Lect 15: Superconductivity — functional field method

$$H=\int dr\, \psi^+_\alpha(r)\left[\frac{1}{2m}(-i\nabla-eA)^2+ie\phi-\mu\right]\psi_\alpha(r)-g\int dr\, \psi^+_\uparrow(r)\psi^+_\downarrow(r)\,\psi_\downarrow(r)\psi_\uparrow(r)$$

this ϕ is $i\otimes$ conventional scaler potential

$$\rightarrow S[\bar\psi,\psi]=\int_0^\beta d\tau\int d\vec r\left\{\left[\bar\psi_\alpha\left(\partial_\tau+ie\phi+\frac{1}{2m}(-i\nabla-eA)^2-\mu\right]\psi_\alpha(r)-g\bar\psi_\uparrow\bar\psi_\downarrow\psi_\downarrow\psi_\uparrow\right\}$$

gauge invariance

$$\psi_\alpha\to e^{i\theta}\psi_\alpha,\qquad \vec A\to\vec A+e^{-1}\nabla\theta=\vec A'$$
$$\bar\psi_\alpha\to e^{-i\theta}\bar\psi_\alpha\qquad \phi\to\phi-e^{-1}\partial_\tau\theta=\phi'$$

minimal coupling

$$(-i\nabla-e\vec A')\psi'=e^{i\theta}(-i\nabla-eA)\psi$$
$$(\partial_\tau+ie\phi')\psi'=e^{i\theta}(\partial_\tau+ie\phi)\psi$$

§ Hubbard - Stratonovich transformation:

$$\exp\left[g\int d\tau\int dr\, \bar\psi_\uparrow\bar\psi_\downarrow\psi_\downarrow\psi_\uparrow\right]=\int D\bar\Delta D\Delta\,\exp\left[-\int d\tau\int dr\left(\frac{|\Delta|^2}{g}-(\bar\Delta\psi_\downarrow\psi_\uparrow+\Delta\bar\psi_\uparrow\bar\psi_\downarrow)\right)\right]$$

define Nambu spinor

$$\bar\psi=(\bar\psi_\uparrow,\psi_\downarrow),\qquad \psi=\begin{pmatrix}\psi_\uparrow\\ \bar\psi_\downarrow\end{pmatrix}$$

$$\mathcal{Z}=\int D\bar\psi D\psi\int D\bar\Delta D\Delta\,\exp\left\{-\int d\tau\, dr\left[\frac{1}{g}|\Delta|^2-\bar\psi\,\mathcal{G}^{-1}\psi\right]\right\},$$

where $$\mathcal{G}^{-1} = \begin{bmatrix} -\partial_\tau - ie\phi - \frac{1}{2m}(-i\nabla - eA)^2 + \mu, & \Delta \\ \bar{\Delta}, & -\partial_\tau + ie\phi + \frac{1}{2m}(i\nabla - eA)^2 - \mu \end{bmatrix}$$

$$G_p^{-1} = -\partial_\tau - H, \qquad G_h^{-1} = -\partial_\tau + H^T, \qquad \text{where } \hat{x}^T = \hat{x}, \ \hat{p}^T = -\hat{p}.$$

$$\Rightarrow Z = \int D\bar{\Delta} D\Delta \ \exp\left[-\frac{1}{g}\int d\tau dr\, |\Delta|^2 + \ln\det -\mathcal{G}^{-1}\right]$$

$$= \int D\bar{\Delta} D\Delta \ \exp\left\{-\frac{1}{g}\int d\tau dr\, |\Delta|^2 + tr\ \ln[-\mathcal{G}^{-1}]\right\}$$

$$\mathcal{G}^{-1} = \mathcal{G}_0^{-1}(\Delta_0) + \delta\Delta\,\tau^+ + \delta\bar{\Delta}\,\tau^-, \qquad \text{where } \Delta = \Delta_0 + \delta\Delta$$

$$= \mathcal{G}_0^{-1}\left[1 + \mathcal{G}_0(\delta\Delta\tau^+ + \delta\bar{\Delta}\tau^-)\right]$$

$$\Rightarrow tr[\ln(-\mathcal{G}^{-1})] = tr[\ln(-\mathcal{G}_0^{-1}(1 + \mathcal{G}_0(\delta\Delta\tau^+ + \delta\bar{\Delta}\tau^-)))]$$

$$= tr\{\ln(-\mathcal{G}_0^{-1})\} + tr[\ln(1 + \mathcal{G}_0(\delta\Delta\tau^+ + \delta\bar{\Delta}\tau^-))]$$

$$= tr\{\ln(-\mathcal{G}_0^{-1})\} + tr[\mathcal{G}_0(\delta\Delta\tau^+ + \delta\bar{\Delta}\tau^-)]$$

$$- \frac{1}{2} tr[\mathcal{G}_0(\delta\Delta\tau^+ + \delta\bar{\Delta}\tau^-)\mathcal{G}_0(\delta\Delta\tau^+ + \delta\bar{\Delta}\tau^-)]$$

$$+ \cdots\cdots$$

$$|\Delta + \delta\Delta|^2 = |\Delta|^2 + \bar{\Delta}_0\delta\Delta + \Delta_0\delta\bar{\Delta} + \delta\bar{\Delta}\delta\Delta$$

$\Rightarrow$ Search for the saddle point to the linear order

$$\left\{ -\frac{1}{g}\bar{\Delta}_0 + tr_{x,\tau}\left[\mathcal{G}_0(\Delta_0)(x\tau, x\tau) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right] \right\} \delta\Delta = 0$$

$$-\frac{1}{g}\bar{\Delta}_0 + \frac{1}{V\beta}\sum_{k, i\omega_n} \begin{bmatrix} i\omega_n - \xi_k, & \Delta_0 \\ \bar{\Delta}_0, & i\omega_n + \xi_k \end{bmatrix}^{-1}_{21} = 0$$

$$\Rightarrow -\frac{1}{g}\bar{\Delta}_0 + \frac{1}{V}\sum_k \frac{1}{\beta}\sum_{i\omega_n} \frac{1}{(i\omega_n)^2 - E_k^2} \begin{bmatrix} i\omega_n + \xi_k, & -\Delta \\ -\bar{\Delta}_0, & i\omega_n - \xi_k \end{bmatrix}_{21} = 0$$

$$\Rightarrow -\frac{1}{g} + \int \frac{d^3k}{(2\pi)^3} \frac{1}{\beta}\sum_{i\omega_n} \frac{-1}{(i\omega_n)^2 - E_k^2} = 0$$

$$\frac{1}{\beta}\sum_{i\omega_n} \frac{1}{(i\omega_n)^2 - E_k^2} = +\frac{1}{\beta}\frac{1}{2E_k}\sum_{i\omega_n}\left\{ \frac{1}{i\omega_n - E_k} - \frac{1}{i\omega_n + E_k} \right\}$$

$$= \frac{1}{2E_k}\left[n_F(E_k) - n_F(-E_k) \right] = -\frac{1}{2E_k}\tanh\frac{\beta}{2}E_k$$

$$\Rightarrow \text{gap Equation:} \quad \boxed{\frac{1}{g} = \int \frac{d^3k}{(2\pi)^3} \frac{\tanh\frac{\beta}{2}E_k}{2E_k}}.$$

§ Expansion close to T_c —— Ginzburg - Landau free energy

Let us set the free $\mathcal{G}_0$ as $\Delta = 0$, and expand the free energy around it. $\mathcal{G}_0$ is propagator of the free system.

$$\Rightarrow \mathrm{tr}\ln\left[-\mathcal{G}_0^{-1}(1+\mathcal{G}_0\hat{\Delta})\right] = \underbrace{\mathrm{tr}\ln -\mathcal{G}_0^{-1}}_{\text{const}} + \underbrace{\mathrm{tr}\ln(1+\hat{\mathcal{G}}_0\hat{\Delta})}_{-\sum_{n=0}^{\infty}\frac{1}{2n}\mathrm{tr}[(\hat{\mathcal{G}}_0\hat{\Delta})^{2n}]} \leftarrow \boxed{\text{even power}}$$

where $\mathcal{G}_0(k, i\omega_n) = \begin{bmatrix} i\omega_n - \xi_k & 0 \\ 0 & i\omega_n + \xi_k \end{bmatrix}$, $\hat{\Delta} = \begin{bmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{bmatrix}$.

$$\Rightarrow -\frac{1}{2}\mathrm{tr}[(\hat{\mathcal{G}}_0\hat{\Delta})^2] = -\frac{1}{2}\mathrm{tr}\left[\mathcal{G}_0\begin{bmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{bmatrix}\mathcal{G}\begin{bmatrix} 0 & \Delta \\ \bar{\Delta} & 0 \end{bmatrix}\right]$$

$$= -\mathrm{tr}[\mathcal{G}_{0,11}\Delta\mathcal{G}_{0,22}\bar{\Delta}] = -\sum_q\sum_p[\mathcal{G}_{0,11}(p)\Delta(q)\mathcal{G}_{0,22}(p-q)\bar{\Delta}(q)]$$

$$\langle p|\mathcal{G}_{0,11}|p\rangle\langle p|\Delta|p-q\rangle\langle p-q|\mathcal{G}_{0,22}|p-q\rangle\langle p-q|\bar{\Delta}|p\rangle$$

$$\Rightarrow -\frac{1}{2}\mathrm{tr}[(\hat{\mathcal{G}}_0\hat{\Delta})^2] = -\sum_q\bar{\Delta}(q)\Delta(q)\sum_{\vec{p}}\frac{1}{\beta}\sum_{ip_n}\mathcal{G}_{0,11}(p)\mathcal{G}_{0,22}(p-q)$$

$$\Rightarrow S^2[\Delta,\bar{\Delta}] = \sum_q \Gamma_q^{-1}|\Delta(q)|^2$$

$$= \sum_q\left(\frac{1}{g} + \frac{1}{V\beta}\sum_{\vec{p}}\sum_{ip_n}\frac{1}{ip_n - \xi_p}\frac{1}{ip_n - iq_n + \xi_{p-q}}\right)$$

$$= \sum_q\left(\frac{1}{g} - \frac{1}{V\beta}\sum_{\vec{p}}\sum_{ip_n}\frac{1}{ip_n - \xi_{\vec{p}}}\frac{1}{iq_n - ip_n - \xi_{-\vec{p}+\vec{q}}}\right)$$

$$-\frac{1}{\beta}\sum_{ip_n}\frac{1}{ip_n - \xi_{\vec{p}}}\frac{1}{iq_n - ip_n - \xi_{-\vec{p}+\vec{q}}} = \frac{1 - n_f(\xi_p) - n_f(\xi_{-p+q})}{iq_n - \xi_p - \xi_{-p+q}}$$

$$\Gamma^{-1}(q=0) = \frac{1}{g} + \frac{1}{V}\sum_{\vec{p}} \frac{(1-2n_f(\xi_p))}{-2\xi_p} = \frac{1}{g} - \int \frac{d^3k}{(2\pi)^3} \frac{\tanh\frac{\beta}{2}\xi_p}{2\xi_p}$$

$\Gamma^{-1}(q=0) = 0$ means the instability, i.e. T_c is determined by

$$\frac{1}{g} = N(0)\int_{-\hbar\omega_0}^{+\hbar\omega_0} d\xi \frac{\tanh\frac{\beta}{2}\xi_p}{2\xi_p} .$$

Expand $\Gamma^{-1}(q=0, T)$ around T_c $\Rightarrow$

$$\tanh\frac{\beta}{2}\xi_p = \tanh\frac{\beta_c}{2}\xi_p + \frac{-\frac{\beta_c}{2}\xi_p}{ch^2\frac{\beta_c}{2}\xi_p}\frac{\Delta T}{T_c}$$

$$\Rightarrow \Gamma^{-1}(q=0,T) = \left[\int\frac{d^3k}{(2\pi)^3}\frac{\frac{\beta_c}{2}}{ch^2\frac{\beta_c}{2}\xi_p}\right]\frac{\Delta T}{T_c} = N(0)\left[\int d\xi \frac{\frac{\beta_c}{2}}{ch^2\frac{\beta_c\xi}{2}}\right]\frac{\Delta T}{T_c}$$

$$= N(0)\left[\int_{-\infty}^{+\infty} dx \frac{1}{ch^2 x}\right]\frac{\Delta T}{T_c} = N(0)\frac{\Delta T}{T_c} .$$

Similarly for higher order: set $(q, q_n) = 0$ $\Rightarrow$

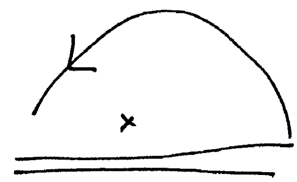
$$S^{(2n)} = \frac{1}{2n} tr\{[\mathcal{G}_0\hat{\Delta}]^{2n}\} = \frac{1}{2n} tr\left\{\left[\begin{matrix} G_p & \\ & -G_{-p}\end{matrix}\right]\left[\begin{matrix} 0 & \Delta \\ \bar{\Delta} & 0\end{matrix}\right]\right\}^{2n}$$

$$= \frac{1}{2n} tr\left\{\left[\begin{matrix} 0, & G_p\Delta \\ -G_{-p}\bar{\Delta}, & 0\end{matrix}\right]^2\right\}^n = \frac{(-)^n}{2n} tr\left[\begin{matrix} G_p\Delta\, G_{-p}\bar{\Delta}, & 0 \\ 0 & G_{-p}\bar{\Delta}\, G_p\Delta\end{matrix}\right]^n$$

$$= \frac{(-)^n}{n}\sum_p \frac{1}{\beta}\sum_{ip_n}\left(\frac{1}{ip_n - \xi_p}\;\frac{1}{-ip_n - \xi_{-p}}\right)^n |\Delta|^{2n}$$

$$= \frac{(-)^n}{n} \frac{N(0)}{\beta} \sum_{ip_n} \int_{-\omega_D}^{\omega_D} d\xi \left(\frac{1}{p_n^2 + \xi^2} \right)^n \quad |\Delta|^{2n}$$

$$\int_{-\infty}^{+\infty} d\xi \left(\frac{1}{p_n^2 + \xi^2} \right)^n = |p_n|^{-(2n-1)} \int_{-\infty}^{+\infty} dx \left(\frac{1}{x^2+1} \right)^n$$



$$\int_{-\infty}^{+\infty} dx \left(\frac{1}{x^2+1} \right)^n = \frac{2\pi i}{(n-1)!} \left[\frac{d}{dx^{n-1}} \left(\frac{1}{x+i} \right)^n \right] \Bigg|_{x=i}$$

$$= \frac{2\pi}{(n-1)!} \frac{(2n-1)!}{(n-1)!} \left(\frac{1}{2i} \right)^{2n-1} (-)^{n-1} \cdot i = \frac{2\pi\,(2n-1)!}{[(n-1)!]^2\, 2^{2n-1}}$$

$$\Rightarrow \quad S^{(2n)} = (-)^n N(0)\, T \left(\frac{|\Delta|}{T} \right)^{2n} \cdot \text{const of } (n).$$

$$\Rightarrow \quad \underline{S_{GL}[\Delta, \bar{\Delta}] = \int d^d r \quad r|\Delta|^2 + \beta|\Delta|^4 \quad + \text{gradient term}}.$$

§6 Action of the Goldstone mode (in the ordered state $\Delta \neq 0$)

$$\mathcal{G}^{-1} = \begin{pmatrix} -\partial_\tau - i\phi - \frac{1}{2m}(-i\nabla - A)^2 + \mu, & \Delta_0 e^{i2\theta} \\ \Delta_0 e^{-i2\theta}, & -\partial_\tau + i\phi + \frac{1}{2m}(i\nabla - A)^2 - \mu \end{pmatrix}$$

where $e\phi \to \phi$, $eA \to A$, "e" has been absorbed. Further, we can absorb "θ" by the transform $\hat{u} = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$, then

$$\mathcal{G}^{-1} \to u\mathcal{G}^{-1}u^\dagger = \begin{pmatrix} -\partial_\tau - i\tilde{\phi} - \frac{1}{2m}(-i\nabla - \tilde{A})^2 + \mu, & \Delta_0 \\ \Delta_0, & -\partial_\tau + i\tilde{\phi} + \frac{1}{2m}(i\nabla - \tilde{A})^2 - \mu \end{pmatrix}$$

where $\tilde{A} = A - \nabla\theta$, $\tilde{\phi} = \phi + \partial_\tau\theta$.

$$\text{Now } \mathcal{G}^{-1} = \underbrace{\begin{pmatrix} -\partial_\tau + \frac{\nabla^2}{2m} + \mu, & \Delta_0 \\ \Delta_0, & -\partial_\tau - \frac{\nabla^2}{2m} - \mu \end{pmatrix}}_{\mathcal{G}_0^{-1}} \underbrace{- i\tilde{\phi}\tau_3 - \frac{i}{2m}[\nabla, \tilde{A}]_+}_{-\hat{\chi}_1} \underbrace{- \frac{\tilde{A}^2}{2m}\tau_3}_{-\hat{\chi}_2}$$

$$S[\tilde{A}, \tilde{\phi}] = -\mathrm{tr}\left[\ln\left(-\mathcal{G}^{-1}\right)\right] = -\mathrm{tr}\{\ln[-\mathcal{G}_0^{-1}(1 - \mathcal{G}_0(\hat{\chi}_1 + \hat{\chi}_2))]\}$$

$$= \text{const} + \underbrace{\mathrm{tr}[\mathcal{G}_0\hat{\chi}_1]}_{\text{1st order } \tilde{A}, \tilde{\phi}} + \mathrm{tr}[\underbrace{\mathcal{G}_0\hat{\chi}_2 + \frac{1}{2}\hat{\mathcal{G}}_0\hat{\chi}_1\hat{\mathcal{G}}_0\hat{\chi}_1}_{\text{2nd order in } \tilde{A}, \tilde{\phi}}]$$

the liner term

$$S^{(1)}[\tilde{A}, \tilde{\phi}] = \frac{1}{V\beta}\sum_{\vec{p}, ip_n} [\underbrace{\mathcal{G}_0(p)\hat{\chi}_1(p,p)}_{\langle p|\mathcal{G}_0|p\rangle\langle p|\chi_1|p\rangle}] = \frac{1}{V\beta}\mathrm{tr}\sum_{\vec{p}, ip_0}[\mathcal{G}_0(p)(i\tilde{\phi}_0\tau_3 + \frac{i}{m}\vec{p}\cdot\tilde{\vec{A}}_0)]$$

the zero frequency uniform part

the term propto to $\vec{p}$ should vanish because $\mathcal{G}(p)$ is even.

$$\frac{1}{\beta}\sum_{ip_n} tr[\mathcal{G}(p)\tau_3] = \frac{1}{\beta}\sum_{ip_n}[\mathcal{G}_{11}(p,ip_n) - \mathcal{G}_{22}(p,ip_n)]$$

according to $\mathcal{G}(k,\tau-\tau') = -\begin{pmatrix} \langle | T_\tau c_k(\tau) c_k^\dagger(\tau') | \rangle, & \langle | T_\tau c_k(\tau) c_{-k}(\tau') | \rangle \\ \langle | T_\tau c_{-k}^\dagger(\tau) c_k^\dagger(\tau') | \rangle & \langle | T_\tau c_{-k}^\dagger(\tau) c_{-k}(\tau') | \rangle \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} \mathcal{G}(k, ik_n), & \mathcal{F}(k, ik_n) \\ \mathcal{F}^\dagger(k, ik_n), & -\mathcal{G}(-k,-ik_n) \end{pmatrix}$$

$$\Rightarrow \frac{1}{\beta}\sum_{ip_n} tr[\mathcal{G}(p)\tau_3] = \frac{1}{\beta}\sum_{ip_n}\left[\mathcal{G}_{\uparrow\uparrow}(p, ip_n) + \mathcal{G}_{\downarrow\downarrow}(-p,-ip_n)\right] = \langle \psi_\uparrow^\dagger(p)\psi_\uparrow(p)\rangle + \langle \psi_\downarrow^\dagger(p)\psi_\downarrow(p)\rangle$$

$\rightarrow S^{(1)}[\tilde{A}] = iN\phi_0 \leftarrow$ Should be compensated by the background, we can set to zero.

② the first term in the second order

$$tr[\mathcal{G}_0 \hat{\chi}_2] = \frac{1}{2m}\frac{1}{V\beta}\sum_p \left[\mathcal{G}_{11}(p,ip_n) - \mathcal{G}_{22}(p,ip_n)\right]\tilde{A}^2(q=0)$$

$$= \frac{n}{2m}\int d\tau \int dr \; A^2(r,\tau)$$ — diamagnetic term.

Now let us consider the second term, where $\hat{\chi}_1 = i\tilde{\phi}\tau_3 + \frac{i}{2m}[\nabla, \tilde{A}]_+$

$\frac{1}{2} tr[\mathcal{G}_0 \hat{\chi}_1 \mathcal{G}_0 \hat{\chi}_1]$,

$$= \frac{1}{2V}\frac{1}{\beta}\sum_{p,q} tr[-\mathcal{G}_{0,p-q/2}\tilde{\phi}_q \tau_3 \mathcal{G}_{0,p+q/2}\tilde{\phi}_{-q} + \mathcal{G}_{0,p-q/2}\langle p-q/2|\tfrac{i}{2m}[\nabla,\tilde{A}]_+|p+\tfrac{q}{2}\rangle \mathcal{G}_{0,p+\frac{q}{2}}\langle p+\tfrac{q}{2}|\tfrac{i}{2m}[\nabla,\tilde{A}]_+|p-q/2\rangle$$

* we will neglect the dependence of $\mathcal{G}_{0,p+q}$ on "q", because $\tilde{A}$, $\tilde{\phi}$ already contains information of $\nabla\theta$ and $\partial_z\theta$.

* Let us check $\langle p-q/2|\frac{i}{2m}[\nabla,\tilde{A}]_+|p+\frac{q}{2}\rangle = \frac{i}{2m}[i(p+\frac{q}{2})A(q) - (-i(p-q/2))A(q)]$

$$= \frac{-1}{m}\vec{p}\cdot\vec{A}(q)$$

similarly $\langle p+q/2|\frac{i}{2m}[\nabla,\hat{A}]_+|p-q/2\rangle = \frac{-1}{m}\vec{p}\cdot\vec{A}(-q)$

$\Rightarrow$ * Crossing term between $\tilde{\phi}$, $\vec{p}\cdot\tilde{A}$, vanishes as $q\to 0$, because it is odd in momenta.

$$\mathcal{G}_{0,p} = [i\omega_n - (\tau_3\xi_p - \Delta_0\tau_1)]^{-1} = \frac{1}{\omega_n^2+\xi_p^2+\Delta_0^2}[-i\omega_n - \tau_3\xi_p + \tau_1\Delta_0]$$

$$\Rightarrow \frac{1}{2}tr[\mathcal{G}_0\hat{\chi}_1\mathcal{G}_0\hat{\chi}_1] = \frac{1}{2V\beta}tr\Big\{\Big[\frac{-i\omega_n-\tau_3\xi_p+\tau_1\Delta_0}{\omega_n^2+\xi_p^2+\Delta_0^2}(-\tilde{\phi}_q\tau_3]\frac{-i\omega_n-\tau_3\xi_p+\tau_1\Delta_0}{\omega_n^2+\xi_p^2+\Delta_0^2}[-\tilde{\phi}_{-q}\tau_3]\Big\}$$

$$+ \frac{1}{2V\beta} tr\left\{ \frac{-i\omega_n - \tau_3\xi_p + \tau_1\Delta_0}{\omega_n^2 + \xi_p^2 + \Delta_0^2} \left[\frac{-1}{m}\vec{P}\cdot\vec{A}(q)\right] \frac{-i\omega_n - \tau_3\xi_p + \tau_1\Delta_0}{\omega_n^2 + \xi_p^2 + \Delta_0^2} \left[\frac{-1}{m}\vec{P}\cdot\vec{A}(-q)\right]\right\}$$

$$= \frac{1}{V\beta} \sum_{P,q} \frac{1}{(\omega_n^2+\lambda_p^2)^2} \left[\tilde{\phi}_q \tilde{\phi}_{-q} \left[-\omega_n^2 + \xi_p^2 - \Delta_0^2\right] + \frac{(\vec{P}\cdot\vec{A}(q))(\vec{P}\cdot\vec{A}(-q))}{m^2} (-\omega_n^2 + \xi_p^2 + \Delta_0^2)\right]$$

set $\lambda_p^2 = \xi_p^2 + \Delta_0^2$

$$= \frac{1}{V\beta} \sum_{P,q} \frac{1}{(\omega_n^2+\lambda_p^2)^2} \left[\tilde{\phi}_q \tilde{\phi}_{-q} (-\omega_n^2 + \lambda_p^2 - 2\Delta_0^2) + \frac{p^2 \vec{A}(q)\cdot\vec{A}(-q)}{3m^2} (-\omega_n^2 + \lambda_p^2)\right]$$

$\longrightarrow$ translate back to real space of $\tilde{\phi}$, $\tilde{A}$ $\begin{bmatrix} \tilde{\phi} = \phi + \partial_z\theta \\ \tilde{A} = A - \nabla\theta \end{bmatrix}$.

$$S[\hat{A}] = \int dz\, dr \left[\underbrace{\frac{1}{V\beta}\sum_P \frac{-\omega_n^2 + \lambda_p^2 - 2\Delta_0^2}{(\omega_n^2+\lambda_p^2)^2}}_{C_1} \tilde{\phi}^2(r,z) + \underbrace{\left(\frac{n}{2m} - \frac{1}{3m^2}\frac{1}{V\beta}\sum_P \frac{p^2(-\omega_n^2+\lambda_p^2)}{(\omega_n^2+\lambda_p^2)}\right)}_{C_2} \tilde{A}^2(r,z)\right]$$

do frequency summation $\leftarrow$ encercise

$$\frac{1}{\beta}\sum_n \frac{-\omega_n^2 + \lambda_p^2 - 2\Delta_0^2}{(\omega_n^2+\lambda_p^2)^2} = -\frac{1}{2\lambda_p}\left[n_F(-\lambda_p)\left(\frac{\Delta_0}{\lambda_p}\right)^2 + n_F'(-\lambda_p)\frac{\xi_p^2}{\lambda_p}\right] + (\lambda_p \to -\lambda_p)$$

$$\approx -\Delta_0^2/(2\lambda_p^3)$$

$$\frac{1}{\beta}\sum_n \frac{-\omega_n^2 + \lambda_p^2}{(\omega_n^2+\lambda_p^2)^2} = -\beta\left[n_F(\lambda_p)(1 - n_F(\lambda_p)\right]$$

$$\Rightarrow \quad C_1 = -\frac{1}{V}\sum_p \frac{\Delta^2}{\lambda_p^3} = -\frac{N_0}{2}\int d\xi \, \frac{\Delta_0^2}{(\xi^2+\Delta_0^2)^{3/2}} = -\frac{N_0}{2}\int dx \, \frac{1}{(x^2+1)^{3/2}}$$

$$= -N_0$$

p^2 is peaked at k_F

$$C_2 = \frac{n}{2m} - \frac{N_0}{3m^2} k_F^2 \int d\xi \; \beta \, n_F(\lambda)(1-n_F(\lambda)) \, .$$

at $T \ll \Delta_0$; $\quad C_2 \simeq \frac{n}{2m}$, i.e. the response is dominated by the diamagnetic term.

at $T > \Delta_0$. $\quad \int d\xi \; \beta \, n_F(\lambda)(1-n_F(\lambda)) = -\int_{-\infty}^{\infty} d\xi \; \partial_\xi n_F(\xi) = n_F(-\infty) - n_F(\infty)$
$\Rightarrow \lambda = \xi$

$$= 1$$

$$n_F(\xi) = \frac{1}{e^{\beta\xi}+1} \qquad \partial_\xi n_F(\xi) = \frac{-e^{\beta\xi}\cdot\beta}{(e^{\beta\xi}+1)^2} = \beta \, n_F(1-n_F) \, .$$

$\Rightarrow \quad C_2 = 0$. The diagramtic response should be canceled by the paramagnetic response.

in the middle $\Rightarrow \quad C_2 = \frac{n_s}{2m} = n - \frac{2N_0 k_F^2}{3m}\int d\xi \; \beta \, n_F(\lambda)(1-n_F(\lambda))$

$$= n - n_s$$

the second term is the normal fluid density.

Lect 10 Superconductivity

- The superfluidity theory we have discussed is for neutral systems.

How about for charged systems, say, superconductors?

The neutral boson Lagrangian

$$\mathcal{L}(\varphi) = \frac{i}{2}(\varphi^* \partial_t \varphi - \varphi \partial_t \varphi^*) - \frac{1}{2m} \partial_x \varphi^* \partial_x \varphi + \mu |\varphi|^2 - \frac{V_0}{2} |\varphi|^4$$

has the $\underbrace{\text{U(1)}}_{\text{global}}$ symmetry $\varphi \to e^{if}\varphi$, but it does not have the

symmetry $\varphi(x,t) \to e^{if(x,t)} \varphi(x,t)$.

For charged bosons (Cooper pairs of electrons)

- $$\mathcal{L}(\varphi, A_\mu) = \frac{i}{2}\left(\varphi^*(\partial_0 + iA_0)\varphi - \varphi(\partial_0 - iA_0)\varphi^*\right) - \frac{1}{2m} |(\partial_i + iA_i)\varphi|^2$$

$$+ \mu|\varphi|^2 - \frac{V_0}{2}|\varphi|^4 + \frac{1}{8\pi e^2}\left(\frac{1}{c}E^2 - cB^2\right)$$

where $E_i = \partial_0 A_i - \partial_i A_0 = F_{0i}$, $B_i = \epsilon_{ijk} \partial_j A_k = \frac{1}{2} \epsilon_{ijk} F_{jk}$

which is invariant as

$$\varphi \to \tilde{\varphi} = e^{if(x,t)} \varphi, \quad \tilde{A}_\mu = A_\mu - \partial_\mu f$$

If φ_c, ~~A_c~~ satisfy the classical equation of motion (fix A_μ).

$\Rightarrow$ do variation respected to $\delta\varphi$: $\varphi' = e^{if(x,t)}\varphi \Rightarrow \delta\varphi' = \varphi(1 + if(x,t))$

$J_i = \frac{-i}{2m}(\varphi^* \partial_i \varphi - \partial_i \varphi^* \cdot \varphi) + A_i|\varphi|^2$

- $\Rightarrow \mathcal{L}[\tilde{\varphi}, A_\mu] = \mathcal{L}[\varphi, A_\mu - \partial_\mu f]$

$\Rightarrow \delta\mathcal{L} = \int d^dx\, dt\; \partial_\mu f \cdot J^\mu \Rightarrow \partial_\mu J^\mu = 0$ where $J_0 = \varphi^* \varphi$,

* Current correlation function & E-M responses

$$\mathcal{L}[\varphi, A_\mu] = \mathcal{L}[\varphi] - A_0 j^0 - A_i j^i - \frac{1}{2m}(A_i)^2 \rho$$

$$j_i = -\frac{i}{2m}[\varphi^*(\partial_i\varphi) - (\partial_i\varphi^*)\varphi]$$

we use the linear response theory

$$\langle J^\mu(x,t)\rangle = \langle j^\mu(x,t)\rangle + (1-\delta_{\mu,0}) A_\mu \rho$$

$$= \int d^d x\, dt\; \Pi^{\mu\nu}(xt; x't')\, A_\nu(x',t')$$

The response function ~~read~~ reads

$$\Pi^{00}(xt; x't') = -i\theta(t-t')\langle[\rho(x,t), \rho(x',t')]\rangle$$

$$\Pi^{0i}(xt; x't') = -i\theta(t-t')\langle[\rho(xt), j^i(x't')]\rangle$$

$$\Pi^{i0}(xt; x't') = -i\theta(t-t')\langle[j^i(xt)\; \rho(x't')]\rangle$$

$$\Pi^{ij}(xt; x't') = -i\theta(t-t')\langle[j^i(xt), j^j(x't')]\rangle + \delta^{ij}\delta(x-x')\delta(t-t')\frac{\langle\rho\rangle}{m}$$

$\Pi^{\mu\nu}$ satisfies $\boxed{(\Pi^{\mu\nu}(xt; x't'))^* = \Pi^{\nu\mu}(x't'; xt)}$

$$\Rightarrow (\Pi^{\mu\nu}(k))^* = \Pi^{\nu\mu}(-k)$$

$\Pi^{\mu\nu}$ satisfies continuity & gauge invariance conditions.

$$\partial_\mu J^\mu = 0 \Rightarrow \boxed{k_\mu \Pi^{\mu\nu}(k) = 0}$$

$$A \to A + \partial f \to \boxed{\Pi^{\mu\nu}(k)\, k_\nu = 0}$$

we can decompose $\Pi^{ij}(k)$ into transverse & longitudinal parts

spatial part
$$\Pi^{ij}(k) = \frac{k_i k_j}{k^2}\Pi^{\parallel}(k) + \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\Pi^{\perp}(k)$$

$$\Pi^{0i}(k) = (\Pi^{i0}(-k))^{\dagger} = -\frac{k_j}{\omega}\Pi^{ji}(k) \Leftarrow k_\mu \Pi^{\mu\nu}(k) = 0$$

$$\Pi^{00}(k) = -\frac{k_j}{\omega}\Pi^{0j} = \frac{k_i k_j}{\omega^2}\Pi^{ij}(k)$$

$$\Rightarrow \boxed{\begin{aligned}\Pi^{0i}(k) &= -k_i \Pi^{\parallel}(k) \\ \Pi^{00}(k) &= \frac{k^2}{\omega^2}\Pi^{\parallel}(k).\end{aligned}}$$

the time component is only related to the ~~longitudinal~~ longitudinal component.

For the correlation in the spatial diretion, we define the paramagnetic contribution as

$$\Pi^{ij}_{para}(xt;x't') = -i\Theta(t-t')\langle[j^i(xt), j^j(x't')]\rangle,$$

then
$$\Pi^{\parallel}(k) = \Pi^{\parallel}_{para}(k) + \frac{\langle\rho\rangle}{m}$$
$$\Pi^{\perp}(k) = \Pi^{\perp}_{para}(k) + \frac{\langle\rho\rangle}{m}$$
diamagnetic contribution.

we know $\delta\rho(k) = \Pi^{00}(k)A_0(k)$,

Since $A_0(k)$ is just external potential, thus $-\Pi^{00}(k)$ is just the compressbility $\chi(k) = -\Pi^{00}(k)$. Let us assume that $\chi(k) \to$ Const $(k\to 0$

then $\boxed{\Pi^{\parallel}(k) = -\chi(k)\frac{\omega^2}{k^2}}$, thus at the static limit ($\omega \to 0$ first; $k \to 0$ second), $\Pi^{\parallel}(k)$ goes to zero (longitudinal field has no effect!), thus $\Pi^{\parallel}_{para}(k)$ must exactly cancel the diamagnetic contribution.

On the other hand, if $k \to 0$ first and $\omega \to 0$ second $\boxed{(k \ll \omega)}$.

$-i\omega A(k,\omega)$ is the electric field

$$J^i(\omega) = \lim_{k\to 0} \frac{\Pi^{ij}}{-i\omega} \underbrace{(-i\omega) A_j(k,\omega)}_{E(\omega,k)}$$

$$\Rightarrow \quad \sigma(\omega) = \frac{\Pi^{\parallel}(\omega,0)}{-i\omega} = \frac{\Pi^{\perp}(\omega,0)}{-i\omega}$$

in the normal state, as $k \to 0$, there's no difference between transverse and longitudinal modes.

$$\nabla\cdot\vec{D} = 4\pi\rho \Rightarrow \vec{D} = \vec{E} + 4\pi\vec{P} = (1+4\pi\chi)\vec{E} = \epsilon\vec{E}$$

$$\vec{j} = \partial_t P \Rightarrow j(\omega) = -i\omega P(\omega) = -i\omega\chi(\omega)\vec{E}$$

$$\vec{j}(\omega) = \frac{-i}{4\pi}(\epsilon - 1)\omega \vec{E}(\omega)$$

i.e. $\sigma(\omega) = \frac{-i}{4\pi}(\epsilon - 1)\omega$ i.e $\boxed{\epsilon(\omega) = 1 + \frac{i 4\pi\sigma(\omega)}{\omega}}$

how about in the static limit $\omega << k$, $\Pi^{\perp}(k,\omega)$?

From $\nabla \times \vec{M} = \vec{j}$

$$\Rightarrow j_i = i\,\epsilon^{ijk} k_j M_k = -\Pi^{ij} A_k = -(\delta_{ij} k^2 - k^i k^j) A_j \frac{\Pi^{\perp}}{k^2}$$

$$= \epsilon^{ij'k'} k_{j'} \epsilon^{k'i'j} k_{i'} A_j \frac{\Pi^{\perp}(k,\omega)}{k^2} = -i\,\epsilon^{ijk} k_j B_k \frac{\Pi^{\perp}(k,\omega)}{k^2}$$

$$\Rightarrow \boxed{M_i = -\frac{\Pi^{\perp}(k,0)}{k^2} B_i}\,,$$ i.e. $\frac{\Pi^{\perp}(k,0)}{k^2}$ is magnetic susceptibility.

Summarize: Compressibility $\lim_{k\to 0} \chi_c(k) = -\lim_{k\to 0} k^2 \lim_{\omega\to 0} \frac{\Pi^{\parallel}(k,\omega)}{\omega^2}$

orbital magnetic susceptibility $\lim_{k\to 0} \chi_m(k) = -\lim_{k\to 0} \frac{1}{k^2} \Pi^{\perp}(k,0)$

* E-M:

let us calculate $\Pi^{\mu\nu}$ for bosonic field:

$$\mathcal{L} = \frac{\chi}{2}\left((\partial_0 \theta + A_0)^2 - v^2 (\partial_i \theta + A_i)^2 \right)$$

the paramagnetic current $j_0 = -\chi \partial_0 \theta$, $j_i = \chi v^2 \partial_i \theta$

total current $J_0 = -\chi(\partial_0 \theta + A_0)$ $J_i = \chi v^2 (\partial_i \theta + A_i)$

$$\Pi^{00}_{para}(k,\omega) = \chi^2 (-i\omega)(+i\omega)\langle \theta(\omega k)\theta(-\omega,-k)\rangle = \frac{\chi \omega^2}{\omega^2 - v^2k^2 + i0^+ \mathrm{sgn}(\omega)}$$

↳ retarded green's function

$$\Pi^{0i}_{para}(k,\omega) = \Pi^{i,0}_{para}(k,\omega)$$

$$= \frac{\chi^2}{v^2}(-i\omega)(-ik_i)\langle \theta(\omega,k)\,\theta(-\omega,-k)\rangle = \frac{v^2\chi\,(-)\,k_i\,\omega}{\omega^2 - v^2k^2 + i0^+ \mathrm{sgn}(\omega)}$$

$$\Pi^{ij} = \chi^2 (ik_i)(-ik_j)\langle \theta(\omega k)\theta(-\omega,-k)\rangle = \chi v^4 \frac{k_i k_j}{\omega^2 - v^2k^2 + i0^+ \mathrm{sgn}(\omega)}$$

$$\Rightarrow \Pi^{00}(k,\omega) = \chi\left[\frac{\omega^2}{\omega^2 - v^2k^2 + i0^+\mathrm{sgn}(\omega)} - 1\right] = +\chi\left[\frac{v^2k^2}{\omega^2 - v^2k^2 + i0^+\mathrm{sgn}(\omega)}\right]$$

$$\Pi^{0i}(k,\omega) = \Pi^{i0}(k,\omega) = -\chi v^2 \frac{\omega k_i}{\omega^2 - v^2k^2 + i0^+\mathrm{sgn}(\omega)}$$

$$\Pi^{ij}(k,\omega) = \chi v^2\left(\delta_{ij} + v^2\frac{k_i k_j}{\omega^2 - v^2k^2 + i0^+\mathrm{sgn}(\omega)}\right)$$

$$\Rightarrow \boxed{\begin{aligned} \Pi^{\parallel} &= \chi v^2\left(1 + \frac{v^2k^2}{\omega^2 - v^2k^2 + i0^+\mathrm{sgn}(\omega)}\right) = \chi v^2 \frac{\omega^2}{\omega^2 - v^2k^2 + i0^+\mathrm{sgn}(\omega)} \\ \Pi^{\perp} &= \chi v^2 \end{aligned}}$$

$\Rightarrow$ the Compressbility $\lim_{k\to 0}\lim_{\omega\to 0} -\Pi^{00}(k,\omega) = \lim_{k\to 0} -\Pi^{00}(k,0) = \chi.$

magnetic susceptibility $-\lim_{k\to 0}\frac{1}{k^2}\Pi^{\perp}(k,0)$ divergences.

~~real part of~~ if we choose transverse gauge $\partial_i A_i = 0$

$$\Rightarrow \boxed{J^i = \Pi^{ij}A_j = \Pi^{\perp}A_i = \chi v^2 A_i = +\frac{\rho}{m}A_i}$$ London equation!

optical conductivity $\boxed{Re\,\sigma(\omega) = -Im\,\frac{\Pi''(\omega,0)}{\omega} = Im\,\frac{\chi v^2}{\omega+i0^+} = \frac{\pi\rho}{m}\delta(\omega)}$

which is zero at finite frequency.

* Anderson - Higgs mechnism

In charged superfluid (superconductor), the gapless Goldstone mode is gone. The gauge field A_μ has its own dynamics.

$$Z = \int D\varphi DA_\mu \, e^{i\int d^dx\,dt \; \mathcal{L}(\varphi, A_\mu) + \frac{1}{8\pi e^2}(E^2 - B^2)}$$

in the symmetry breaking ground states $\varphi = |\varphi| e^{i\theta(x,t)}$

we can absorb the phase by doing a gauge transformation

$$\left|(\partial_\mu - iA_\mu)|\varphi| e^{i\theta(x,t)}\right|^2 = \left|(\partial_\mu \underbrace{-iA_\mu - i\partial_\mu\theta}_{i\tilde{A}_\mu})|\varphi|\right|^2$$

which is equavilent to iA_μ, and does not give new physical field configuration.

thus φ can be made real:

$$\varphi = \bar{\varphi} + \delta\varphi$$

$\Rightarrow$

~~to~~ or equivalently we can set $\theta = 0$ in the action

$$\mathcal{L} = \frac{\chi}{2}\left[(\partial_0\theta + A_0)^2 - v^2(\partial_i\theta + A_i)^2\right] = \frac{1}{2V_0}A_0^2 - \frac{\rho}{2m}A_i^2$$

⇒ the Langrangian for A becomes

$$\mathcal{L} = \frac{1}{2V_0} A_0^2 - \frac{\rho}{2m} A_i^2 + \frac{1}{8\pi e^2}\left(E^2 - B^2\right)$$

note that $\frac{1}{8\pi e^2} E^2 = \frac{1}{8\pi e^2}[\partial_0 A_i]^2 + \frac{1}{8\pi e^2}(\partial_i A_0)^2 - \frac{1}{4\pi e^2}\partial_i A_0 \partial_0 A_i$

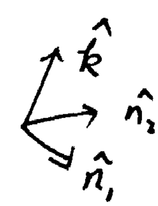
and thus A_0 has no time derivative. Let's integrate out A_0.

$$A_0\left(\frac{1}{2V_0} - \underbrace{\frac{\partial_i^2}{8\pi e^2}}_{\text{neglected}}\right)A_0 + \frac{1}{4\pi e^2} A_0\, \partial_0 \partial_i A_i$$

$$\approx \underbrace{\frac{1}{2V_0}\left(A_0 - \frac{V_0}{4\pi e^2}\partial_0 \partial_i A_i\right)^2}_{\text{const after gaussian integration}} - \frac{1}{2V_0}\frac{V_0^2}{(4\pi e^2)^2}(\partial_0 \partial_i A_i)(\partial_0 \partial_j A_j)$$

$$\Rightarrow \mathcal{L}_{eff} = \partial_0 A_j \Big[\frac{1}{8\pi e^2}\delta_{ij} + \frac{1}{2}\frac{V_0}{(4\pi)^2 e^4}\underbrace{\partial_i \partial_j}_{\text{longitudinal}}\Big]\partial_0 A_i - \frac{\rho}{2m}A_i^2 - \underbrace{\frac{B^2}{8\pi e^2}}_{\text{transverse}}$$

Let us decompose $A_i = \hat{k}_i A^{\parallel} + \hat{n}_a A_a^{\perp}$



$$\mathcal{L}_{eff} = \frac{1}{8\pi e^2}\left[(\partial_0 A_a^{\perp})^2 - (\partial_i A_a^{\perp})^2\right] - \frac{\rho}{2m}(A_a^{\perp})^2$$

$$+ \frac{1}{8\pi e^2}\partial_0 A^{\parallel}\left(1 + \frac{V_0}{4\pi e^2}(\partial_i)^2\right)\partial_0 A^{\parallel} - \frac{\rho}{2m}(A^{\parallel})^2$$

all the 3 modes has the gap $\Delta = e\sqrt{4\pi\rho/m}$ — replace $\partial_0^2 = -\Delta^2$ and $v^2 = V_0 \rho/m$

$$\Rightarrow \mathcal{L}_{eff} = \frac{1}{8\pi e^2}\left[(\partial_0 A_a^{\perp})^2 - (\partial_i A_a^{\perp})^2\right] - \frac{\rho}{2m}(A_a^{\perp})^2 + \frac{1}{8\pi e^2}\left[(\partial_0 A^{\parallel})^2 - v^2(\partial_i A^{\parallel})^2\right] - \frac{\rho}{2m}(A^{\parallel})^2$$

★ Superfluidity & superfluid density:

let's consider a boson system which is invariant under Galileo transformation, and assum an excitation spectrum $\mathcal{E}(k)$. Consider a single excitation $\mathcal{E}_\cdot(k)$, thus $\vec{P}=\vec{k}$ and $E=E_{ground}+\mathcal{E}$. Let us boost the system by velocity $\vec{v}$, ~~excitation has a doppler shift~~ the total energy and momentum change to

$$E = E_{ground} + \epsilon + \frac{1}{2}Nmv^2 + \vec{v}\cdot\vec{k}$$
$$\vec{P} = \vec{k} + Nm\vec{v}$$

Thus compared with boosted ground state, we have

$$\mathcal{E}_v(k) = \mathcal{E}(k) + \vec{k}\cdot\vec{v}, \quad \vec{P} = \vec{k}.$$

In the boosted Superfluid, in the equilibrium, we have distribution according to $\mathcal{E}_v(k)$.

$$E_v = E_g + \sum_k \mathcal{E}_v(k)\, n_B(\mathcal{E}_v(k)) + \frac{1}{2}Nmv^2$$

$$\vec{P} = Nm\vec{v} + \sum_k \vec{k}\, n_B(\mathcal{E}_v(\vec{k}))$$

The momentum can be written as, (expand to first order of $\vec{v}\cdot\vec{k}$).

$$\vec{P} = (Nm - V\rho_n m)\,\vec{v} = V\underline{\rho_s}\, m\vec{v}$$

↑ superfluid density.

where $\rho_n = -\frac{1}{md}\int \frac{d^dk}{(2\pi)^d}\, k^2\, n_B'(\varepsilon(k))$.

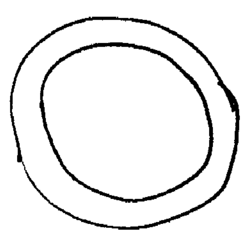
$\frac{1}{d}$ from the angle integral $\vec{v}\cdot\vec{k}$ average

If $\varepsilon(k) \propto |k|$ $\Rightarrow$ $\rho_n \propto T^{d+1}$. ~~and ρ_s~~

Another question is: since super-current carring state is not the lowest energy state, why it is stable?

Let us consider a symmetry breaking state $\varphi = \varphi_0 + \delta\varphi$
(φ_0: condensate, $\delta\varphi$: fluctuation)

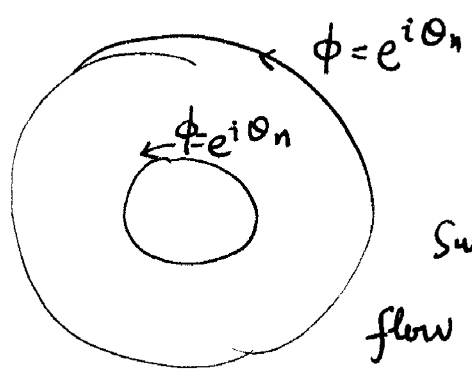
Let us twist φ_0 (boost)



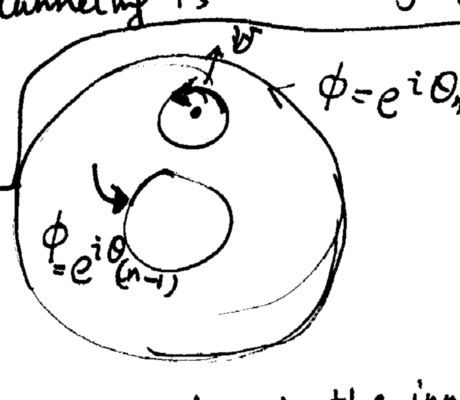
$$\varphi' = \varphi_0 e^{imvx} \quad \text{such that} \quad mvL = 2\pi n .$$

Because $|\varphi_0|$ is fixed, we cannot untwist the condensate without suppressing $|\varphi_0|$ to zero. Thus although $\varphi_0 e^{imvx}$ has a high energy, it cannot relax unless vortex tunneling.

The vortex tunneling is the decaying machanism of superfluidity.



Superfluid won't flow forever, but it is long-lived!



creat a vortex in the inner wall and move it accross the sample to outer wall untwist the superfluid.