Lect 5. mean field theory of Superfluid

§1. Review of second quantization

$\psi(r) \longrightarrow$ field operator $\hat{\psi}(r) = \sum_k \frac{1}{\sqrt{V}} e^{ikr} \hat{a}_k$.

$\rho(r) = \psi^*(r)\psi(r) \quad \rightarrow \quad \rho(r) = \hat{\psi}^+(r)\psi(r)$

$H_0 = \sum_i \frac{-\hbar^2}{2m}\nabla_i^2 \quad \rightarrow \quad H_0 = \int dr\, \hat{\psi}^+(r)\left(-\frac{\hbar^2}{2m}\nabla^2\right)\psi(r)$

$$= \sum_k a_k^+ a_k \varepsilon_k$$

$H_{int} = \frac{1}{2}\int dr\,dr'\, \psi^+(r)\psi^+(r')\psi(r')\psi(r) \quad \rightarrow \quad H_{int} = \frac{1}{2}\int dr\,dr'\, \hat{\psi}^+(r)\hat{\psi}^+(r')\hat{\psi}(r')\psi(r)$

$$= \frac{1}{2V}\sum_{q,p_1p_2} V(q)\, a^+_{p_1+q} a^+_{p_2-q} a_{p_2} a_{p_1}$$

§2. Bosonic system

$$H = \int dr\, \psi^+(r)\Big(\underbrace{-\frac{\hbar^2}{2m}\nabla^2}_{-\mu}\Big)\psi(r) + \int dr\,dr'\, \frac{1}{2}\rho(r)V(r-r')\rho(r')$$

$$= \sum_k (\varepsilon_k - \mu)\, a_k^+ a_k + \frac{1}{2V}\sum_{kk'q} V_q\, a^+_{k'-q} a^+_{k+q} a_k a_{k'}$$

where $V_q = \int dr\, V(r)\, e^{ikr}$

we assume that, there's a macroscopic number of particles occupying $k=0$ state. i.e. $\langle\Omega|a_0^\dagger a_0|\Omega\rangle = N_0 \sim O(N)$. Then we can neglect the commutation relation $[a_0, a_0^\dagger]=1 \to [\frac{a_0}{\sqrt{N_0}}, \frac{a_0^\dagger}{\sqrt{N_0}}] = \frac{1}{N_0} \to 0.$

$$H_{mean} = \sum_k (\varepsilon_k - \mu) a_k^\dagger a_k + \frac{V}{2}\rho_0^2 V_0 + \frac{\rho_0}{2}\sum_{k\neq 0}(2a_k^\dagger a_k V_0$$
$$+ 2a_k^\dagger a_k V_k + V_k a_{-k} a_k + V_k a_{-k}^\dagger a_k^\dagger)$$
$$= \sum_k (\varepsilon_k - \mu_k') a_k^\dagger a_k + \frac{1}{2}\rho_0^2 V_0 \cdot V + \frac{\rho_0}{2}\sum_{k\neq 0} V_k (a_{-k} a_k + a_{-k}^\dagger a_k)$$

$\mu_k' = \mu - \rho_0 V_0 - \rho_0 V_k$, $\rho_0 = N_0/V$. we have neglected term at $(a_{k\neq 0})^3$.

$$H_{mean} = \sum_k{}' (a_k^\dagger \;\; a_{-k}) \begin{pmatrix} \varepsilon_k - \mu_k' & \rho_0 V_k \\ \rho_0 V_k & +(\varepsilon_k - \mu_k') \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}$$
$$-\frac{1}{2}\sum_k{}' (\varepsilon_k - \mu_k') + \frac{V}{2}\rho_0^2 V_0$$

define Bogoliubov transformation $\begin{pmatrix} \alpha_k \\ \alpha_{-k}^\dagger \end{pmatrix} = \begin{pmatrix} \cosh\theta_k & \sinh\theta_k \\ \sinh\theta_k & \cosh\theta_k \end{pmatrix} \begin{pmatrix} a_k \\ a_{-k}^\dagger \end{pmatrix}$

It's easy to check α_k, satisfies the commutator relation

$[\alpha_k, \alpha_{k'}^+] = \delta_{kk'}$, we need to determine the value of

angle of θ_k to diagonalize the matrix.

$$\begin{pmatrix} \cosh\theta_k & \sinh\theta_k \\ \sinh\theta_k & \cosh\theta_k \end{pmatrix}\begin{pmatrix} \varepsilon_k-\mu_k' & \rho_0 V_k \\ \rho_0 k & \varepsilon_k-\mu_k' \end{pmatrix}\begin{pmatrix} \cosh\theta_k & \sinh\theta_k \\ \sinh\theta_k & \cos\theta_k \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\theta_k & \sinh\theta_k \\ \sinh\theta_k & \cosh\theta_k \end{pmatrix}\begin{pmatrix} (\varepsilon_k-\mu_k')\cosh\theta_k + \rho_0 V_k\sinh\theta_k, & (\varepsilon_k-\mu_k')\sinh\theta_k + \rho_0 V_k\cos\theta_k \\ \rho_0 k\cosh\theta_k + (\varepsilon_k-\mu_k')\sinh\theta_k, & \rho_0 k\sinh\theta_k + (\varepsilon_k-\mu_k')\cos\theta_k \end{pmatrix}$$

$$= \begin{pmatrix} (\varepsilon_k-\mu')\cos2\theta_k + \rho_0 V_k\sin2\theta_k. & \rho_0 V_k\cos2\theta_k + (\varepsilon_k-\mu_k')\sin2\theta_k \\ \rho_0 V_k\cos2\theta_k + (\varepsilon_k-\mu')\sin2\theta_k, & (\varepsilon_k-\mu')\cos2\theta_k + \rho_0 V_k\sin2\theta_k \end{pmatrix}$$

we set $\rho_0 V_k \cos h2\theta_k + (\varepsilon_k-\mu_k')\sin h2\theta_k = 0 \Rightarrow \tanh 2\theta_k = -\dfrac{\rho_0 V_k}{\varepsilon_k-\mu_k'}$,

$$\cosh 2\theta = \frac{\varepsilon_k-\mu_k'}{\sqrt{(\varepsilon_k-\mu_k)^2-(\rho_0 V_k)^2}}$$

$$\sinh 2\theta = \frac{-\rho_0 V_k}{\sqrt{(\varepsilon_k-\mu_k)^2-(\rho V_k)^2}}$$

$\Rightarrow$

$$H_{mean} = \sum_k{}' (\alpha_k^+ \ \alpha_{-k})\begin{pmatrix} E_k & 0 \\ 0 & E_k \end{pmatrix}\begin{pmatrix} \alpha_k \\ \alpha_{-k}^+ \end{pmatrix}$$

$$-\frac{1}{2}\sum_k{}'(\varepsilon_k-\mu_k') + \frac{V}{2}\rho_0^2 V_0$$

where $E_k = \sqrt{(\varepsilon_k-\mu_k)^2-(\rho_0 V_k)^2}$, $H_{mean} = \sum_{k\neq0} E_k \alpha_k^+\alpha_k + \underbrace{V(-\mu\rho_0 + \frac{1}{2}\rho_0^2 V_0)}_{\text{contribution at } k=0} - \sum_{k\neq0}\frac{1}{2}(\varepsilon_k-\mu_k' - E_k)$

The value of μ is set by minimizing Ω_g, respect to N_0.

$$\partial\Omega_g/\partial N_0 = 0 \qquad \Omega_g = V\left(-\mu \rho_0 + \tfrac{1}{2}\rho_0^2 V_0\right) \Rightarrow \boxed{\rho_0 = \frac{\mu}{V_0}}$$

keep terms at linear order of V_k

$$N = \langle\Omega| \sum_k a_k^+ a_k |\Omega\rangle = N_0(\mu) + \sum_k (\sinh\theta_k)^2.$$

$$= V\left[\rho_0(\mu) + \int\frac{d^dk}{(2\pi)^d} \sinh^2\theta_k\right] \Rightarrow$$

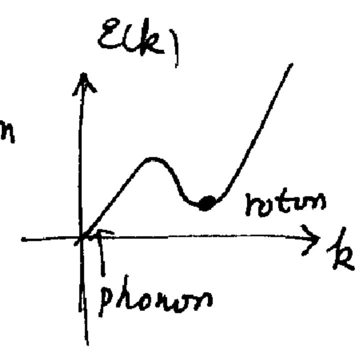
$$\boxed{\rho = \frac{\mu}{V_0} + \int\frac{d^dk}{(2\pi)^d} \sinh^2\theta_k}$$

$$\Rightarrow E_k^2 = (\varepsilon_k + \rho_0 V_k)^2 - (\rho_0 V_k)^2 \Rightarrow E_k = \sqrt{\varepsilon_k(\varepsilon_k + 2\rho_0 V_k)}$$

at small $k \Rightarrow E_k = v|k|$ with $v = \sqrt{\dfrac{\rho_0 V_0}{m}}$

The linear dispersion at $k=0$ in the above calculation is the result of $\rho_0 V(k=0) = \mu$, which is a mean-field result. The linear dispersion relation even holds at aritbary order of interaction strength.

The actural spectrum is Helium-4: phonon & Roton



§ Ground state properties of ^4He

The many-body ground state wavefunction $\varphi(R_1 \cdots R_n)$ is positive definite.

$\varphi(R_1 \cdots R_n)$ satisfies $\left[-\frac{\hbar^2}{2m}\sum_i \nabla_i^2 + \sum_{i<j} V(r_{ij})\right]\varphi = E\varphi$, and symmetry constraint.

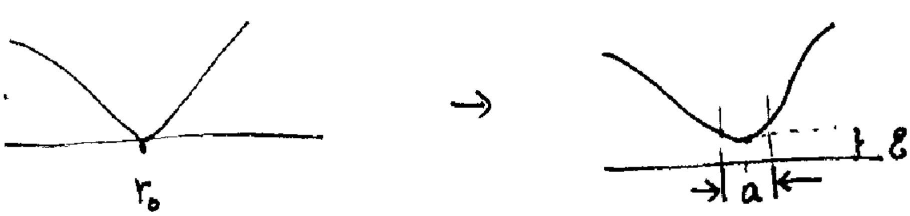
$$\langle E \rangle = \frac{\int \varphi H \varphi \, d^N R}{\int \varphi \varphi \, d^N R} = \frac{\int \frac{1}{2m}\sum_i (\nabla_i \varphi)^2 + \sum V \varphi^2}{\int \varphi^2 \, d^N R}.$$

If φ has nodes,

, we can try $|\varphi(r_1, \cdots r_N)|$,

which will give the same $|\nabla_i \varphi|^2$ and $|\varphi_i|^2$, and thus $\langle E \rangle$.

However, we can soften the kink a litte bit, $\nabla\varphi$ changes from $|\nabla\varphi|_{r_0} \to 0$



$$\rightarrow \frac{\Delta E}{Vol} = -\,|\nabla\varphi|^2_{r_0 \to 0}\,\frac{1}{2m} + \left(|\nabla\varphi|_{r_0} \cdot a\right)^2 \cdot V$$

as "a" goes small,

we can gain energy by making

$\varphi(r_1 \cdots r_n)$ positive definite.

This also showes that Ground state is no degenerate. Because any two positive-definit wavefunction cannot be orthogonal to each other.

§ Single mode approximation

Excition $|\psi_k\rangle = \rho_k |\Phi_0\rangle$, where $\rho_k = \sum_{j=1}^{N} e^{i\vec{k}\cdot\vec{r}_j}$

$$\Rightarrow E_k - E_0 = \frac{\langle \psi_k | H - E_0 | \psi_k \rangle}{\langle \psi_k | \psi_k \rangle} = \frac{\langle \Phi_0 | \rho_{-k} (H - E_0) \rho_k | \Phi_0 \rangle}{\langle \Phi_0 | \rho_{-k} \rho_k | \Phi_0 \rangle}$$

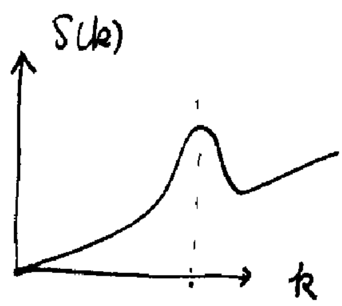
$$\langle \Phi_0 | \rho_{-k} (H - E_0) \rho_k | \Phi_0 \rangle = \langle \Phi_0 | \rho_{-k} [H - E_0, \rho_k] | \Phi_0 \rangle = \frac{1}{2} \langle \Phi_0 | [\rho_{-k}, [H - E_0, \rho_k]] | \Phi_0 \rangle$$

$$[\rho_{-k}, [H - E_0, \rho_k]] = \frac{\hbar^2 k^2}{m}$$

$$\Rightarrow E_k - E_0 = \frac{\hbar^2 k^2}{2m} \frac{1}{S(k)}, \quad \text{where } S(k) = \int dk \, e^{i\vec{k}(\vec{r}-\vec{r}')} \langle \Phi_0 | \rho(r) \rho(r') | \Phi_0 \rangle$$

$$= \langle \Phi_0 | \rho_{-k} \rho_k | \Phi_0 \rangle$$

The neutron scattering shows at $k \to 0$, $S(k)$ is linear $\sim \xi k$ with k, $\Rightarrow E_k - E_0 = \frac{\hbar^2}{2m\xi} k$,



$S(k)$ develops a hump at $k \sim 1/a$, $\to$ roton minimum.

Lecture 6 path integral to superfluid.

For a free boson system $H_0 = \sum_k (\epsilon_k - \mu) a_k^\dagger a_k \longrightarrow$

$$\int D^2[a_k(t)] \; e^{i\int dt \, \sum_k \left(i\frac{1}{2}(a_k^* \dot{a}_k - a_k \dot{a}_k^*) - (\epsilon_k-\mu) a_k^* a_k\right)}$$

in real-space, we have

$$\int D^2 a \;\; e^{i\int d^dx\, dt \left[i\frac{1}{2}\left(\varphi^*(xt)\partial_t\varphi(xt) - \varphi(xt)\partial_t\varphi^*(xt)\right) - \frac{1}{2m}\partial_x\varphi^*(xt)\partial_x\varphi(xt) - \mu\varphi^*(xt)\varphi(xt)\right]}, \quad \varphi(xt) = \sum_k e^{ikr} a(r,t)$$

we can also add term of interaction in the action $\int d^dx\, d^dy\, \frac{1}{2}\varphi^*(xt)\varphi^*(yt)V(x-y)\varphi(yt)\varphi(xt)$

By using the short-range inter-action approximation,

$$Z = \int D\varphi^*(t) D\varphi(t)\; e^{iS}, \quad \text{where}$$

$$S = \int d^dx\, dt \left[i\frac{1}{2}(\varphi^*\partial_t\varphi - \varphi\partial_t\varphi^*) - \frac{1}{2m}\partial_x\varphi^*\partial_x\varphi + \mu|\varphi|^2 - \frac{V_0}{2}|\varphi|^4\right]$$

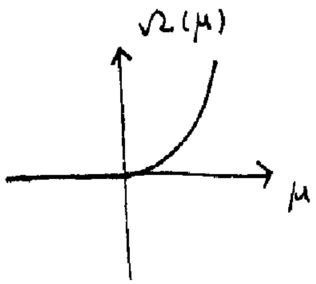
The leading order approximation of thermal potential

$$\Omega = \int d^dx \;\; \frac{1}{2m}\partial_x\varphi^*\partial_x\varphi \;\; - \mu|\varphi|^2 + \frac{V_0}{2}|\varphi|^4$$

§2 symmetry spontaneous breaking

a uniform solution $\frac{\Omega_0}{V} = -\mu|\varphi_0|^2 + \frac{V_0}{2}|\varphi_0|^4$, minimizing Ω_0

$$\Rightarrow \quad \varphi_0 = \begin{cases} \sqrt{\frac{\mu}{V_0}}\, e^{i\theta} & \text{for } \mu > 0 \\ 0 & \text{for } \mu \le 0 \end{cases} \quad \Rightarrow \Omega_0(\mu) = \begin{cases} \frac{\mu^2}{V_0} & \mu > 0 \\ 0 & \mu < 0. \end{cases}$$



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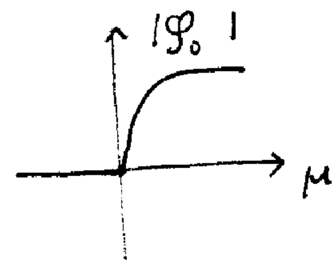
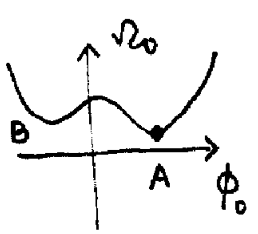


$U(1)$- symmetry : $\varphi \to e^{i\theta}\varphi$, S does not change.

However, the classical ground state has a fixed value of θ, which breaks the $U(1)$ symmetry. A phase transition is related to a singlurarity of the thermo-dynamic potential. v.s. μ.



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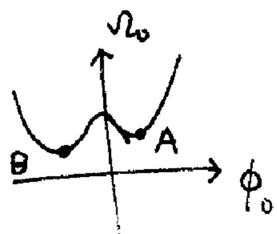
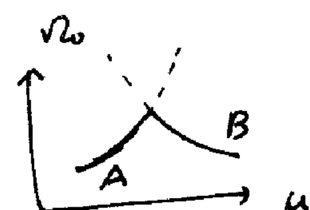
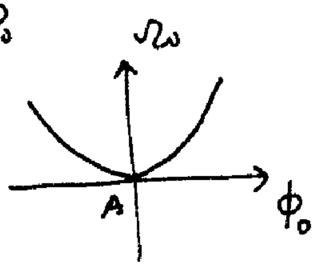




First order

$\phi_0 \to -\phi_0$
symmetry



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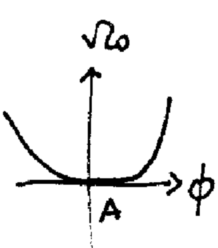

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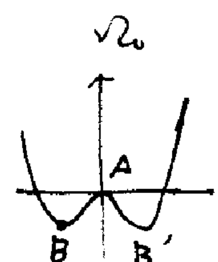


→ second order phase transition
breaking symmetry

Long-range order $\langle \varphi^\dagger(x,t)\, \varphi(0,0) \rangle \Big|_{(x,t)\to\infty} = |\varphi_0|^2 \neq 0$

The symmetry breaking phase can be either represented by a nonzero order-parameter, or a long range correlation.

§ 3. low energy effective theory

$$\varphi = \varphi_0 + \delta\varphi \qquad \text{at } \mu < 0 \Rightarrow \varphi_0$$

$$\Rightarrow \quad S = \int d^dx\, dt \; \Big[i\frac{1}{2}(\delta\varphi^* \partial_t \delta\varphi - c.c.) - \frac{1}{2m}\partial_x \delta\varphi^* \partial_x \delta\varphi + \mu|\varphi|^2 \Big]$$

$\rightarrow$ equation of motion: $\left[i\partial_t \delta\varphi - \frac{1}{2m}(-i\partial_x)^2 + \mu \right] \varphi = 0$

$$\omega = \frac{1}{2m}k^2 - \mu, \quad \text{free boson.}$$

At the symmetry-breaking ground state, we parameterize the the ground state field configuration as $\varphi = \sqrt{\rho_0 + \delta\rho}\; e^{i\theta}, \; (\rho_0 = \mu/V_0)$

$$\Rightarrow \quad S = \int d^dx\, dt\; \Big[-(\rho_0 + \delta\rho)\partial_t\theta - \frac{\rho_0(\partial_x\theta)^2}{2m} - \frac{(\partial_x\delta\rho)^2}{8m\rho_0} - \frac{V_0}{2}(\delta\rho)^2$$

(keep to quadratic order of $\delta\rho$ and θ).

θ descripes the slow, low-frequency mode, $\delta\rho$ describes the massive fluctuations if we set $\delta\rho = 0 \Rightarrow \quad S = \int d^d x\, dt \; -\rho_0 \partial_t \theta - \frac{1}{2m} \rho_0 (\partial_x \theta)^2$

which is not correct, because $\partial_t \theta$ is a total derivative and does not enter equation of motion. We need to integrate out the massive field $\delta\rho$

$$\mathcal{Z} = \int D\theta \, D\delta\rho \; e^{i\int d^d x dt \left[\frac{-V_0}{2}(\delta\rho)^2 - \frac{(\partial_x \delta\rho)^2}{8m\rho_0}\right] - \delta\rho\, \partial_t\theta + \frac{1}{2m}\delta\rho\, (\partial_x^2 \theta)}$$

integral over $\delta\rho$

$$\hookrightarrow \int D\delta\rho \; e^{i\int d^d x\, dt \; -\frac{1}{2}\left(\delta\rho + \frac{\partial_t\theta}{V_0 - \frac{\partial_x^2}{4m\rho_0}}\right)\left[V_0 - \frac{\partial_x^2}{4m\rho_0}\right]\left(\delta\rho + \frac{\partial_t\theta \cdots}{V_0 - \frac{\partial_x^2}{4m\rho_0}}\right)}$$

$$\cdot\; e^{i\int d^d x dt \; \frac{1}{2}\left(\partial_t \theta \cdot \qquad\right) \frac{1}{V_0 - \frac{1}{4m\rho_0}\partial_x^2} \left(\partial_t \theta \qquad\right)}$$

$$\approx \text{Const} \cdot e^{i\int d^d x dt \; \frac{1}{2}[\partial_t\theta] \frac{1}{V_0} (\partial_t\theta)}$$

$$\Rightarrow \boxed{\mathcal{Z} = \int D\theta \; e^{i\int d^d x\, dt \; \frac{1}{2V_0}(\partial_t\theta)^2 - \frac{\rho_0}{2m}(\partial_x\theta)^2}}$$

θ lives on a circle, θ and $\theta + 2\pi$ refer to the same point.

we can introduce $z = e^{i\theta} \Rightarrow S_{eff} = \int d^d x dt \left\{ \frac{1}{2V_0} |\partial_t z|^2 - \frac{\rho_0}{2m} |\partial_x z|^2 \right\}$

X-y model

equation of motion: $\Theta(x,t) = \bar{\theta} + \delta\theta \Rightarrow$

$$\left(-\frac{1}{2V_0}\partial_t^2 + \frac{\rho_0}{2m}\partial_x^2\right)\delta\theta = 0 \Rightarrow \omega = v|k| \text{ with } v^2 = \frac{\rho_0 V_0}{m}.$$

Density & current:

we add source term $-A_0\rho = -A_0(\rho_0 + \delta\rho)$, $\vec{A}\cdot\vec{j} = \vec{A}\cdot Re(\varphi^\dagger \frac{\partial_x}{im}\varphi)$

$$= \vec{A}\,\frac{\rho_0}{m}\nabla_x\theta$$

$$\int D\delta\rho\; e^{i\int d^dx dt\; -\frac{1}{2}\left[\delta\rho + \frac{\partial_t\theta + A_0}{V_0}\right]^2 \frac{1}{V_0}} \cdot \exp\left[i\int d^dx dt\,\frac{1}{2V_0}(\partial_t\theta + A_0)^2\right.$$

$$\rightarrow \quad \frac{1}{V_0}\partial_t\theta\cdot A_0 - A_0\rho_0 \Rightarrow \boxed{\rho = \rho_0 - \frac{\partial_t\theta}{V_0}} \qquad \vec{j} = \frac{\rho_0}{m}\nabla_x\theta$$

§ phonon mode

In the above discusse θ is considered as waves. According to the particle-wave duality, we can also consider them as bosons.

$$\mathcal{L}_{eff} = \sum_k \frac{A_k}{2}\dot{\theta}_{-k}\dot{\theta}_k - \frac{B_k}{2}\theta_{-k}\theta_k, \quad A_k = V_0^{-1}, \; B_k = \frac{\rho_0 k^2}{m}$$

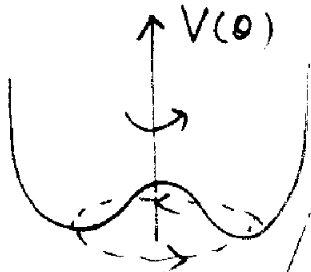
$$\pi_k = \frac{\partial\mathcal{L}_{eff}}{\partial\dot{\theta}_k} = A_k\dot{\theta}_{-k} \rightarrow H = \sum_k\left(\frac{1}{2A_k}\pi_{-k}\pi_k + \frac{B_k}{2}\theta_{-k}\theta_k\right)$$

introduce $\alpha_k = u_k\theta_k + i v_k \pi_{-k}$, $u_k = \frac{1}{\sqrt{2}}(A_k B_k)^{1/4}$, $v_k = \frac{1}{\sqrt{2}}(A_k B_k)^{-1/4}$

we get $H = \sum_k \varepsilon_k \alpha_k^\dagger \alpha_k$

§ Goldstone modes & spontaneous symmetry breaking

U(1) symmetry breaking $\varphi \to \varphi e^{i\theta}$, but the ground state does not have such a symmetry, Nambu, Goldstone



proved that there's a gapless mode, which describe that the system is trying to restore the symmetry, i.e. the fluctuations among degenerate ground states.

Different degenerate ground states are related by symmetry transformation.

One can write low energy theory in terms of Goldstone modes, which is usually called non-linear-σ model.

• Spontaneous symmetry can only happen in infinite systems. At finite systems, quantum fluctuations will restore the full symmetry. But for infinite system, it takes an infinitely long time to fluctuate from one state with one value of order parameter to another state by a different order parameter.

For example, for the superfluid boson state $\varphi_0 = \langle \Omega | a | \Omega \rangle = |\varphi_0| e^{i\theta}$.

Let us define $W = e^{i\hat{N}\theta}$, we know $[W, H] = 0$.

$$W\, a(x)\, W^{\dagger} = a(x)\, e^{i\theta}, \quad \Rightarrow \text{ For finite system, } |\Omega\rangle \text{ is } \hat{N} \text{ eigenstate}$$

$$\langle\Omega| a |\Omega\rangle = \langle\Omega| W a(x) W^{\dagger} |\Omega\rangle = e^{i\theta} \langle\Omega| a |\Omega\rangle = 0.$$

Let us look at as the Volume $\to +\infty$, how the fluctuation slow down and lead to a symmetry breaking ground state. We only consider uniform fluctuations

$$\mathcal{L} = V \cdot \left[\frac{1}{2v_0} (\partial_t \theta)^2 - \rho_0 \partial_t \theta + \mu \rho_0 - \frac{1}{2} V_0 \rho_0^2 \right],$$

which describe a particle moving on a circle with $\theta \in (0, 2\pi)$. The mass of the particle $M = V/V_0$, and $p = \partial_{\dot\theta} \mathcal{L} = M\dot\theta - \frac{\mu V}{V_0} = -N$.

$$\Rightarrow \quad H = p\dot\theta - \mathcal{L} = \frac{p^2}{2M} + \frac{\mu V}{v_0} p \,.$$

As $V \to +\infty$, the particle becomes classic, and does not move i.e. $\dot\theta = 0$.

But as V is finite, p is quantized with momentum $-n$ $\Rightarrow$

$$E_n = \frac{n^2}{2M} - \frac{\mu V n}{V_0 M} = \frac{V_0 n^2}{2V} - \mu n$$

The state with $p = -n = \frac{-\mu V}{V_0}$ has the minimum energy, which Can be written as

$$|\Phi_0\rangle = \int d\theta \; e^{-i\frac{\mu V}{V_0}\theta_0} \, |\theta_0\rangle \,.$$

$\hat{N}$ is the conjugate operator to θ, $\hat{N} = -P_\theta = +i\hbar\frac{\partial}{\partial\theta}$. Thus $\hat{N}$-eigenstate cannot have a well defined θ-value.

But as $V \to +\infty$, the low lying state with different $\hat{N}$, are nearly degenerate

$$|\theta\rangle = \sum_n e^{in\theta} \, |n\rangle,$$ the gap vanishes as $\frac{1}{V}$.

the symmetry breaking state.

Lect 7. Superfluidity at low dimensions.

— At low dimensions, strong quantum fluctuations can destroy long range superfluid order. Let us calculate the correlation function $\langle \varphi^\dagger(t\,x)\, \varphi(00)\rangle$. We assume $\varphi(x t)$ has a non-vanishing amplitude, but fluctuating phase.

$$\mathcal{L} = \frac{\chi}{2}\left((\partial_t \theta)^2 - v^2 (\partial_x \theta)^2 \right), \quad \text{where } \chi = \frac{1}{V_0}, \quad v = \sqrt{\frac{\rho_0 V_0}{m}}.$$

$$\langle \varphi^\dagger(t x)\, \varphi(00)\rangle = |\varphi_0|^2 \langle e^{-i\theta(tx)} e^{i\theta(00)} \rangle = \frac{\int D\theta\, e^{-i\theta(tx)} e^{i\theta(00)} e^{iS}}{\int D\theta\, e^{iS}},$$

Introducing source field J, and

— $$Z[J] = \int D\theta \; e^{iS + i\int dt\, d^dx\, J(x,t)\, \theta(t,x)}$$

$$\langle \varphi^\dagger(t,x)\, \varphi(00)\rangle = Z[-\delta(t-t_0, x-x_0) + \delta(t,x)] / Z[0],$$

$$\frac{Z[J(x t)]}{Z[0]} = \frac{\int D\theta\; e^{\frac{i}{2}\int d^dx\, dt\; \theta(x_1 t_1)\, G_\theta^{-1}(x_1 t_1, x_2 t_2)\, \theta(x_2 t_2) + iJ(x t)\, \theta(x, t)}}{Z(0)}$$

$$= \int D\theta\; e^{\frac{i}{2}\int d^dx_1 dt_1 \int d^dx_2 dt_2 \left\{ [\theta(x_1 t_1) + \int dx_1' dt_1'\, \underbrace{G(x_1' t_1', x_1 t_1)}_{J(x_1' t_1')}]\, G^{-1}(x_1 t_1, x_2 t_2) \cdot \{\theta(x_2 t_2) + \int dx_2' dt_2'\, G(x_2 t_2, x_2' t_2)\, J(x_2' t_2')\}\right\}}$$

$$\cdot \exp\Big[-\frac{i}{2} \int d^dx_1 dt_1\, d^dx_2 dt_2\, d^dx_1' dt_1'\, d^dx_2' dt_2'\; J(x_1' t_1')\, G(x_1' t_1', x_1 t_1)\, G^{-1}(x_1 t_1, x_2 t_2)\, G(x_2 t_2, x_2' t_2')\, J(x_2' t_2')) / Z(0)$$

—

$$= \exp\Big[-\frac{i}{2} \int d^dx_1' dt_1'\, dx_2 dt_2'\; J(x_1' t_1')\, G(x_1' t_1', x_2' t_2')\, J(x_2' t_2') \Big]$$

extra (−) comes from partial integration

where G_θ is the inverse of $-\chi(+\partial_t^2 - v^2\partial_\theta^2)$. As we have shown before

$$\langle T[\theta(t\,x).\,\theta(0,0)\rangle = \frac{\int D\theta.\ \theta(t,x)\,\theta(00)\ e^{\frac{i}{2}\int dx_1 dt_1 dx_2 dt_2\ \theta(x_1 t_1)\,G^{-1}(x_1 t_1; x_2 t_2)\,\theta(x_2 t_2)}}{\int D\theta\ \ e^{\frac{i}{2}\int dx_1 dt_1 dx_2 t_2\ \theta(x_1 t_1)\,G^{-1}(x_1 t_1; x_2 t_2)\,\theta(x_2 t_2)}}$$

$$= (-i)^{-1}G^{+1}(t,x;00) = i\,G(tx;00)$$

$$\Rightarrow \langle e^{-i\theta(t_0 x_0)}\,e^{i\theta(00)}\rangle = e^{-i\frac{1}{2}\int dt_1 dx_1 \int dt_2 x_2 \{[-\delta(t_1-t_0,\,x_1-x_0)+\delta(t_1,x_1)] \times G(t_1-t_2,\,x_1-x_2)\,(-\delta(t_2-t_0,\,x_2-x_0)+\delta(t_2\,x_2)]\}}$$

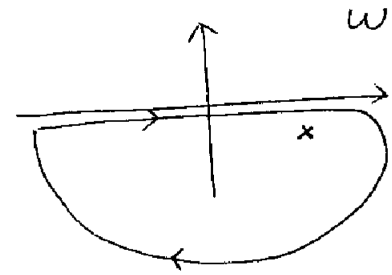
$$= e^{-\frac{i}{2}[2G(0,0) - G(t_0,x_0;00) - G(-t_0 - x_0;00)]} = e^{i[G(tx) - G(00)]}$$

$$G(\omega,k) = \frac{\chi^{-1}}{\omega^2 - v^2k^2 + i0^+} = \frac{\chi^{-1}}{2vk}\left[\frac{1}{\omega - vk + i0^+} - \frac{1}{\omega + vk - i0^+}\right]$$

* at 1D

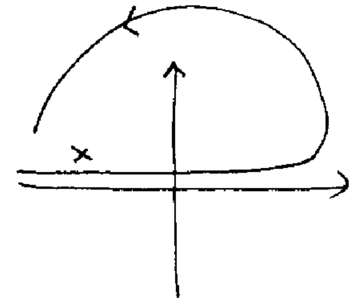
$$G(tx;00) = \int\frac{dk\,d\omega}{(2\pi)^2}\ G(\omega,k)\,e^{-i\omega t + ikx}$$

$$= \chi^{-1}\int\frac{dk\,d\omega}{(2\pi)^2}\ \frac{1}{2v|k|}\left[\frac{1}{\omega - vk + i0^+} - \frac{1}{\omega + vk - i0^+}\right] e^{-i\omega t + ikx}$$



if $t>0$

$$\chi^{-1}\int\frac{dk}{(2\pi)^2}\,\frac{-i\pi 2}{2v|k|}\ e^{-i(v|k| - i0^+)t + ikx}$$



if $t<0$

$$-\quad \chi^{-1}\int\frac{dk}{(2\pi)^2}\frac{-i\pi 2}{2v|k|}\, e^{i(v|k|t)+ikx}$$

$$\Rightarrow \boxed{G(tx;00) = \chi^{-1}\int\frac{dk}{(2\pi)^2}\frac{-i\pi}{v|k|}\, e^{i(-v|k||t|+kx)-0^+|t|}}$$

For finite system with length L, $k=\frac{2\pi}{L}\times$ integer, $\int\frac{dk}{2\pi}\to L^{-1}\sum_k$,

we get

$$G(tx;00) = \chi^{-1} L^{-1}\sum_k \frac{-i}{2v|k|}\, e^{i(-v|k||t|+kx)-0^+|t|}$$

$$= \frac{i}{4\pi v\chi}\left[\ln\left(1-e^{2\pi i\frac{-v|t|+x}{L}-0^+}\right)+\ln\left(1-e^{2\pi i\frac{-v|t|-x}{L}-0^+}\right)\right]$$

$$-\quad = \frac{i}{4\pi v\chi}\ln\left[\left(1-e^{2\pi i\frac{x-v|t|}{L}-0^+}\right)\left(1-e^{2\pi i\frac{-x-v|t|}{L}-0^+}\right)\right]$$

$$= \frac{i}{4\pi v\chi}\ln\left(-\left(\frac{2\pi i}{L}\right)^2 (x-v|t|+i0^+)(+x+v|t|-i0^+)\right)$$

$$= \frac{i}{4\pi v\chi}\ln\left(\frac{4\pi^2}{L^2}(x^2-v^2t^2+i0^+)\right)].$$

$$\Rightarrow \langle e^{-i\theta(tx)}\, e^{i\theta(00)}\rangle = e^{iG(xt)}\, e^{-iG(00)} = e^{-iG(00)}\left(\frac{L^2}{4\pi^2(x^2-v^2t^2+i0^+)}\right)^{\frac{1}{4\pi v\chi}}$$

$$-\quad = e^{-iG(00)}\, e^{-i\frac{\pi}{4\pi v\chi}\Theta(v^2t^2-x^2)}\left(\frac{L^2}{4\pi^2|x^2-v^2t^2|}\right)^{\frac{1}{4\pi v\chi}}$$

The $G(00)$ is actually a short length scale cut off "l"

$$\Rightarrow G(00) \to G_\theta(t=0, x=l) \Rightarrow \langle e^{-i\theta(t-x)} e^{i\theta(00)} \rangle = \left(\frac{l^2}{x^2 - v^2t^2 + i0^+} \right)^{\frac{1}{4\pi v\chi}}$$

point splitting

$$= e^{-i\frac{\pi}{4\pi v\chi}\Theta(v^2t^2 - x^2)} \left(\frac{l^2}{|x^2 - v^2t^2|} \right)^{\frac{1}{4\pi v\chi}} .$$

Thus we do not have true long range order, but a quasi-long range order with power law decay.

* at 2D:

We first calculate in the imaginary time representation

$$\mathcal{G}(x^i) = +T_\tau \langle \theta(x^i)\,\theta(0) \rangle , \quad i=1,2 \text{ for spatial coordinates}, \quad i=3 \text{ for imaginary time}$$

$$Z = \int D\theta \; e^{-\int d^3x \, \frac{\chi}{2}\left[v^2(\partial_x\theta)^2 + v^2(\partial_y\theta)^2 + (\partial_\tau\theta)^2\right]}$$

$$T_\tau\langle\theta(x^i)\,\theta(0)\rangle = \frac{\int D\theta \;\; \theta(x_1 x_2 \tau)\,\theta(000)\; e^{-\int d^3x_1 d^3x_2 \frac{\chi}{2}\theta(x_1^i)\,G^{-1}(x_1^i, x_2^i)\,\theta(x_2^i)}}{\int D\theta \;\; e^{-\int d^3x_1 d^3x_2 \frac{\chi}{2}\theta(x_1^i)\,G^{-1}(x_1^i, x_2^i)\,\theta(x_2^i)}}$$

$$= \qquad \chi^{-1} G(x^i; 0)$$

where $(-\partial_\tau^2 - v^2\partial_x^2)\, G(x^i; 0) = \delta^{(3)}(x)$, From electromagnetism we know

$$(-\partial_\tau^2 - v^2\partial_x^2)\left(\frac{1}{\sqrt{\tau^2 + \left(\frac{x}{v}\right)^2}} \right) = 4\pi\,\delta(\tau)\,\delta(x_{1}/v)\,\delta\!\left(\frac{x_2}{v}\right)$$

$$\Rightarrow -(\partial_\tau^2 + v^2\partial_x^2)\left(\frac{1}{4\pi}\frac{1}{v\sqrt{x^2+(v\tau)^2}}\right) = \delta(\tau)\delta(x_1)\delta(x_2)$$

$$\Rightarrow \mathcal{G}(x^i) = T_\tau\langle\theta(x^i)\theta(0)\rangle = \frac{1}{4\pi\chi v}\frac{1}{\sqrt{x^2+(v\tau)^2}}$$

$$i\,G(t\,x_1,x_2) = \mathcal{G}(x_1 x_2;\, e^{i\pi/2}t) = \frac{1}{4\pi\chi v}\frac{1}{\sqrt{x^2-(vt)^2}}\,,$$

at $|x|<|vt|$, how to decide the phase angle, we need define $\mathcal{G}(x_1 x_2;\, e^{i\theta}t)$

and let θ from $0\to\frac{\pi}{2}$, and we continuously from $\mathcal{G}\to iG$. as

Thus $$i\,G(t x_1, x_2) = \frac{1}{4\pi\chi v}\frac{1}{\sqrt{|x^2-(vt)^2|}}\,e^{-i\frac{\pi}{2}\Theta(v|t|-x)}\,.$$

$$\langle e^{-i\theta(tx)}e^{i\theta(00)}\rangle = e^{iG(tx)-iG(00)} \quad \text{as } t;x\to+\infty,$$

we have $G(t.x)\to 0$; we also set a point-splitting for $G(00)$

$$\to G(0,\ell) = \frac{-i}{4\pi\chi v\ell}$$

$$\Rightarrow \langle e^{-i\theta(tx)}e^{i\theta(00)}\rangle \to e^{-\frac{1}{4\pi v\chi\ell}} = \langle e^{-i\theta(tx)}\rangle\langle e^{i\theta(00)}\rangle$$

Thus at 2+1 dimension. Quantum fluctuation does not destroy the long range order. but reduce the value of long range order.

§ Short-distance cut-off and upper critical dimension

We showed that quantum fluctuations will suppress the amplitude of order parameter, which depends on the short range length scale cut off l.

If $l \gg (v\chi)^{-1}$, then the correction will be small. How to define "l"? It should be no smaller than the "healing length" which describes the amplitude mode fluctuations: $\frac{(\partial_x \delta\rho)^2}{8m\rho_0} + \frac{V_0}{2}(\delta\rho)^2$

$$\Rightarrow \xi^2 = (4m\rho_0 V_0)^{-1}.$$

Below this length scale, we enter a high energy region characterized by fluctuation of amplitude mode, which cannot be described by XY-model

$$\text{Seating } l = \xi \Rightarrow e^{-\frac{1}{4\pi v\chi l}} = e^{-\frac{mV_0}{\sqrt{2}\pi}} = e^{-\sqrt{2}\frac{E_{int}}{E_{qua}}}.$$

(check dimension at 2+1 d, V_0 has the unit of $[E]^{-1}[L]^{-2}$, which is correct.

$$E_{int} = \frac{1}{2} V_0 \rho_0 \qquad E_{qua} = \frac{N}{d^2 m} \sim \rho_0 m$$

the energy scale below which Boson statistic is important.

Let us do a scale transformation:

$$S = \int d^d x\, dt \left\{ \left[i\frac{1}{2}(\varphi^* \partial_t \varphi - \varphi \partial_t \varphi^*) \right] - \frac{1}{2m} \partial_x \varphi^* \partial_x \varphi - \frac{V_0}{2} (|\varphi|^2 - \rho_0)^2 \right.$$

let us rescale $t, x, \varphi, \cdot$

$$x = \xi \tilde{x}, \quad t = (V_0 \rho_0)^{-1} \tilde{t}, \quad \varphi = \sqrt{\rho_0}\, \tilde{\varphi}$$

$$\Rightarrow S = g^{-1} \int d^d\tilde{x}\, d\tilde{t}\, [i \tfrac{1}{2} (\tilde{\varphi}^* \partial_{\tilde{t}} \tilde{\varphi} - \tilde{\varphi} \partial_t \tilde{\varphi}^*] - 2 \partial_x \tilde{\varphi}^* \partial_x \tilde{\varphi} - \tfrac{1}{2}(|\tilde{\varphi}|^2 - 1)^2$$

where $g = (\rho_0 \xi^d)^{-1} = [\rho_0^{d-2} (4 m V_0)^d]^{1/2}$.

If g is small, the potential in $\int D^2\tilde{\varphi}\, e^{i \tilde{S}/g}$ is steep, and fluctuations are small. Otherwise, if g is large, fluctuations are large.

Thus as $d \geq 2$. as $\rho_0 \to 0$, we have weak coupling theory at quantum critical point.

as $d < 2$, as $\rho_0 \to 0$, we have to include strong quantum fluctuations.