilon}\right]^{-1}$, and $v_F = \frac{k_F}{m^*}$.

Let us introduce two-body Green's functions

$$G^{(2)}(x_1x_2;x_3x_4) = \langle G_0 | T(\psi(x_1)\psi(x_2)\psi^\dagger(x_3)\psi^\dagger(x_4)) | G_0\rangle$$

$$= G(1,3)G(2,4) - G(1,4)G(2,3) + i\int d1'd2'd3'd4'\; G(11')G(22')\,\Gamma^{(2)}(1'2'|3'4')\;G(3',3)G(4',4)$$

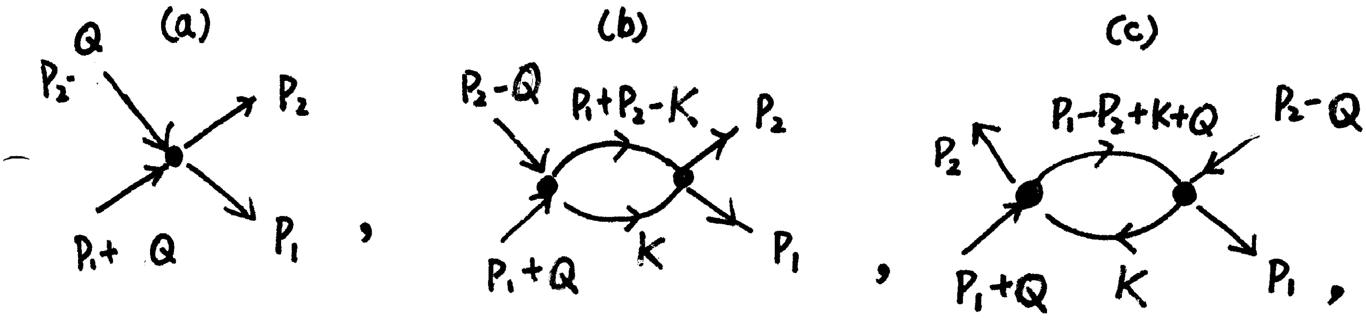


$\rightarrow$

$$G^{(2)}(P_1P_2;P_3P_4) = (2\pi)^8\, G(P_1)G(P_2)\left[\delta(P_1-P_3)\delta(P_2-P_4) - \delta(P_1-P_4)\delta(P_2-P_3)\right]$$

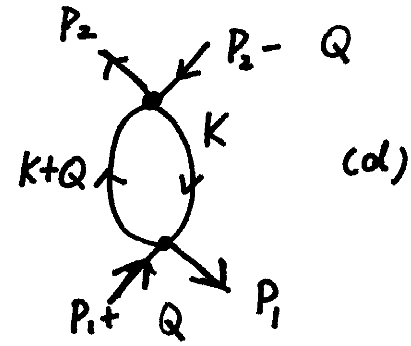
$$+ i\; G(P_1)G(P_2)G(P_3)G(P_4)\;\Gamma^{(2)}(P_1P_2;P_3P_4)$$

define $\Gamma^{(2)}(P_1P_2;P_3P_4) = (2\pi)^4\,\delta(P_1+P_2-P_3-P_4)\;\Gamma(P_1P_2;P_3P_4)$ ← energy momentum conservation!



First and second order graphs

to Γ ← vertex function.



Consisid the vertex function for small values of $(\omega_3 - \omega_1, \vec{P}_3 - \vec{P}_1)$, or Q

Define $P_3 = P_1 + Q$, $P_4 = P_2 - Q$, and $\Gamma(P_1 P_2 ; Q) \equiv \Gamma(P_1 P_2 ; P_3 P_4)$.

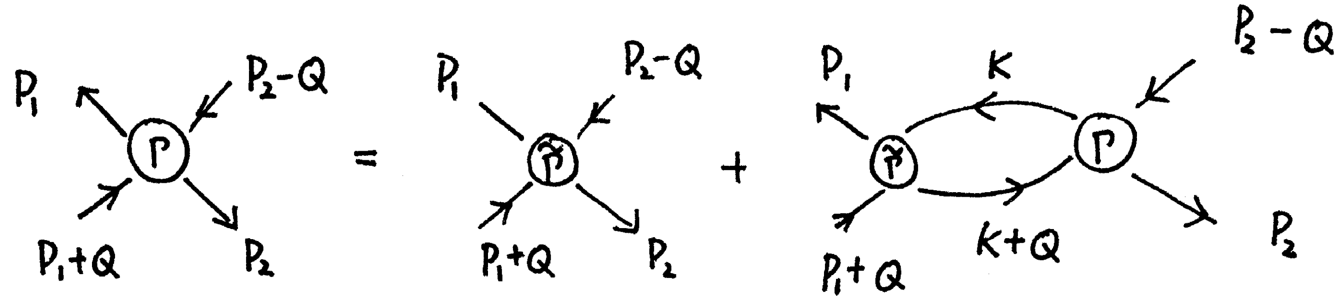
where $Q = (q, \omega)$ is a small 4-vector. This corresponds to nearly forward scattering!

(a) → antisymmetrized matrix elements $v(\vec{Q}) - v(P_1 + P_2 + \vec{Q})$

(b), (c) have no singularity as $Q \to 0$, because the poles of the two fermion lines do not coalesce. We can safely set $Q = 0$. (internal lines describe large angle scatterings).

However, the diagram d, as $Q \to 0$, the poles of $G(K)G(K+Q)$ are close, we need ~~treat~~ it with care.

We denote $\tilde{\Gamma}$ as the part of Γ, which is no singular as $Q \to 0$, and use $\tilde{\Gamma}(P_1 P_2) = \tilde{\Gamma}(P_1 P_2 ; Q=0)$.



$$\Gamma(P_1 P_2 ; Q) = \tilde{\Gamma}(P_1 P_2) - i \int \frac{d^4 k}{(2\pi)^4} \tilde{\Gamma}(P_1 ; K)\, G(K)\, G(K+Q)\, \Gamma(K P_2 ; Q)$$

Comes from $(-i)$ interaction $(i)^2$ each fermi line $(-)$ fermion loop.

$$K = (k, \varepsilon_k)$$

$$G(K)\, G(Q+K) \approx \frac{Z}{\varepsilon_k - v_F(k - k_F) + i \operatorname{sgn} \varepsilon_k} \; \frac{Z}{\omega + \varepsilon_k - v_F(|k+q| - k_F) + i \operatorname{sgn}(\omega + \varepsilon_k)}$$

As $q, \omega \to 0$, the singularity approaches $\varepsilon_k = 0$ and $k = k_F$.

We approximate

$$G(K)\, G(Q+K) \xrightarrow{q \to 0} A(\theta)\, \delta(\varepsilon_k)\, \delta(k - k_F) + \phi(K)$$

$A(\theta)$: the angle between $\vec{k}$ & $\vec{q}$

$\phi(K)$: background

$$A(\theta) = \int d\varepsilon_k \int dk \; \frac{Z}{\varepsilon_k - v_F(k - k_F) + i \operatorname{sgn} \varepsilon_k} \; \frac{Z}{\omega + \varepsilon_k - v_F(|k+q| - k_F) + i \operatorname{sgn}(\omega + \varepsilon_k)}$$

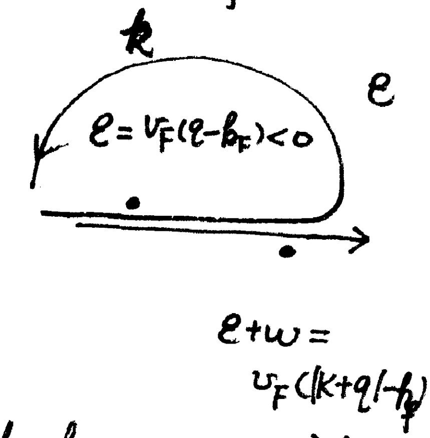
$\boxed{\text{no angular integral}}$

if ε_k & $\varepsilon_k+\omega$ have the same sign $\rightarrow \int d\epsilon = 0$

$|\vec{k}+\vec{q}\,| \approx k + q\cos\theta$. 1. for $\cos\theta>0 \Rightarrow k-k_F<0<k-k_F+q\cos\theta$

i.e. $k_F - q\cos\theta < k < k_F$

$$A(\theta) \underset{q\to 0}{\sim} \int_{k_F-q\cos\theta}^{k_F} dk \, \frac{2\pi i z^2}{\omega - v_F(|\vec{k}+\vec{q}\,|-k)}$$

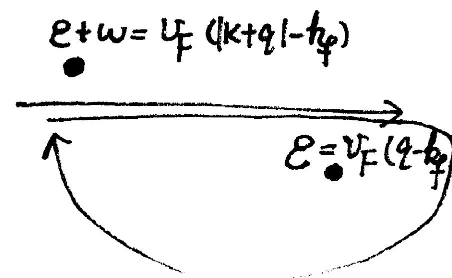


$$= \frac{2\pi i z^2 q\cos\theta}{\omega - v_F q\cos\theta}$$

2. for $\cos\theta<0 \Rightarrow \quad k+\cos\theta\cdot q - k_F < 0 < k-k_F$

$$\Rightarrow \quad k_F < k < k_F - q\cos\theta$$

$$A(\theta) \underset{q\to 0}{\sim} \int_{k_F}^{k_F-q\cos\theta} dk \, \frac{-2\pi i z^2}{\omega - v_F(|\vec{k}+\vec{q}\,|-k)}$$



$$= \frac{2\pi i z^2 q\cos\theta}{\omega - v_F q\cos\theta}$$

$$\Rightarrow G(K)\,G(Q+K) = \frac{2\pi i z^2\, \vec{q}\cdot\hat{k}}{\omega - v_F\vec{q}\cdot\hat{k}}\,\delta(\varepsilon_k-\mu)\,\delta(k-k_F) + \phi(K)$$

$$\Rightarrow \Gamma(\vec{P}_1,\vec{P}_2;Q) = \tilde{\Gamma}(P_1 P_2) - i\int\frac{d^4k}{(2\pi)^4}\,\tilde{\Gamma}(P_1 K)\,\phi(K)\,\Gamma(K P_2;Q)$$

$$+ \frac{z^2 k_F^2}{(2\pi)^3}\int d\Omega_K\,\tilde{\Gamma}(P_1 K)\,\frac{\vec{q}\cdot\hat{k}}{\omega - v_F\hat{k}\cdot\vec{q}}\,\Gamma(K P_2;Q)$$

The integral kenel $\dfrac{\hat{k}\cdot\vec{q}}{\omega - \hat{k}\cdot\vec{q}}$ has non-analytical behavior as $\begin{matrix} q\to 0 \\ \omega \to 0 \end{matrix}$

define the limit

$$\Gamma^{\omega}(P_1 P_2) = \lim_{\omega\to 0}\lim_{q\to 0} \Gamma(P_1 P_2 ; Q),$$ under this limit

the kernal $\to 0$. $\Rightarrow$

$$\boxed{\Gamma^{\omega}(P_1 P_2) = \tilde{\Gamma}(P_1 P_2) - i\int \frac{d^4 k}{(2\pi)^4} \tilde{\Gamma}(P_1 K)\, \phi(K)\, \Gamma^{\omega}(K, P_2)}$$

we want to represent $\Gamma(P_1 P_2 ; Q)$ in terms of $\Gamma^{(\omega)}(P_1 P_2)$ which

is also non-singular as $Q \to 0$.

Let us write the above Eq in a compact form

$$\Gamma^{\omega} = \tilde{\Gamma} - i\tilde{\Gamma}\phi\Gamma^{\omega} = \tilde{\Gamma} - i\Gamma^{\omega}\phi\tilde{\Gamma} = (1 - i\Gamma^{\omega}\phi)\tilde{\Gamma}$$

need a little excercise

Γ satisfies

$$\Gamma = \tilde{\Gamma} - i\tilde{\Gamma}\phi\Gamma + \tilde{\Gamma}\frac{\vec{q}\cdot\hat{K}}{\omega - v_F\hat{k}\cdot\vec{q}}\Gamma$$

$$\Rightarrow (1 - i\Gamma^{\omega}\phi)\Gamma = (1 - i\Gamma^{\omega}\phi)\tilde{\Gamma} - i(1 - i\Gamma^{\omega}\phi)\tilde{\Gamma}\phi\Gamma + (1 - i\Gamma^{\omega}\phi)\tilde{\Gamma} \cdot \frac{\vec{q}\cdot\hat{k}}{\omega - v_F\hat{k}\cdot\vec{q}}\Gamma$$

$$= \Gamma^{\omega} - i\Gamma^{\omega}\phi\Gamma + \Gamma^{\omega}\frac{\vec{q}\cdot\hat{k}}{\omega - v_F\hat{k}\cdot\vec{q}}\Gamma$$

$$\Rightarrow \Gamma = \Gamma^{\omega} + \Gamma^{\omega}\frac{\vec{q}\cdot\hat{k}}{\omega - v_F\hat{k}\cdot\vec{q}}\Gamma,$$ i.e

$$\boxed{\Gamma(P_1 P_2 ; Q) = \Gamma^{\omega}(P_1 P_2) + \frac{z^2 k_F^2}{(2\pi)^3}\int d\Omega\, \Gamma^{\omega}(P_1 K)\frac{\vec{q}\cdot\hat{k}}{\omega - v_F\vec{q}\cdot\hat{k}}\Gamma(K P_2 ; Q)}$$

we are also interested in the other limit $\lim_{k\to 0}\lim_{\omega\to 0}\frac{\hat{k}\cdot q}{\omega - v_F\hat{k}q} = -\frac{1}{v_F}$.

$$\Gamma^{(k)}(P_1 P_2) = \lim_{k\to 0}\lim_{\omega\to 0}\Gamma(P_1 P_2; K) \Rightarrow$$

$$\boxed{\Gamma^{(k)}(P_1 P_2) = \Gamma^{\omega}(P_1 P_2) - \frac{z^2}{(2\pi)^3}\frac{k_f^2}{v_F}\int d\Omega\, \Gamma^{(\omega)}(P_1\ K)\,\Gamma^{(k)}(K, P_2)}$$

we are interested on the singular behavior (poles) of $\Gamma(P_1 P_2; Q)$, thus

we neglect $\Gamma^{\omega}(P_1 P_2) \Rightarrow$

$$\Gamma(P_1 P_2;\ _Q) \underset{k\to 0}{\simeq} \frac{z^2 k_f^2}{(2\pi)^3}\int d\Omega_K\, \Gamma^{(\omega)}(P_1, K)\frac{\hat{k}\cdot\vec{q}}{\omega - v_F\hat{k}\cdot\vec{q}}\Gamma(K P_2; Q)$$

let us fix P_2, and let P_1, P_2 at Fermi surface.

$$\Gamma(\Omega_{P_1}, \Omega_{P_2}; Q) \underset{k\to 0}{\simeq} \frac{z^2 k_f^2}{(2\pi)^2}\int d\Omega_K\, \Gamma^{(\omega)}(\Omega_{P_1}, \Omega_K)\frac{\hat{k}\cdot\vec{q}}{\omega - v_F\hat{k}\cdot\vec{q}}\Gamma(\Omega_K, \Omega_{P_2}; Q)$$

define $\quad u(\hat{p}) = \dfrac{\hat{p}\cdot\vec{q}}{\omega - \hat{p}\cdot\vec{q}\, v_F}\Gamma(\Omega_p, \Omega_{P_2}; Q)$

$$\Rightarrow (\omega - v_F\hat{p}\cdot\hat{q})\, u(\hat{p}) \simeq (\hat{p}\cdot\vec{q})\frac{z^2 k_f^2}{(2\pi)^3}\int d\Omega_K\, \Gamma^{(\omega)}(\Omega_p, \Omega_K)\, u(\hat{k})$$

$$\left[\frac{\omega}{v_F q} - \cos\theta\right] u(\hat{p}) = \cos\theta\,\frac{m^* k_f}{\pi^2}\frac{1}{2}\int\frac{d\Omega_K}{4\pi}\boxed{z^2\,\Gamma^{(\omega)}(\Omega_p, \Omega_K)}\, u(\hat{k})$$

Compared with zero sound eigen-equation $\Longrightarrow$ Landau interaction parameter

$$\boxed{f(P_1, P_2) = z^2\,\Gamma^{(\omega)}(\hat{P}_1\ \hat{P}_2)}$$

P^{ω} represents virtual excitation with small energy transfer. P^{k} describes the physical forward scattering amplitude $(P_1 P_2 \to P_1 P_2)$

From

$$P^{k}(P_1, P_2) = P^{\omega}(P_1, P_2) - \frac{z^2}{(2\pi)^3} \frac{k_F^2}{v_F} \int d\Omega_K \, P^{\omega}(P_1, K) \; P^{(k)}(K, P_2)$$

$$\Rightarrow z^2 P^{k}(P_1, P_2) = f(P_1, P_2) - \frac{k_F m^*}{\pi^2} \frac{1}{2} \int \frac{d\Omega}{4\pi} f(P_1, K) \; z^2 P^{k}(K, P_2)$$

restore spin

$$z^2 P^{k}_{\sigma\sigma'}(P_1, P_2) = f(P_1\sigma_1 ; P_2\sigma_2) - N_0 \int \frac{d\Omega}{4\pi} f(P_1\sigma_1 ; K\sigma_3) \; z^2 P^{k}(K\sigma_3 ; P_2\sigma_2)$$

$\Rightarrow$ harmonics decomposition

$$N_0 \left(z^2 P^{k}_{\sigma\sigma'}(P_1, P_2) \right) = \sum_{\ell} \left[B_\ell + C_\ell \, \sigma\cdot\sigma' \right] P_\ell(\cos\theta) \qquad \cos\theta = \hat{P}_1 \cdot \hat{P}_2$$

$$\Rightarrow \quad B_\ell = \frac{F_\ell^s}{1 + \frac{F_\ell^s}{2\ell+1}} , \qquad C_\ell = \frac{F_\ell^a}{1 + \frac{F_\ell^a}{2\ell+1}}$$

Sum rule

$$\boxed{\lim_{p' \to p} P^{k}(p'\sigma, p\sigma) = 0 \quad \text{same spin}}$$

$$\Rightarrow \boxed{0 = \sum_L (B_L + C_L) = \sum_L \left(\frac{F_\ell^s}{1 + \frac{F_\ell^s}{2L+1}} + \frac{F_\ell^a}{1 + \frac{F_\ell^a}{2L+1}} \right)}$$

Lecture 11. Landau Fermi liquid (IV)

§ Ward identities — Abriksov et al. P158

Consider an external small pertarbation $H'_{(t)} = \int \delta u(t)\, \psi^{+}_{\alpha}(r)\, \psi_{\alpha}(r)\, dr$.

we study the variation of green's function:

$$i\, G_{\alpha\beta}(x\,;x') = \int D\bar{\psi}D\psi\ \psi_{\alpha}(x)\bar{\psi}_{\beta}(x')\, e^{i\int dt\,(\mathcal{L}+\mathcal{L}')} \Big/ \int D\bar{\psi}D\psi\, e^{i\int dt\, \mathcal{L}+\mathcal{L}'}$$

where $\mathcal{L}' = -H'(t)$. correct to the linear order of $\delta u(t)$, we have

$$i\, G_{\alpha\beta}(x,x') = \frac{\int D\bar{\psi}D\psi\ \psi_{\alpha}(x)\bar{\psi}_{\beta}(x')\, e^{i\int dt\,\mathcal{L}}\,[1+i\int dt\,\mathcal{L}'] \Big/ \int D\bar{\psi}D\psi\, e^{i\int dt\,\mathcal{L}}}{\int D\bar{\psi}D\psi\ e^{i\int dt\,\mathcal{L}}\,[1+i\int dt\,\mathcal{L}'] \Big/ \int D\bar{\psi}D\psi\, e^{i\int dt\,\mathcal{L}}}$$

$$= \int D\bar{\psi}D\psi\ \psi_{\alpha}(x)\bar{\psi}_{\beta}(x')\, e^{i\int dt\,\mathcal{L}} \Big/ \int D\bar{\psi}D\psi\, e^{i\int dt\,\mathcal{L}}$$

$$+ \int D\bar{\psi}D\psi\, \left[\psi_{\alpha}(x)\bar{\psi}_{\beta}(x')\ i\int dt\,\mathcal{L}'\right] e^{i\int dt\,\mathcal{L}} \Big/ \int D\bar{\psi}D\psi\ e^{i\int dt\,\mathcal{L}}$$

$$- \frac{\int D\bar{\psi}D\psi\, [\psi_{\alpha}(x)\bar{\psi}_{\beta}(x')\, e^{i\int dt\,\mathcal{L}}]}{\int D\bar{\psi}D\psi\ e^{i\int dt\,\mathcal{L}}} \cdot \frac{\int D\bar{\psi}D\psi\,(i\int dt\,\mathcal{L}')\, e^{i\int dt\,\mathcal{L}}}{\int D\bar{\psi}D\psi\ e^{i\int dt\,\mathcal{L}}}$$

$$\to i\,\delta G_{\alpha\beta}(x,x') = \langle \psi_{\alpha}(x)\,\bar{\psi}_{\beta}(x')\,(-i)\int dt_y\ \delta u(t_y)\,\bar{\psi}_{\gamma}(y)\,\psi_{\gamma}(y)\rangle$$

$$- \langle \psi_{\alpha}(x)\bar{\psi}_{\beta}(x')\rangle\, \langle (-i)\int dy\ \delta u(t_y)\,\bar{\psi}_{\gamma}(y)\,\psi_{\gamma}(y)\rangle$$

Path integral average, or time-ordered in the operator language

According to the definition of the vertex function

$$-\langle \psi(x_1)\psi(x_2)\psi^\dagger(x_3)\psi^\dagger(x_4)\rangle = -\langle\psi(x_1)\psi^\dagger(x_3)\rangle\langle\psi(x_2)\psi^\dagger(x_4)\rangle + \langle\psi(x_1)\psi^\dagger(x_4)\rangle\langle\psi(x_2)\psi^\dagger(x_3)\rangle$$

$$+\ i\int dx_1'\,dx_2'\,dx_3'\,dx_4' \quad \langle\psi(x_1)\psi^\dagger(x_1')\rangle\langle\psi(x_2)\psi^\dagger(x_2')\rangle\langle\psi(x_3')\psi^\dagger(x_3)\rangle\langle\psi(x_4')\psi^\dagger(x_4)\rangle$$

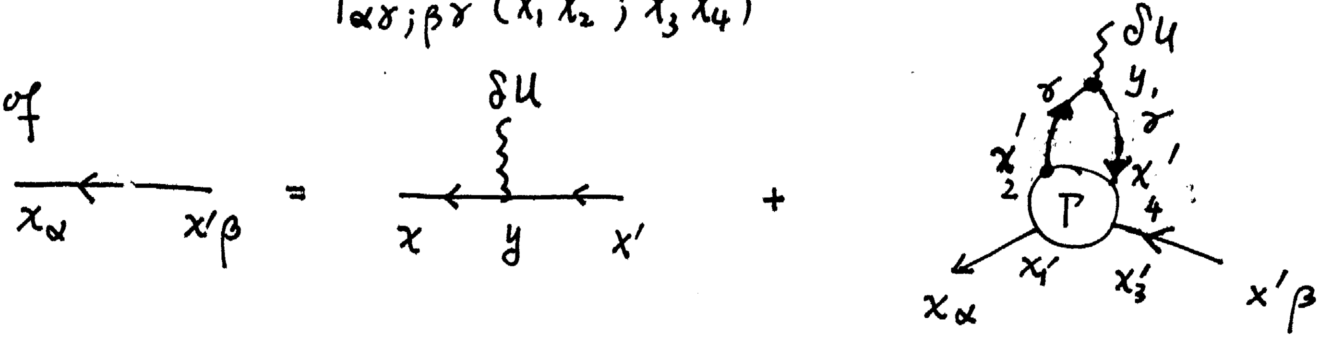
$$\Gamma(x_1'x_2';x_3'x_4') \qquad \leftarrow \qquad x_1 = x \quad x_3 = x' \quad x_2 = y \quad x_4 = y$$

$$\Rightarrow \delta G_{\alpha\beta}(x,x') = \delta_{\alpha\beta}\int d^4y\ \delta u(t_y)\ G(x-y)\,G(y-x')$$

$$-i\int d^4y\ dx_1'dx_2'dx_3'dx_4' \quad \delta u(t_y)\ G(x-x_1')\,G(y-x_2')\ G(x_3'-x')\ G(x_4'-y)$$

$$\Gamma_{\alpha\gamma;\beta\gamma}(x_1'x_2';x_3'x_4')$$

Change of

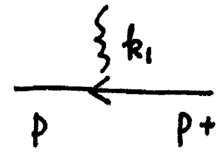


→ Fourier

$$\delta G_{\alpha\beta}^{(p;p+k_1)} = \delta_{\alpha\beta}\,G(p)\ \delta u(\omega_1)\ G(p+k_1) - i\,G(p)\,G(p+k_1)$$

$$\times\int \Gamma_{\alpha\gamma;\beta\gamma}(p\,q;k_1)\ G(q)\ \delta u(\omega_1)\ G(q-k_1)\ \frac{d^4q}{(2\pi)^4}$$

where $k_1 = (0, \omega_1)$



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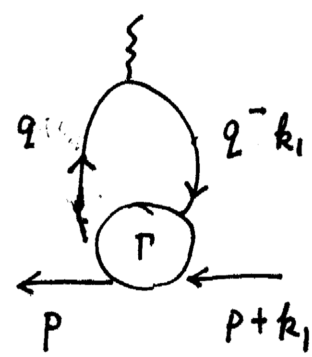


taking 1/2 of the trace

$$\delta G = G(p)\,\delta u(\omega_1)\,G(p+k_1) - i\,G(p)\,G(p+k_1)\,\frac{1}{2}\int \Gamma_{\alpha\beta;\alpha\beta}(pq;k_1)$$

$$\otimes\, G(q)\,\delta u(\omega_1)\,G(q-k_1)\,\frac{d^4q}{(2\pi)^4}$$

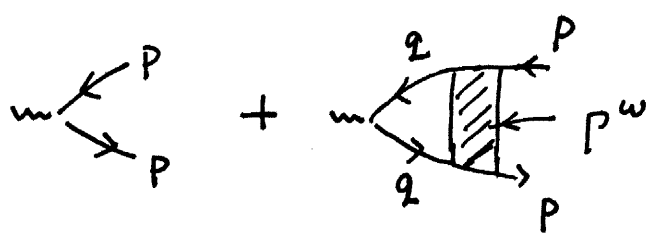
Set $\omega_1 \to 0$, which means we add $-\delta u$ into the frequency of

$$G(p) \quad \Rightarrow \quad \frac{\delta G}{\delta u} = -\frac{\partial G}{\partial \omega}$$

$$\Rightarrow \frac{\partial G}{\partial \omega} = -\left[G^2(p)\right]_\omega \left[1 - \frac{i}{2}\int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(pq)\left[G^2(q)\right]_\omega \frac{d^4q}{(2\pi)^4}\right.$$

• We are considering the limit of $\lim\limits_{\omega\to 0}\lim\limits_{k\to 0}$

$$\Rightarrow \boxed{\frac{\partial G^{-1}(p)}{\partial \omega} = \frac{1}{Z} = 1 - \frac{i}{2}\int \frac{d^4q}{(2\pi)^4}\,\Gamma^{\omega}_{\alpha\beta;\alpha\beta}(pq)\left(G^2(q)\right]_\omega}$$



(2) Now let's consider the particle has charge

$$\frac{(\vec{P} - \frac{e}{c}\vec{A})^2}{2m}$$

$\rightarrow H' = -\frac{e}{mc}\int \psi^\dagger_\alpha(r)\,\hat{p}\,\psi_\alpha(r)\,d^3r$. Let's consider the static but slowly varying limit $\lim\limits_{k\to 0}\lim\limits_{\omega\to 0}$ $\leftarrow$ Static small B field.

$\Rightarrow$ Similarly, we get the variation of

$$\delta G = -G(p)\ \frac{e}{mc}\,(\vec{p}\cdot\vec{A})\,G(p+k_2) + \frac{i}{2}\,G(p)G(p+k_2)\ \times$$

$$\int\frac{d^4q}{(2\pi)^4}\ \Gamma_{\alpha\beta;\alpha\beta}(p\,q\,;k_2)\,\frac{e}{mc}\,(\vec{q}\cdot\vec{A})\ G(q)\ G(q+k_2)$$

where $k_2 = (\vec{k},0)$. Set $\vec{A}$ to be small, this means

$$p \to p - \frac{e}{c}A$$

$$\Rightarrow\quad \delta G = \frac{\partial G}{\partial \vec{p}}\left(-\frac{e}{c}\vec{A}\right) \Rightarrow$$

$$-\frac{\partial G}{\partial \vec{p}} = -\left(G^2(p)\right)_k \frac{\vec{p}}{m} + \frac{i}{2}\left\{G^2(p)\right\}_k \int\frac{d^4q}{(2\pi)^4}\ \Gamma^k_{\alpha\beta;\alpha\beta}(pq)\ G^2(q)\,\frac{\vec{q}}{m}$$

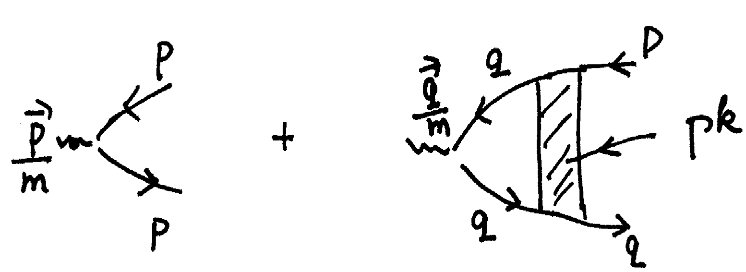
$$\|$$

$$\Rightarrow\ \frac{\partial G^{-1}(p)}{\partial \vec{p}} = -\frac{\vec{p}}{m} + \frac{i}{2}\int\frac{d^4q}{(2\pi)^4}\ \Gamma^k_{\alpha\beta;\alpha\beta}(pq)\left[G^2(q)\right]_k\frac{\vec{q}}{m}$$

$$\uparrow$$

$$-\frac{\vec{v}}{Z} = -\frac{\vec{p}}{m^* Z}\,,\quad \text{i.e}$$

$$\boxed{\frac{\vec{p}}{m^* Z} = \frac{\vec{p}}{m} - \frac{i}{2}\int\frac{d^4q}{(2\pi)^4}\ \Gamma^k_{\alpha\beta;\alpha\beta}(pq)\left[G^2(q)\right]_k\frac{\vec{q}}{m}}$$



Current vertex

③ Let us consider a boost $H' = -\vec{v}\cdot\vec{P} = -\vec{v}\cdot\int \psi^{+}_{\alpha}(r)(-i\hbar\nabla)\psi_{\alpha}(r)\,d\vec{r}$

$\Rightarrow$ the variation of Green's function. We consider the $\lim\limits_{\omega\to 0}\lim\limits_{k\to 0}$

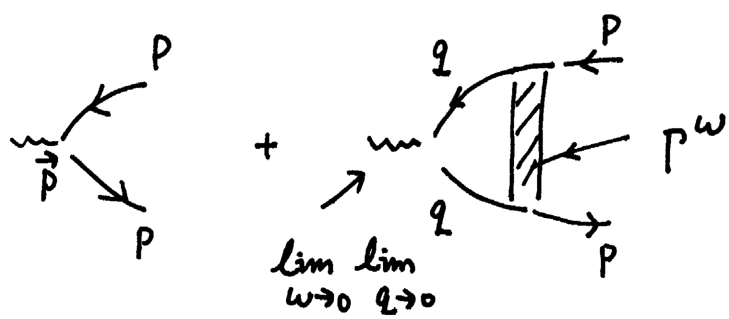
$\to$ small E-field

$$\delta G = -G(p)(\vec{p}\cdot\vec{v})G(p+k_1)$$
$$+\frac{i}{2}G(p)G(p+k_1)\int \Gamma_{\alpha\beta;\alpha\beta}(pq;k_1)(\vec{q}\cdot\vec{v})G(q)G(q-k_2)\frac{d^4q}{(2\pi)^4}$$

where $k_1 = (0,\omega_1)$. The doppler shift the spectra $\varepsilon_p \to \varepsilon_p - \vec{p}\cdot\vec{v}$, which is equalivlent to $\omega \to \omega + \vec{p}\cdot\vec{v}$. Set $\lim\limits_{\omega_1\to 0}\lim\limits_{k_1\to 0}$

$$\delta G = \frac{\partial G}{\partial\omega}\vec{p}\cdot\vec{v} \Rightarrow -G^{-2}(p)\frac{\partial G}{\partial\omega}\vec{p} = \vec{p} - \frac{i}{2}\int \Gamma^{\omega}_{\alpha\beta}(pq)\,\vec{q}\,[G(q)]^2_{\omega}$$

$$\Rightarrow \underbrace{\frac{\partial G^{-1}}{\partial\omega}\vec{p}}_{\frac{\vec{p}}{Z}} = \boxed{\vec{p} - \frac{i}{2}\int\frac{d^4q}{(2\pi)^4}\Gamma^{\omega}_{\alpha\beta}(pq)\,\vec{q}\,[G(q)]^2_{\omega} = \frac{\vec{P}}{Z}}$$



④ Let us consider a variation of $\mathcal{U}(r)$ in the limit of $\lim\limits_{k\to 0}\lim\limits_{\omega\to 0}$ (static limit). $\Rightarrow$ $\underline{H' = \int dr\,\delta\mathcal{U}(r)\psi^{+}(r)\psi(r)}$

$$\delta G = G(p)\,\delta\mathcal{U}(\vec{k})\,G(p+k_2) - \frac{i}{2}G(p)G(p+k_2)\int\frac{d^4q}{(2\pi)^4}\Gamma_{\alpha\beta;\alpha\beta}(pq;k_2)$$
$$G(q)\,\delta\mathcal{U}(\vec{k})\,G(q+k_2)$$

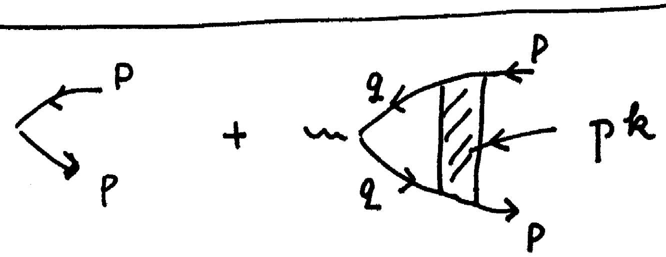
where $k_2 = (\vec{k}, 0)$

In the limit of $k \to 0$, this corresponds to $\mu + \delta U = \text{const}$

thus $\delta\mu = -\delta U$. $\Rightarrow \delta G = \frac{\partial G}{\partial \mu}(-\delta U)$

$$\Rightarrow -\frac{\partial G}{\partial \mu} = G^2(p) \mp \frac{i}{2} G^2(p) \int \frac{d^4 q}{(2\pi)^4} \Gamma^k_{\alpha\beta;\alpha\beta}(p\, q) \left[G^2(q)\right]_k$$

$$\Rightarrow \boxed{\frac{\partial G^{-1}}{\partial \mu} = 1 - \frac{i}{2} \int \frac{d^4 q}{(2\pi)^4} \Gamma^k_{\alpha\beta;\alpha\beta}(p\, q) \left[G^2(q)\right]_k}$$

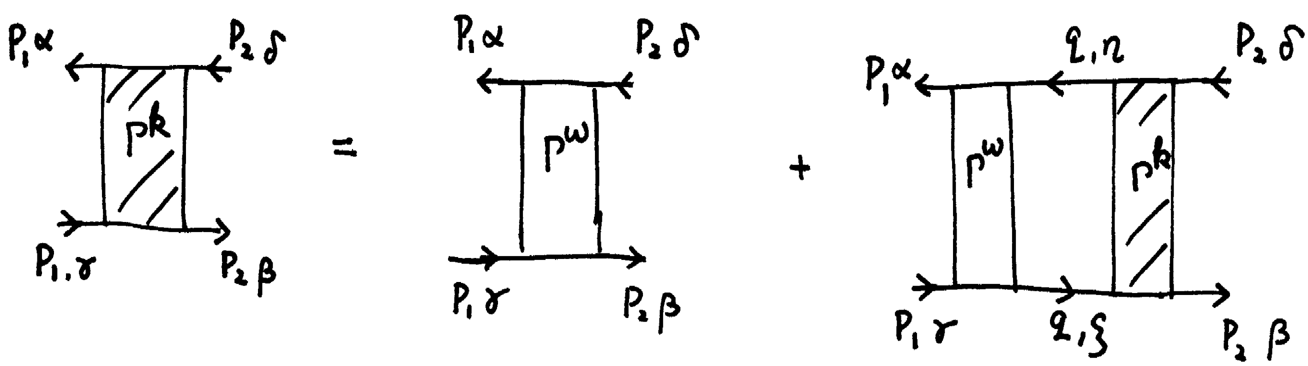


scalar vertex

§ Fermi surface volume

correct close to Fermi Surface

We need to use $\Gamma^k_{\alpha\beta;\gamma\delta}(P_1, P_2) = \Gamma^\omega_{\alpha\beta;\gamma\delta}(P_1; P_2) - \frac{k_F^2 Z^2}{(2\pi)^3 v_F} \int \Gamma^\omega_{\alpha\xi;\gamma\eta}(P_1 q)\, \Gamma^k_{\eta\beta;\xi\delta}(q, P_2)$



Plug in this Eq into

$$\frac{\vec{p}}{m^* Z} = \frac{\vec{p}}{m} - \frac{i}{2} \int \frac{d^4 q}{(2\pi)^4} \Gamma^k_{\alpha\beta;\alpha\beta}(p\, q) \left[G^2(q)\right]_k \frac{\vec{q}}{m}$$

$$\Rightarrow \frac{\vec{p}}{m^* Z} - \frac{\vec{p}}{m} = -\frac{i}{2}\int \frac{d^4q}{(2\pi)^4}\, \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q)\, \frac{\vec{q}}{m}\, \left[G^2(q)\right]_k$$

$$+\frac{1}{2}\frac{k_F^2 Z^2}{(2\pi)^3 v_F}\int d\Omega\; \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q)\left(-\frac{\vec{q}}{m^* Z}+\frac{\vec{q}}{m}\right)$$

by using the previous Eq again.

plug in $\left[G^2(p)\right]_k = \left[G^2(p)\right]_\omega - \frac{2\pi i Z^2}{v_F}\,\delta(\omega)\,\delta(|p|-k_F)$

$$\Rightarrow \frac{\vec{p}}{m^* Z} = \frac{\vec{p}}{m} - \frac{i}{2}\int \frac{d^4q}{(2\pi)^4}\, \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(pq)\, \frac{\vec{q}}{m}\, \left[G^2(p)\right]_\omega$$

$$-\frac{\pi Z^2}{v_F}\int \frac{d^4q}{(2\pi)^4}\, \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p,q)\,\delta(\omega)\,\delta(|q|-k_F)\,\frac{\vec{q}}{m}$$

$$+\frac{1}{2}\frac{k_F^2 Z^2}{(2\pi)^3 v_F}\int d\Omega\; \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p,q)\left(\frac{-\vec{q}}{m^* Z}+\frac{\vec{q}}{m}\right)$$

The first line $= \dfrac{\vec{p}}{Zm}$

The second line $= -\dfrac{\pi Z^2}{v_F}\dfrac{1}{(2\pi)^4}\cdot k_F^2\displaystyle\int d\Omega\, \Gamma^{\omega}_{\alpha\beta\,\alpha\beta}(p,q)\frac{\vec{q}}{m}$

$$\Rightarrow \frac{\vec{p}}{m^* Z} = \frac{\vec{p}}{Zm} - \frac{1}{2}\frac{k_F^2 Z}{(2\pi)^3 v_F}\int d\Omega\; \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p,q)\,\frac{\vec{q}}{m^*}$$

$\Rightarrow$

$$\boxed{\frac{1}{m^*} = \frac{1}{m} - \frac{k_F^2}{2(2\pi)^3}\int d\Omega\; Z^2\, \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\cdot q)\,\frac{\vec{q}}{k_F}}$$

$$\frac{z^2}{2}\Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p,q) = 2f^S(p,q)$$

$$\Rightarrow \frac{1}{m^*} = \frac{1}{m} - \frac{\tilde{N}_0}{m}\frac{1}{3}f_1^S$$

where $f_1^S = \frac{1}{2}\int_{-1}^{1} d\cos\theta\, f^S(\cos\theta)\cos\theta$

and $\tilde{N}_0 = \frac{2\cdot k_f^2\cdot 4\pi}{(2\pi)^3 v_f^0} = \frac{2k_f m\cdot 4\pi}{(2\pi)^3}$

where $\tilde{N}_0$ is the DOS of free Fermi gas.

or $\frac{1}{m^*} = \frac{1}{m} - \frac{N_0}{m^*}\frac{1}{3}f_1^S \Rightarrow \frac{1+\frac{F_1^S}{3}}{m^*} = \frac{1}{m}$ or $\boxed{\frac{m^*}{m} = 1+\frac{F_1^S}{3}}$

<u>N_0 is the DOS of Fermi liquid.</u>

§ Luttinger theorem:

we prove the Fermi wave vector does not change in Fermi liquid.

Consider $G(p,\omega) = \frac{z}{\omega - v_F(|p|-k_F) + i\,\mathrm{sgn}\,\omega\,\eta} + \cdots$

all of z, k_F, v_F depends on μ. However, the contribution to $\frac{\partial G}{\partial \mu}$ mainly comes from $\partial k_F/\partial\mu$:

$$\frac{\partial G}{\partial\mu} = \frac{\partial z/\partial\mu}{\omega - v_F(|p|-k_F)+i\,\mathrm{sgn}\,\omega} + \frac{-z}{[\omega - v_F(|p|-k_F)+i\,\mathrm{sgn}\,\omega]^2}\left[-\frac{\partial v_F}{\partial\mu}(|p|-k_F) + v_F\frac{\partial k_F}{\partial\mu}\right]$$

as $|p|\to k_F$ and $\omega\to 0$, the last term domainates

$$\Rightarrow \frac{\partial G}{\partial\mu} \approx -G^2\frac{v_F}{z}\frac{dk_f}{d\mu} \quad \text{or} \quad \boxed{v_F\frac{dk_f}{\partial\mu} = z\left(\frac{\partial G^{-1}}{\partial\mu}\right)\Bigg|_{\substack{\omega=0\\ |p|=k_F}}}$$

Plug in $\frac{\partial G^{-1}(p)}{\partial \mu} = 1 - \frac{i}{2}\int \Gamma^{k}_{\alpha\beta;\alpha\beta}(pq)\,\{G^2(q)\}_k \frac{d^4q}{(2\pi)^4}$

and $\Gamma^{k}_{\alpha\beta;\alpha\beta}(pq) = \Gamma^{\omega}_{\alpha\beta\alpha\beta}(pq) - \frac{k_F^2 z^2}{(2\pi)^3 v_F}\int d\Omega_{q'}\, \Gamma^{\omega}_{\alpha\xi;\alpha\eta}(pq')\, \Gamma^{k}_{\eta\beta;\xi\beta}(q',q)$

$$\Rightarrow \frac{v_F}{z}\frac{dk_f}{\partial\mu} = \left.\frac{\partial G^{-1}(p)}{\partial\mu}\right|_{\substack{\omega=0\\ |p|=k_F}} = 1 - \frac{i}{2}\int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(pq)\{G^2(q)\}_k \frac{d^4q}{(2\pi)^4}$$

$$+ \frac{i}{2}\frac{k_F^2 z^2}{(2\pi)^3 v_F}\int d\Omega_{q'}\int\frac{d^4q}{(2\pi)^4}\, \Gamma^{\omega}_{\alpha\xi;\alpha\eta}(pq')\, \Gamma^{k}_{\eta\beta;\xi\beta}(q',q)\{G^2(q)\}_k$$

the second line: after integrate $\frac{d^4q}{(2\pi)^4}$

$$\rightarrow -\frac{k_F^2 z^2}{(2\pi)^3 v_F}\int d\Omega_{q'}\, \Gamma^{\omega}_{\alpha\xi;\alpha\eta}(pq')\,\frac{1}{2}\Big[\underbrace{\frac{\partial G^{-1}(q')}{\partial\mu}}_{\left(\frac{v_F}{z}\frac{dk_F}{d\mu}-1\right)} - 1\Big]\,\delta_{\eta\xi}$$

(η has to be equal to ξ, otherwise, vertex $\Gamma_{\alpha\xi;\alpha\eta}$ and $\Gamma_{\eta\beta;\xi\beta}$ are not spin conserved

$$\Rightarrow \frac{v_F}{z}\frac{dk_f}{d\mu} = 1 - \frac{i}{2}\int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(pq)\{G^2(q)\}_k\frac{d^4q}{(2\pi)^4}$$

$$- \frac{1}{2}\frac{k_F^2 z^2}{(2\pi)^3 v_F}\int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(pq)\left(\frac{v_F}{z}\frac{dk_f}{d\mu}-1\right)d\Omega_q$$

using $\{G^2(q)\}_k = \{G^2(q)\}_\omega - \frac{2\pi i z^2}{v_F}\delta(\varepsilon)\,\delta(|q|-k_F)$

The first line: $1 - \frac{i}{2}\int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p.q)\{G^2(q)\}_\omega \frac{d^4q}{(2\pi)^4} + \frac{i}{2}\frac{2\pi i z^2}{v_F}$

$$\otimes\int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(pq)\,\delta(\omega_q)\,\delta(|q|-k_F)\frac{d^4q}{(2\pi)^4} = \frac{1}{z} - \frac{2\pi z^2}{2v_F}\frac{k_F^2}{(2\pi)^4}\int \Gamma^{\omega}_{\alpha\beta}(pq)\,d\Omega_q$$

$$\frac{v_F}{z}\frac{dk_F}{d\mu} = \frac{1}{z} - \frac{k_F^2 z^2}{2(2\pi)^3 v_F}\,\frac{v_F}{z}\frac{dk_F}{d\mu}\int \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p,q)\, d\Omega_q$$

$$\Rightarrow \boxed{v_F\frac{dk_F}{d\mu} = \Big(1 + \underbrace{\frac{k_F^2}{2(2\pi)^3 v_F}}_{\frac{N_0}{4}}\int \underbrace{z^2\Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p,q)}_{4 f_0^s}\, d\Omega_q\Big)^{-1}} = \frac{1}{1+F_0}$$

$$v_F = \frac{d\epsilon_F}{dk_F} \Rightarrow \frac{d\epsilon_F}{d\mu} = \frac{1}{1+F_0} \quad \text{i.e.} \quad \boxed{\frac{dn}{d\mu} = \frac{dn}{d\epsilon}\frac{d\epsilon}{d\mu} = \frac{N_0}{1+F_0}}$$

From the definition of Green's function

$$n(k) = \langle \psi^\dagger(k)\psi(k)\rangle = -\langle T\,[\psi(k,t)\psi^\dagger(k,t+0^+)]\rangle = -i\,G(k,-0^+)$$

$$= -i\int\frac{d\omega}{2\pi}\, G(p,\omega)\, e^{i\omega\eta} \quad \text{where } \eta = 0^+$$

Taking into account spin $\Rightarrow$

$$n = \frac{N}{V} = -2i\int\frac{d^3p}{(2\pi)^3}\frac{d\omega}{2\pi}\, G(p,\omega)\, e^{i\omega\eta} \leftarrow \eta\to 0^+$$

$$\frac{dn}{d\mu} = -2i\int\frac{\partial G(p)}{\partial\mu}\frac{d^4p}{(2\pi)^4} = 2i\int\frac{\partial G^{-1}(p)}{\partial\mu}\{G^2(p)\}_k\frac{d^4p}{(2\pi)^4}$$

Plug in $\displaystyle \frac{\partial G^{-1}(p)}{\partial\mu} = 1 - \frac{i}{2}\int \Gamma^{k}_{\alpha\beta;\alpha\beta}(p\,q)\{G^2(q)\}_k\frac{d^4q}{(2\pi)^4}$

$$\Rightarrow \frac{dn}{d\mu} = 2i\int\{G^2(p)\}_k\frac{d^4p}{(2\pi)^4} + \int \Gamma^{k}_{\alpha\beta;\alpha\beta}(p\,q)\{G^2(q)\}_k\frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}\,\{G^2(p)\}_k$$

Again, using

$$\Gamma^{k}_{\alpha\beta\,\alpha\beta}(p\,q) = \Gamma^{\omega}_{\alpha\beta\,\alpha\beta}(p\,q) - \frac{k_F^2 z^2}{(2\pi)^3 v_F}\int d\Omega_{q'}\,\Gamma^{\omega}_{\alpha\xi;\alpha\eta}(p\,q')\,\Gamma^{k}_{\eta\beta;\xi\beta}(q',q)$$

the second term goes

$$\int \{G^2(p)\}_k \, \Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q)\,\{G^2(q)\}_k \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4} - \frac{k_F^2 z^2}{(2\pi)^3 v_F}\int\frac{d^4p}{(2\pi)^4}\{G^2(p)\}_k$$

$$\otimes \int d\Omega_{q'}\, \Gamma^{\omega}_{\alpha\xi;\alpha\eta}(p\,q')\, \underbrace{\int\frac{d^4q}{(2\pi)^4}\,\Gamma^{k}_{\eta\beta;\xi\beta}(q',q)\,\{G^2(q)\}_k}_{\left[\frac{\partial G^{-1}(q')}{\partial\mu}-1\right]2i\,\frac{\delta_{\eta\xi}}{2}}$$

$$\Rightarrow \frac{dn}{d\mu} = 2i\int\{G^2(p)\}_k\frac{d^4p}{(2\pi)^4} + \int\{G^2(p)\}_k\,\Gamma^{\omega}_{\alpha\beta,\alpha\beta}(p\,q)\{G^2(q)\}_k\frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4}$$

$$- \frac{i k_F^2 z^2}{(2\pi)^3 v_F}\int\frac{d^4p}{(2\pi)^4}\{G^2(p)\}_k\,\overbrace{\Gamma^{\omega}_{\alpha\beta\,\alpha\beta}(p,q')}^{d\Omega_{q'}}\left[\frac{v}{z}\frac{dk_F}{d\mu}-1\right]$$

us $\{G^2(p)\}_k = \{G^2(p)\}_\omega - \frac{2\pi i z^2}{v_F}\,\delta(\omega)\,\delta(|p|-k_F)$

The first term $\Rightarrow$

$$2i\int\{G^2(p)\}_\omega\frac{d^4p}{(2\pi)^4} + \frac{4\pi z^2}{v_F}\frac{4\pi k_F^2}{(2\pi)^4} = 2i\int[G^2(p)]_\omega\frac{d^4p}{(2\pi)^4} + \frac{8\pi z^2 k_F^2}{(2\pi)^3 v_F}$$

the second term

$$\int\{G^2(p)\}_\omega\,\Gamma^{\omega}_{\alpha\beta\alpha\beta}(p\,q)\{G^2(q)\}_\omega\frac{d^4q}{(2\pi)^4}\frac{d^4p}{(2\pi)^4} - 2\times\frac{2\pi i z^2}{v_F}\,\underbrace{\frac{k_F^2}{(2\pi)^4}\int}_{\int d\Omega_p}\frac{d^4q}{(2\pi)^4}\Gamma^{\omega}_{\alpha\beta,\alpha\beta}(p,q)\{G^2(q)\}_\omega \quad (*)$$

$$+\left(\frac{2\pi i z^2}{v_F}\right)^2\frac{(k_F^2)^2}{(2\pi)^4(2\pi)^4}\int d\Omega_p\, d\Omega_q\,\Gamma^{\omega}_{\alpha\beta,\alpha\beta}(p,q) \quad (**)$$

the third one

$$-\frac{-ik_F^2 z^2}{(2\pi)^3 v_F}\int\frac{d^4p}{(2\pi)^4}\int d\Omega_{q'}\,\{G^2(p)\}_\omega\, \Gamma^\omega_{\alpha\beta\,\alpha\beta}(p\,q')\left[\frac{v}{z}\frac{dk_F}{d\mu}-1\right] \quad (*)$$

$$-\frac{2\pi k_F^2 z^4}{(2\pi)^3 v_F^2}\frac{k_F^2}{(2\pi)^4}\int d\Omega_p\int d\Omega_{q'}\;\Gamma^\omega_{\alpha\beta,\alpha\beta}(p\,q')\left[\frac{v}{z}\frac{dk_F}{d\mu}-1\right] \quad (**)$$

two $(*)$ terms combine together

$$\frac{-ik_F^2 z^2}{(2\pi)^3 v_F}\int\frac{d^4p}{(2\pi)^4}\int d\Omega_{q'}\;\underbrace{\{G^2(p)\}_\omega\,\Gamma^\omega_{\alpha\beta\,\alpha\beta}(p\,q')}_{2i\left(\frac{1}{z}-1\right)}\left[\frac{v}{z}\frac{dk_F}{d\mu}+1\right]$$

$$=\frac{2k_F^2 z^2}{(2\pi)^3 v_F}\int d\Omega_{q'}\left(\frac{1}{z}-1\right)\left(\frac{v}{z}\frac{dk_F}{d\mu}+1\right)=\frac{8\pi k_F^2 z(1-z)}{(2\pi)^3 v_F}\left(\frac{v}{z}\frac{dk_F}{d\mu}+1\right)$$

two $(**)$ terms Combine together

$$\frac{-k_F^4 z^4}{(2\pi)^6 v_F^2}\int d\Omega_p\, d\Omega_{q'}\;\Gamma^\omega_{\alpha\beta\alpha\beta}(p\,q')\left[\frac{v}{z}\frac{dk_F}{d\mu}\right]=-4\pi\left[\frac{k_F^2 z^2}{(2\pi)^3 v_F}\right]^2$$

$$\otimes\int\Gamma^\omega_{\alpha\beta\alpha\beta}(p\,q')\,d\Omega_{q'}\left[\frac{v_F}{z}\frac{dk_F}{d\mu}\right]$$

$$\Rightarrow\ \frac{d(n)}{d\mu}=2i\int\{G^2(p)\}_\omega\frac{d^4p}{(2\pi)^4}+\int[G^2(p)]_\omega\,\Gamma^\omega_{\alpha\beta;\alpha\beta}(p\,q)\,[G^2(q)]_\omega\frac{d^4p\,d^4q}{(2\pi)^4(2\pi)^4}$$

$$+\frac{8\pi k_F^2 z}{(2\pi)^3 v_F}+\frac{8\pi k_F^2(1-z)}{(2\pi)^3 v_F}\,v_F\frac{dk_F}{d\mu}-4\pi\left(\frac{k_F^2 z^2}{(2\pi)^3 v_F}\right)^2\int\Gamma^\omega_{\alpha\beta\alpha\beta}(p\,q)\,d\Omega_q\left(\frac{v_F}{z}\frac{dk_F}{d\mu}\right)$$

only this term survives

The first two terms : $\Rightarrow 0.$

$$2i\int\frac{d^4p}{(2\pi)^4}\left[G^2(p)\right]_\omega\int\frac{d^4q}{(2\pi)^4}\left(1-\frac{i}{2}\int\frac{d^4q}{(2\pi)^4}\,\Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q)\left[G^2(q)\right]_\omega\right]$$

$$=-2i\int\frac{d^4p}{(2\pi)^4}\frac{\partial G}{\partial\omega}=-2i\int\left[G(\vec{p},\omega=+\infty)-G(\vec{p},\omega=-\infty)\right]\frac{d^3p}{(2\pi)^3}=0$$

this equals to the shift of total particle number due to shift of energy zero. $\left(\varepsilon\to\varepsilon+u\Rightarrow\frac{\partial G}{\partial\varepsilon}=-\frac{\partial G}{\partial u}\right)$

The other 3-terms can be organized as

$$\frac{8\pi k_F^2 z}{(2\pi)^3 v_F}\left\{1-\left(1+\frac{k_F^2 z^2}{2(2\pi)^3 v_F}\int\Gamma^{\omega}_{\alpha\beta;\alpha\beta}(p\,q)\,d\Omega_q\right)\left[\underbrace{v_F\frac{dk_F}{d\mu}}_{=1}\right]\right\}\ \text{``}=0\text{''}$$

$$+\frac{8\pi k_F^2}{(2\pi)^3 v_F}v_F\frac{dk_F}{d\mu}=\frac{8\pi k_F^2}{(2\pi)^3}\frac{dk_F}{d\mu}$$

$$\Rightarrow\quad\frac{dn}{d\mu}=\frac{8\pi k_F^2}{(2\pi)^3}\frac{dk_F}{d\mu}\quad\Rightarrow\quad n=\frac{8\pi}{3}\frac{k_F^3}{(2\pi)^3}$$

$\Rightarrow k_F$ is the same as no-interacting system!

Lecture 11.5 Landau Fermi liquid (V)

§ More on Luttinger theorem — Luttinger & Ward

Another expression: $$\frac{N}{V} = 2i\int\left[\frac{\partial}{\partial\omega}\ln G(p) - G(p)\frac{\partial}{\partial\omega}\Sigma(p)\right]e^{i\omega\eta}\frac{d^4p}{(2\pi)^4}$$

Proof:

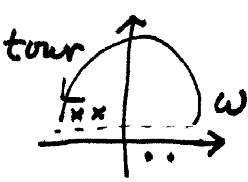
$$\frac{\partial}{\partial\omega}\ln G(p) - G(p)\frac{\partial}{\partial\omega}\Sigma(p) = G^{-1}(p)\frac{\partial}{\partial\omega}G(p) - G\frac{\partial}{\partial\omega}\Sigma = (\omega-\xi-\Sigma)\frac{\partial}{\partial\omega}G$$

$$-G\frac{\partial}{\partial\omega}\Sigma = \omega\frac{\partial}{\partial\omega}G - \xi\frac{\partial}{\partial\omega}G - \frac{\partial}{\partial\omega}(\Sigma G) = \frac{\partial}{\partial\omega}(\omega G) - \frac{\partial}{\partial\omega}(\Sigma G) - \frac{\partial}{\partial\omega}(\xi G) - G$$

$$= \frac{\partial}{\partial\omega}[(\omega-\xi-\Sigma)G] - G \Rightarrow \boxed{\frac{N}{V} = -2i\int G\, e^{i\omega\eta}\frac{d^4p}{(2\pi)^4}}$$

— The second term of the integral can be proved to be zero. Let us postpone its proof for a while. $$\frac{N}{V} = 2i\int\frac{\partial}{\partial\omega}\ln G(p)\, e^{i\omega\eta}\frac{d^4p}{(2\pi)^4}$$

Contour



G is not analytic. $G_R = \begin{cases} G(\epsilon) & \epsilon>0 \\ G^*(\epsilon) & \epsilon<0, \end{cases}$ G_R is analytic for ω in the upper plan

So

$$\frac{N}{V} = 2i\left[\int_0^\infty\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial}{\partial\omega}\ln G_R(p) + \int_{-\infty}^0\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial}{\partial\omega}\ln G(p)\right]$$

$$= 2i\left[\underbrace{\int_{-\infty}^{+\infty}\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial}{\partial\omega}\ln G_R(p)}_{=0} + \int_{-\infty}^0\frac{d\omega}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{\partial}{\partial\omega}\ln\frac{G(\vec{p},\omega)}{G^*(\vec{p},\omega)}\right]$$

$$= \frac{2i}{2\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}\ln\frac{G(p,\omega)}{G^*(p,\omega)}\Bigg|_{-\infty}^{0} = -\frac{2}{\pi}\int\frac{d^3\vec{p}}{(2\pi)^3}[\varphi(0)-\varphi(-\infty)],$$

where φ denotes the phase of G. We need to consider the variation of φ from $\omega=-\infty$ to $\omega=0$. From the Lehmann Rep. we have

$$G(\omega) = e^{\beta\Omega}\sum_{n,m}\langle n|\psi|m\rangle\langle m|\psi^{\dagger}|n\rangle\left\{\frac{e^{-\beta E_n}}{\omega+E_n-E_m+i\eta}+\frac{e^{-\beta E_m}}{\omega+E_n-E_m-i\eta}\right\}$$

$$\xrightarrow{T=0K} = \sum_m \frac{\langle 0|\psi|m\rangle\langle m|\psi^{\dagger}|0\rangle}{\omega-E_m+i\eta}+\sum_n\frac{\langle 0|\psi^{\dagger}|n\rangle\langle n|\psi|0\rangle}{\omega+E_n-i\eta}$$

$$=\int_0^{+\infty} dE\ \frac{A(E)}{\omega-E+i\eta}+\frac{B(E)}{\omega+E-i\eta}\,,$$

$\boxed{A, B \text{ are positive}}$

where
$$A(E)=\sum_m\langle 0|\psi|m\rangle\langle m|\psi^{\dagger}|0\rangle\,\delta(E-E_m)$$
$$B(E)=\sum_n\langle 0|\psi^{\dagger}|n\rangle\langle n|\psi|0\rangle\,\delta(E-E_n)\,.$$

As $\omega\to\infty \Rightarrow G(\omega)\to\frac{1}{\omega}\int_0^{+\infty}(A(E)+B(E))\,dE$

$$=\sum_{m,n}\frac{1}{\omega}\left\{\langle 0|\psi|m\rangle\langle m|\psi^{\dagger}|0\rangle+\langle 0|\psi^{\dagger}|n\rangle\langle n|\psi|0\rangle\right\}=\frac{1}{\omega}\langle 0|\psi\psi^{\dagger}+\psi^{\dagger}\psi|0\rangle$$

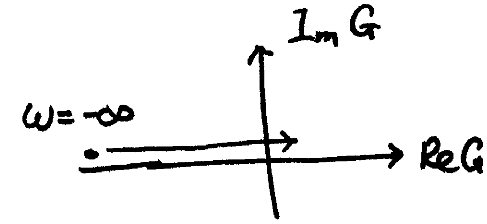
$$=\frac{1}{\omega}\quad \text{i.e.}\quad \boxed{G(\omega)\to\frac{1}{\omega}\,,\ \text{as } \omega\to\infty}\,.$$

$$\mathrm{Re}\,G(\omega)=P\int_0^{+\infty}dE\ \frac{A(E)}{\omega-E}+\frac{B(E)}{\omega+E}$$

$$\mathrm{Im}\,G(\omega)=\begin{cases}-\pi A(\omega) & \text{for } \omega>0\\ \pi B(-\omega) & \text{for } \omega<0\end{cases}\,.$$

As $\omega\to-\infty$, $\mathrm{Re}\,G(\omega)\to-\frac{1}{\omega}<0$. $\mathrm{Im}\,G(\omega)$ at $\omega<0$, describe the hole excitation below the Fermi surface. If we assume that

the hole-excitation is bounded from below, then $\mathrm{Im}\, G(\omega) > 0$ decays faster. $\Rightarrow \varphi(\omega \to -\infty) = \pi$.
towards 0



We set $\mathrm{Im}\, G(\omega) = 0$ at $\omega = 0$, because all the Z_n, E_m in the Lenhmann Rep > 0. The φ of $G(\omega=0)$ is determined by the real part $\mathrm{Re}\, G(\omega=0)$. If $\mathrm{Re}\, G(\omega=0) < 0 \Rightarrow \varphi = \pi$,

$\mathrm{Re}\, G(\omega=0) > 0 \Rightarrow \varphi = 0$.

$$\Rightarrow \boxed{\frac{N}{V} = 2 \int_{\mathrm{Re}\, G(p,\omega=0)>0} \frac{d^3p}{(2\pi)^3}} \longleftarrow$$

The region $G(p, \omega=0)$ is bound by a surface of either zero or diveregence.

Vanishing of $G(p, \omega=0)$ corresponds to $\Sigma \to \infty$, which corresponds to superconductivity $G = \frac{u_k^2}{\omega - \sqrt{\varepsilon_k^2+\Delta^2}} + \frac{v_k^2}{\omega + \sqrt{\varepsilon_k^2+\Delta^2}}$.

$\Rightarrow G(0,k) = \frac{-u_k^2 + v_k^2}{E_k}$, (we do not consider this possibility here).

On the other hand, $G(p, \omega=0) = \frac{Z}{\omega - \xi_p + i\eta} + \cdots \to +\infty$

corresponds to location of $\xi_p = 0$, i.e. <u>the location of FS.</u>

$$G(p, \omega=0) \to -\frac{Z}{\xi_p} \to \begin{cases} +\infty & \xi_p < 0 \\ -\infty & \text{for } \xi_p > 0 \end{cases}.$$

This proof does not assume the isotropy of FS. Interaction may deform the shape but not its volume!