A theoretical description of the new phases of liquid ^{3}He

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This paper reviews the theory of anisotropic superfluid phases and its application to the new A and B phases of liquid ^{3}He. It is tutorial in nature and advanced formal techniques are avoided; even the formalism of second quantization is not required. After an initial discussion of the Fermi-liquid theory of Landau and its application to the normal phase of liquid ^{3}He, the idea of instability against formation of Cooper pairs is introduced. The effective interaction in liquid ^{3}He is considered, with emphasis on the spin-dependent interaction arising from virtual spin polarization of the medium ("spin fluctuation exchange"). Next, a self-contained discussion of the "weak-coupling" BCS theory as applied to anisotropic superfluids is given, with special attention to the "Ginzburg–Landau" region close to the transition temperature. Formulas are derived for the specific heat, spin susceptibility, normal density tensor, and static spin-dependent correlation properties of superfluids with both singlet and triplet pairing: In the triplet case the ideas of "spin superfluid velocity" and "spin superfluid density" are also introduced. After a preliminary comparison of the weak-coupling theory with experiment, it is shown that feedback effects due to the modification, by formation of Cooper pairs, of the effective interaction connected with spin fluctuation exchange can produce results which are qualitatively different from those of the weak-coupling theory. An attempt is made to reformulate recent graph-theoretical treatments of this phenomenon in a more elementary language, and considerations based on possible invariant forms of the free energy are also introduced. The properties of the so-called Anderson–Brinkman–Morel and Balian–Werthamer states, which are commonly identified with ^{3}He–A and B, respectively, are studied in detail. Next, the effects which tend to orient the Cooper pair wave function in a given experimental situation are discussed; in this context the form of the free energy terms arising from spatial variation of the wave function is explored. A semiphenomenological theory of the nuclear magnetic resonance properties is developed and applied in particular to the case of unsaturated cw resonance; the analogy with the Josephson effect is emphasized. The question of relaxation and linewidths is also briefly discussed. A partial account is given of the theory of finite-wavelength collective oscillations, with particular reference to first, second, and fourth sound and spin waves. The splitting of the A-normal transition in a magnetic field is considered, with special attention to the possibility it offers of testing theories of the "spin fluctuation" type. Finally, a brief assessment is made of the extent to which the current experimental data support the conventional identification of ^{3}He–A and B and the spin fluctuation theory, and some outstanding problems and possibilities are outlined. Subjects not discussed include "first-principles" theories of the effective interaction in ^{3}He collective excitations in the "collisionless" regime, and the problem of ultrasonic absorption, "orbit waves," and the theory of the kinetic coefficients.

CONTENTS

I. INTRODUCTION

The discovery of the new phases of ^{3}He described in the accompanying paper by Wheatley (1975)[1] did not burst into a theoretical vacuum. Very soon after Bardeen, Cooper, and Schrieffer (1957) and Bogoliubov (1958) proposed what is now the almost universally accepted microscopic theory of superconductivity, it was realized that the phenomenon of Cooper pairing which lay at its base might not be restricted to electrons in metals but could well occur in other highly degenerate systems of fermions—in particular, in liquid ^{3}He. It was quickly appreciated that whereas the electrons in metals form pairs with relative angular momentum zero, in ^{3}He this would be prevented by the strong hard-core repulsion, and that therefore Cooper pairing, if it occurred, was likely to be in a state with finite angular momentum. Such a state would have an anisotropic pair wave function and hence presumably anisotropic properties. Moreover, since the ^{3}He atom is neutral, it could of course not show anomalous electrical conduction properties like the BCS state of electrons in a superconductor; but it would presumably show the corresponding anomalous mass flow properties, that is, superfluidity. Consequently, this proposed low-temperature phase of liquid ^{3}He, at that time completely hypothetical, became known as an "anisotropic superfluid" state.

In the years between 1960 and 1972 the theory of anisotropic superfluidity was developed in some depth, and many specific predictions were made for the behavior of liquid ^{3}He should it ever enter such a condensed phase.[2] When therefore the new phases ^{3}He–A and B were discovered in 1972, it was almost universally assumed from the very beginning that they were indeed the long-predicted anistropic superfluid phase, and almost all theoretical work in the last two years has been based on this assumption: that is, that both the new phases are essentially similar to the "BCS state" of electrons in a superconductor, but with the Cooper pairs forming in a state of nonzero relative angular momentum.

The present author must be one of a tiny minority in still harboring any doubts at all of the correctness of this identification (the reasons for such lingering doubts are touched on briefly in the Conclusion). Nevertheless, the theory based on it has in many ways proved extremely successful in accounting for the properties of ^{3}He–A and B, and no alternative theory has as yet got off the ground. This review, then, will discuss the theory of anisotropic superfluidity and its applications to the new phases.

This is very definitely a "tutorial" type of review, and in writing it I have tried to bear in mind the needs of someone (e.g., a starting theoretical or experimental graduate student) coming fresh into the topic without any great experience of theoretical work in related areas of many-body physics. I have therefore tried to use elementary methods as much as possible, and have rigorously eschewed Green's functions, diagrammatic perturbation theory, and superfluid kinetic equations; nor have I wandered away from the real axis into the complex plane. I have even set myself the task of deriving all results without explicit use of the second-quantization formalism (though I do sometimes add alternative derivations which use it, for the benefit of those readers already fluent in this language). This in itself has the consequence of restricting somewhat the areas which can be covered, and there are several important topics which I have made a conscious decision to omit entirely. In particular no discussion at all is given of first-principles calculations of the effective pairing interaction, nor of the problems concerned with ultrasonic attenuation: to do more than graze the surface of these topics without using advanced formal techniques would, I believe, be pointless. I have also omitted all discussion of "orbit waves" and related topics, not so much because they require advanced techniques but because it seems to me that there are a number of rather fundamental conceptual difficulties in this area which have not yet received any agreed resolution. Finally I have had to leave out any discussion of the kinetic coefficients, largely because of sheer shortage of time. (This, in any case, is an area in which the difficulties seem to be primarily mathematical rather than conceptual.)

The plan of the paper is as follows. In Sec. II I discuss the Landau theory of a normal Fermi liquid and its applica-

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[1] This paper is throughout referred to simply as "Wheatley."

[2] This account is deliberately ultra-brief and nonhistorical. A parallel review paper by P. W. Anderson and W. F. Brinkman (to be published) discusses the history of the subject in some detail. A partial bibliography of early work, including some less well-known references, is to be found in Galasiewicz, 1974.

tion to normal liquid ^{3}He; this is a necessary preliminary to discussion of the new phases, since many of the factors which differentiate normal ^{3}He from a weakly interacting Fermi gas also play a vital role in the new A and B phases. Section III is a brief and rather naive introduction to the "Cooper instability," that is, the possible instability of a normal Fermi system against formation of Cooper pairs; this is intended mainly to motivate the discussion, which follows in Sec. IV, of the effective pairing interaction between quasiparticles in ^{3}He.

Sections V–VII are an account of the standard "weak-coupling" theory of anisotropic superfluids, that is, roughly speaking, the generalization of the BCS theory of superconductivity to the case of pairing with nonzero angular momentum. In an effort to make the presentation self-contained, I have not assumed a previous knowledge of superconductivity theory (though it will obviously help) and have therefore started from scratch. It would, of course, have been possible to refer the reader to the pre-1972 literature, in particular to the classic papers of Anderson and Morel (1961) and Balian and Werthamer (1963) for most of the results, but I have preferred as far as possible to derive them explicitly. One reason for doing this (apart from wanting to make the paper self-contained) is that the pre-1972 papers often tend to concentrate on the behavior at zero temperature, whereas in the recent literature much more emphasis has been placed on the region near the critical temperature. In view of this I have also tended to focus primarily on this region, and have for instance included an explicit discussion of the Ginzburg–Landau theory (without gradient terms) for anisotropic superfluids and its derivation from BCS-type theory. In Sec. V the BCS theory for the case of spin singlet pairing is presented, in Sec. VI I derive expressions for the specific heat, spin susceptibility, normal density tensor, and static correlation properties for this case, and in Sec. VII I generalize the theory to the case of spin triplet pairing. Most of the results of these sections are to be found in the pre-1972 literature, although there are one or two such as the notion of, and expression for, the "spin superfluid density" which I believe had not been explicitly derived.

In Sec. VIII we pause for breath and try to see how far the theory so far developed fits the experiments on ^{3}He–A and B. It is clear that the very existence of ^{3}He-A is a major puzzle. In Sec. IX, therefore, I present the ideas of Anderson and Brinkman on "spin fluctuation feedback" which have proved so brilliantly successful in resolving this anomaly, together with more general considerations on possible invariant forms of the Ginzburg–Landau free energy. In Subsection IX.D, I discuss in detail the properties of the so-called ABM and BW states, which are now generally believed to correspond to ^{3}He–A and B, respectively (this could as well have formed a separate section).

In Sec. X I discuss the various factors which in a practical experimental situation are likely to determine the orientation of the Cooper pair wave function, introducing in the course of this discussion the gradient terms in the Ginzburg–Landau free energy. This leads on in a natural way (Sec. XI) to the nuclear magnetic resonance properties, which by probing the anisotropy and orientation of the wave function in a rather delicate way have been one of the most valuable keys to the identifications of the new phases. Section XII is devoted to the propagating collective oscillations, mainly those of "hydrodynamic" type which can be treated by relatively simple techniques: first, second and fourth sound and spin waves of various types are discussed, but not orbit waves (see above). Section XIII deals with the A transition in a magnetic field, and in particular its connection with the "spin fluctuation feedback" theory. In the Conclusion, Sec. XIV, I review the status of the "orthodox" identification of ^{3}He–A and B and of the spin fluctuation theory, and then make some general comments. An Appendix gives a formal proof of the formula for correlations in a spin triplet state established by naive arguments in Sec. VII.D, and also of the expression for the dipole energy used in Sec. X.A. The first part of this Appendix is the only part of the paper which requires some fluency in the technique of second quantization.

As far as possible I have tried to present basic physical ideas rather than mathematics (though no doubt some readers will find this hard to believe as they struggle through the several hundred equations!). In particular, I have usually tended to cut short the explicit discussion of a problem once it has been reduced to a well-defined mathematical form; e.g., I have not presented explicitly calculations of particular "textures", nor of the NMR behavior in "unusual" geometries (although I have discussed the theory of cw resonance under bulk conditions, since historically speaking this has been an important key to the identification of the new phases). I have also tried not to duplicate more than necessary the discussion given by Wheatley of the extent to which existing experimental data fits the theory, although it is clearly impossible to ignore this topic completely.

In an effort to ensure that this paper appeared in the same issue as the experimental review by Wheatley, I have had to complete it to a tight deadline and to make some compromises which, given a more leisurely timetable, I would have hoped to avoid. Apart from tidying up the notation in some places, I should have liked to rewrite Sec. III and perhaps also the end of Sec. V, and especially Subsection IX.C. This last is a source of some embarrassment to me: wishing to give some account of the "sophisticated" spin fluctuation theory of Kuroda and Brinkman, Serene, and Anderson, I was faced with the alternatives of presenting it in the diagrammatic perturbation-theoretic language of the original papers (which might have made it quite inaccessible to many readers) or of trying to reformulate it in more elementary language. Having decided to do the second, I found a method of approach which seemed to give just the original results by a much simpler method. Unfortunately, a few days before the final deadline for submission, I discovered that I had made a trivial algebraic mistake and that in fact the results of the "elementary" method as given are *not* equivalent to those of the original papers, although the difference is only an overall numerical factor. I *think* that the source of at least most of the discrepancy can be pinpointed [see the remark on Eq. (9.33)] but I have simply not had time to work this out in detail before submitting the paper. Such is the nemesis visited by field-theoretic methods on those that scorn them! Although the subsection in question may be quantitatively quite wrong, I have decided to leave it in with appropriate warning signs, as I believe it may nevertheless convey the basic physical ideas behind the graph-theoretic calculations.

One major headache for anyone writing a theoretical review paper is of course the question of notation. In this connection my first concern has been to coordinate my own notation as closely as possible to that of the companion paper by Wheatley, even at the expense of divergence from other theoretical papers. I have appended a list of symbols which are likely to cause confusion, with references to the defining equations, etc.: here I just comment on one or two of the most troublesome points. $\boldsymbol{\sigma}$ generally denotes a *spin* operator in units of $\hbar$: at well-defined points in Sec. VII and the Appendix, however, the notation σ_i and $\sigma_{\alpha\beta}{}^i$ is used for (elements of) the ith *Pauli* matrix. I have taken care always to comment explicitly when this is done and hope it will not cause confusion.[3] The quantities $\mathbf{S}$ and $\mathbf{S}(\mathbf{n})$ are the actual total spin and differential spin polarization (see Sec. VII.D), respectively, that is, they are *not* in units of $\hbar$. The quantity $\mathbf{d}(\mathbf{n})$ is the *normalized* gap parameter (or order parameter: see Section V.D). The quantity $\chi_{\mathrm{sp}}(q\omega)$ in Sec. II is the genuine dynamic *magnetic* susceptibility, but the quantity $\chi(q\omega)$ used from Sec. IV onwards is the dynamic *spin* susceptibility in units of $\hbar^2$, i.e., χ_{sp} divided by $\gamma^2\hbar^2$. The static quantities χ and χ_{ij} are everywhere the *magnetic* susceptibilities.

Finally, in the context of the spin fluctuation theory and the A transition, I have distinguished three quantities δ, δ', δ''; all of these have at one time or other been called δ in the literature. I am very conscious that the notation could usefully have been tidied up had time allowed.

The figures are meant only to give a qualitative indication of behavior and should not be taken seriously from a quantitative point of view. Where numerical estimates of quantities in ^{3}He are quoted, unless otherwise stated they are meant to be appropriate to the melting pressure: generally speaking, the corresponding numbers at lower pressures may differ by factors of order two or three.

I do not regard it as part of the function of a review of this type to assign credits or priorities, a task for which I have in any case neither the qualifications nor the inclination. The citation of a particular reference in connection with a given result is meant only to imply that further useful information or discussion may be found there, not that it was necessarily the first paper, historically speaking, in which that result was presented. Where the discussion given is self-contained I have often cited references, if at all, only in connection with whole sections or subsections; this is especially the case in the earlier chapters dealing with the weak-coupling BCS theory, the results of which are by now in some sense standard. Despite this, the choice of references has inevitably been somewhat arbitrary. If any authors feel their contributions have been slighted, I beg them to attribute it to ignorance, stupidity, laziness, or haste rather than to downright malice!

II. THE NORMAL STATE OF LIQUID ^{3}He- FERMI-LIQUID THEORY

A. Landau's theory

Between about 100 mK and the onset of the new phases below 3 mK, liquid ^{3}He behaves in many ways very like a weakly interacting degenerate Fermi gas. For instance, the specific heat is proportional to T, the spin susceptibility is temperature-independent, and the viscosity is proportional to T^{-2}. This is at first sight somewhat puzzling when one realizes that the mean interatomic spacing is quite comparable to the hard-core radius, so that one would expect collisions to be extremely important. A theory which takes this into account and nevertheless reproduces the experimental behavior was formulated by Landau—the theory of a "Fermi liquid." This theory is generally believed to give a very good account of the normal phase of ^{3}He below about 100 mK. In this section I shall briefly review the main elements of the theory, concentrating on those points which are important for an understanding of the new phases. For a more comprehensive discussion see, for example, Abrikosov and Khalatnikov, 1960; Pines and Nozierès, 1966 (Ch. i), or Baym and Pethick (1974).

In a free gas of N Fermi particles of spin $\frac{1}{2}$, the single-particle eigenstates are plane-wave states of wave vector $\mathbf{k}$ (momentum $\hbar\mathbf{k}$) and spin projection $\sigma_z = \pm\frac{1}{2}$ with energy $\epsilon_k = \hbar^2k^2/2m$. If we impose the usual periodic boundary conditions in unit volume, the ground state is the so-called "Fermi sea": all single-particle states are filled up to a limiting wave vector $k_F = (3\pi^2N)^{1/3}$. One defines the Fermi momentum, energy, and velocity by

$$p_F \equiv \hbar k_F = \hbar(3\pi^2N)^{1/3}, \qquad \epsilon_F \equiv p_F{}^2/2m,$$
$$v_F \equiv (d\epsilon/dp)_{p_F} = p_F/m. \tag{2.1}$$

Then the density of states per unit energy and unit volume (of both spins) at the Fermi surface is given by

$$\left(\frac{dn}{d\epsilon}\right) = \frac{2}{(2\pi\hbar)^3}\frac{4\pi p_F{}^2}{(d\epsilon/dp)_{p_F}} \equiv \frac{p_F{}^2}{\pi^2\hbar^3 v_F} = \frac{3N}{p_F v_F} = \frac{3Nm}{p_F{}^2}. \tag{2.2}$$

The low-temperature static properties are determined entirely by the density of states. For instance the specific heat and Pauli paramagnetic susceptibility can be written

$$C_v = (\pi^2/3)k_B{}^2(dn/d\epsilon)T, \qquad \chi = \tfrac{1}{4}\gamma^2\hbar^2(dn/d\epsilon), \tag{2.3}$$

where $\gamma \equiv 2\mu/\hbar$ is the gyromagnetic ratio of the ^{3}He nucleus (μ the nuclear magnetic moment).

For the purposes for which we shall need the Fermi-liquid theory, it is adequate to think of it in the following, somewhat nonrigorous way. We imagine that each ^{3}He atom collects around itself a "screening cloud" of other atoms, thereby becoming a "quasiparticle" with some effective mass m^*. The number of quasiparticles is equal to the number of N of ^{3}He atoms; like them, the quasiparticles occupy plane-wave states of momentum $\mathbf{p} = \hbar\mathbf{k}$ and spin projection $\pm\frac{1}{2}$, and must obey the Pauli principle. Consequently in the ground state the *quasiparticles* fill the Fermi sea up to the Fermi momentum, which is still given in terms of N by Eq. (2.1). Excited states are formed by taking a quasiparticle out of a filled state and putting it in an empty state: we can describe any state by specifying for each state $\mathbf{p}, \sigma$ the number of quasiparticles $n(\mathbf{p}\sigma)$ in that state or equivalently the deviation $\delta n(\mathbf{p}\sigma)$ of $n(\mathbf{p}\sigma)$ from its groundstate value (see [C1]).

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[3] This is done to preserve as far as possible a familiar form of Eq. (7.32) and related equations.

The energy of a single quasiparticle state is just

$$\epsilon(p) = p^2/2m^*. \tag{2.4}$$

Consequently the Fermi velocity and density of states at the Fermi surface are found from Eqs. (2.1) and (2.2) simply by replacing m by m^*:

$$v_F = p_F/m^*, \qquad dn/d\epsilon = 3N/p_F v_F = 3Nm^*/p_F^2 \equiv 3N/m^* v_F^2. \tag{2.5}$$

Were this all that there is to the Fermi-liquid theory, it would be a trivial extension of the theory of a degenerate Fermi gas, and one would be able to obtain all properties by simply replacing m by m^*, and hence the density of states (2.2) by (2.5).

However, a second very important feature of the theory is that it introduces an *effective interaction* between quasiparticles. Let us suppose for the moment that this interaction is spin-independent. Then it can depend only on the momenta of the two quasiparticles involved, $\mathbf{p}$ and $\mathbf{p}'$: we write it $f(\mathbf{p}, \mathbf{p}')$. From its definition, obviously $f(\mathbf{p}, \mathbf{p}') \equiv f(\mathbf{p}', \mathbf{p})$. Then we can write the total energy of the system in the form

$$E = E_0 + \sum_{\mathbf{p}\sigma} \epsilon(p)\delta n(\mathbf{p}\sigma) + \tfrac{1}{2} \sum_{\mathbf{p}\mathbf{p}';\sigma\sigma'} f(\mathbf{p}, \mathbf{p}')\delta n(\mathbf{p}\sigma)\delta n(\mathbf{p}'\sigma'). \tag{2.6}$$

(Slightly different formal expressions may be sometimes found in the literature: as we shall see, however, the end results are identical.) An equivalent expression for E may be obtained by defining the "true" quasiparticle energy $\tilde{\epsilon}(\mathbf{p})$, a functional of the distribution of the other quasiparticles, as the variational derivative of E with respect to $n(\mathbf{p}\sigma)$:

$$\tilde{\epsilon}(\mathbf{p}) \equiv \delta E/\delta n(\mathbf{p}\sigma) = \epsilon(p) + \sum_{\mathbf{p}'\sigma'} f(\mathbf{p}\mathbf{p}')\delta n(\mathbf{p}'\sigma'). \tag{2.7a}$$

E is then given by

$$E = E_0 + \sum_{\mathbf{p}\sigma} \int \tilde{\epsilon}(\mathbf{p})\delta n(\mathbf{p}\sigma). \tag{2.7b}$$

If the interaction is spin-dependent we have a slight complication. A quasiparticle in a plane-wave state $\mathbf{p}$ need not necessarily be in an eigenstate of σ_z with respect to the particular combination of axes we have chosen: it may be in a linear combination of the states $\sigma_z = \pm\frac{1}{2}$, i.e., it may have its spin $\boldsymbol{\sigma}$ oriented in some arbitrary direction. A formal way of dealing with this possibility is to replace $n(\mathbf{p}\sigma)$ by a 2×2 matrix $\hat{n}(\mathbf{p})$ and the sum over σ by a trace. Alternatively, what is equivalent and perhaps easier to grasp intuitively, one can completely describe the state of occupation of the plane-wave state $\mathbf{p}$ by specifying the total number of particles in it, $\tilde{n}(\mathbf{p})$, and the expectation value of (components of) spin associated with the state, $\sigma_i(\mathbf{p})$. This is equivalent to specifying the four components of the 2×2 matrix $\hat{n}(p)$. In fact we have

$$\delta\tilde{n}(\mathbf{p}) \equiv \mathrm{Tr}\,\delta\hat{n}(\mathbf{p})\Big[\equiv \sum_{\sigma} \delta n(\mathbf{p}\sigma)\Big] \quad \sigma_i(\mathbf{p}) = \tfrac{1}{2}\,\mathrm{Tr}\,\hat{\sigma}_i\delta\hat{n}(\mathbf{p}), \tag{2.8}$$

where $\hat{\sigma}_i$ is a Pauli matrix.

Now if the forces in the system are invariant under spin rotation, the only possibilities for the form of the effective interaction energy between quasiparticles in states $\mathbf{p}$ and $\mathbf{p}'$ are $\delta\tilde{n}(\mathbf{p})\delta\tilde{n}(\mathbf{p}')$ and $\boldsymbol{\sigma}(\mathbf{p}) \cdot \boldsymbol{\sigma}(\mathbf{p}')$, i.e., the most general form of the energy is, apart from the constant E_0

$$E = \sum_{\mathbf{p}} \epsilon(p)\delta\tilde{n}(\mathbf{p}) + \tfrac{1}{2}\sum_{\mathbf{p}\mathbf{p}'} \{ f(\mathbf{p}\mathbf{p}')\delta\tilde{n}(\mathbf{p})\delta\tilde{n}(\mathbf{p}') + \zeta(\mathbf{p}\mathbf{p}')\,\boldsymbol{\sigma}(\mathbf{p}) \cdot \boldsymbol{\sigma}(\mathbf{p}') \} \tag{2.9}$$

or equivalently

$$E = \sum_{\mathbf{p}} \int \tilde{\epsilon}(\mathbf{p})\delta n(\mathbf{p}) + \int \mathbf{K}(\mathbf{p}) \cdot \delta\boldsymbol{\sigma}(\mathbf{p}) \tag{2.10}$$

$$\tilde{\epsilon}(\mathbf{p}) \equiv \epsilon(\mathbf{p}) + \sum_{\mathbf{p}'} f(\mathbf{p}\mathbf{p}')\delta\tilde{n}(\mathbf{p}'),$$

$$\mathbf{K}(\mathbf{p}) \equiv \sum_{\mathbf{p}'} \zeta(\mathbf{p}\mathbf{p}')\,\boldsymbol{\sigma}(\mathbf{p}'). \tag{2.11}$$

The "true" energy of a quasiparticle of momentum $\mathbf{p}$ and spin $\boldsymbol{\sigma}$ is therefore, intuitively speaking

$$\tilde{\epsilon}(\mathbf{p}, \boldsymbol{\sigma}) = \tilde{\epsilon}(\mathbf{p}) + \boldsymbol{\sigma} \cdot \mathbf{K}(\mathbf{p}) \tag{2.12}$$

(there is no need, here, to go into the matrix notation needed to make this statement precise). The quantities f and ζ have the dimensions of energy; it is conventional to define dimensionless quantities F and Z by multiplying them by the density of states $dn/d\epsilon$. Moreover, we are generally interested in values of $\mathbf{p}$ and $\mathbf{p}'$ lying near the Fermi surface: $p \sim p' \sim p_F$. In that case the invariance of the system under spatial rotation allows F and Z to be functions only of the angle θ between $\mathbf{p}$ and $\mathbf{p}'$, so that we can expand them in Legendre polynomials:

$$(dn/d\epsilon)f(\mathbf{p}, \mathbf{p}') \equiv F(\mathbf{p}, \mathbf{p}') = \sum_l F_l P_l(\cos\theta),$$

$$(dn/d\epsilon)\zeta(\mathbf{p}, \mathbf{p}') \equiv Z(\mathbf{p}, \mathbf{p}') = \sum_l Z_l P_l(\cos\theta). \tag{2.13}$$

Thus the two-particle interaction is completely described in terms of the infinite set of "Landau parameters" F_l and Z_l. [In the literature one often finds the alternative notation $F_l \equiv F_l^s$, $Z_l \equiv 4F_l^a$; also, the Landau parameters are sometimes defined to include an extra factor $(2l + 1)^{-1}$.] Anticipating somewhat, we quote the result that the first four parameters at least (F_0, F_1, Z_0, Z_1) can be found from experiment: in general F_0 is very large and positive, F_1 large and positive, Z_0 negative and about -3, and Z_1 probably near zero. (See below for more details.)

The Landau theory is, then, a semiphenomenological description of the system which introduces one phenomenological parameter, m^*, for the single-particle energy spectrum and an infinite set for the two-particle interaction. It should be emphasized that it was invented for a specific purpose, namely to describe the *low-energy* excited states of the system and hence its low-temperature properties: we will get nonsense if we try to push it beyond this region. Moreover, although the "quasiparticle" concept is an intuitively attractive one, considerable care is needed in applying it quantitatively. For instance, it is true that the total spin and momentum of the system are given in terms of the quantities $\delta\tilde{\mathbf{n}}(\mathbf{p})$ and $\boldsymbol{\sigma}(\mathbf{p})$ by

$$\mathbf{S} = \hbar \sum_{\mathbf{p}} \boldsymbol{\sigma}(\mathbf{p}), \qquad \mathbf{p} = \sum_{\mathbf{p}} \mathbf{p}\delta\tilde{n}(\mathbf{p}) \tag{2.14}$$

just as for a gas of free particles, and one might therefore think that the total spin current (a dyadic) would be given also by the same expression as for the free gas, namely

$$\mathbf{J}_{\mathrm{spin}} = \sum_{p} \frac{\mathbf{p}}{m} \hbar \boldsymbol{\sigma}(\mathbf{p}). \tag{?}$$

This however is *not* true: there is an asymmetry between the spin and momentum, which are conserved quantities, and the spin current, which is not [for a discussion of this point, see Leggett (1970)]. The correct expression is in fact

$$\mathbf{J}_{\mathrm{spin}} = \hbar \sum_{p} \frac{\mathbf{p}}{m^*} (1 + \tfrac{1}{12}Z_1)\, \boldsymbol{\sigma}(\mathbf{p}). \tag{2.15}$$

B. Molecular fields

With a view to applications to the superfluid phases, it is helpful to interpret the "Landau" terms (the effective quasiparticle interaction) in terms of a set of *molecular fields*. Consider for example the term in Z_0. If we keep only this term in $\zeta(\mathbf{p}\cdot\mathbf{p}')$, i.e., put $Z(\mathbf{p}\cdot\mathbf{p}') = Z_0$, we find [from Eq. (2.14)] that the quantity $\mathbf{K}(\mathbf{p})$ is just $\hbar^{-1}(dn/d\epsilon)^{-1}Z_0\mathbf{S}$. Hence we can rewrite Eq. (2.12) in the form

$$\tilde{\epsilon}(\mathbf{p}, \boldsymbol{\sigma}) = \tilde{\epsilon}(\mathbf{p}) - \gamma\hbar\boldsymbol{\sigma}\cdot\mathbf{H}_{\mathrm{mol}} \tag{2.16}$$

$$\mathbf{H}_{\mathrm{mol}} = -\gamma^{-1}\hbar^{-2}(dn/d\epsilon)^{-1}Z_0\mathbf{S} \tag{2.17}$$

so that the term in Z_0 generates a "molecular field" as in the Weiss theory of ferromagnetism. (We have inserted the factor $\gamma\hbar$ only so that $\mathbf{H}_{\mathrm{mol}}$ does have the normal dimensions of a magnetic field.) A similar argument can be carried out for the term involving F_0, which generates a Hartree-type field proportional to the number of particles. Let us consider more explicitly the term involving F_1, since this will play an important role in determining some properties of the superfluid state. If we keep only this term in $f(\mathbf{p}\cdot\mathbf{p}')$, we have

$$\begin{aligned}\tilde{\epsilon}(\mathbf{p}) &= \epsilon(p) + (dn/d\epsilon)^{-1}F_1 \sum_{\mathbf{p}'} (\mathbf{p}\cdot\mathbf{p}'/p_F^2)\delta\tilde{n}(\mathbf{p}')\\ &= \epsilon(p) + (dn/d\epsilon)^{-1}(F_1/p_F^2)\,\mathbf{p}\cdot\mathbf{P}.\end{aligned} \tag{2.18}$$

Now, a charged particle in a vector potential $\mathbf{A}/e$ would have an energy shift equal to $-\mathbf{p}\cdot\mathbf{A}$. Consequently, the term in F_1 may be regarded as producing a "molecular vector potential" proportional to the total momentum $\mathbf{P}$ of the system (we ignore here the factor of e since we are dealing with neutral particles)

$$\begin{aligned}\tilde{\epsilon}(\mathbf{p}) &= \epsilon(p) - \mathbf{p}\cdot\mathbf{A}_{\mathrm{mol}},\\ \mathbf{A}_{\mathrm{mol}} &= -(dn/d\epsilon)^{-1}(F_1/p_F^2)\mathbf{P}.\end{aligned} \tag{2.19}$$

One may proceed in a similar way with the other F_l's and Z_l's.

In fact, let us sum $\delta\tilde{n}(\mathbf{p})$ and $\boldsymbol{\sigma}(\mathbf{p})$ over the *magnitudes* of $\mathbf{p}$ and then expand over the Fermi surface in spherical harmonics, defining

$$N_{lm} = \sum_{\mathbf{p}} \delta\tilde{n}(\mathbf{p})Y_{lm}(\hat{\mathbf{p}}), \qquad \mathbf{S}_{lm} \equiv \sum_{\mathbf{p}} \boldsymbol{\sigma}(\mathbf{p})Y_{lm}(\hat{\mathbf{p}}). \tag{2.20}$$

Note that N_{lm} and $\mathbf{S}_{lm}$ are *macroscopic* quantities.

Then we find using the properties of spherical harmonics,

$$\begin{aligned}E = E_0 + \sum_{\mathbf{p}} \{\textstyle\int \tilde{\epsilon}(\mathbf{p})\delta\tilde{n}(\mathbf{p}) + \mathbf{K}(\mathbf{p})\cdot\delta\boldsymbol{\sigma}(\mathbf{p})\} \quad \text{or} \quad \epsilon(\mathbf{p}\boldsymbol{\sigma})\\ = \tilde{\epsilon}(\mathbf{p}) + \boldsymbol{\sigma}\cdot\mathbf{K}(\mathbf{p}),\end{aligned} \tag{2.21}$$

where

$$\begin{aligned}\tilde{\epsilon}(\mathbf{p}) &= \epsilon(p) - \sum_{lm} Y_{lm}^*(\hat{\mathbf{p}})Q_{lm},\\ \mathbf{K}(\mathbf{p}) &= -\sum_{lm} Y_{lm}^*(\hat{\mathbf{p}})\mathbf{R}_{lm}\end{aligned} \tag{2.22}$$

$$\begin{aligned}Q_{lm} &\equiv -(dn/d\epsilon)^{-1}[F_l/(2l+1)]N_{lm},\\ \mathbf{R}_{lm} &\equiv -(dn/d\epsilon)^{-1}[Z_l/(2l+1)]\mathbf{S}_{lm}\end{aligned} \tag{2.23}$$

and hence whenever the macroscopic "polarization" $N_{lm}(\mathbf{S}_{lm})$ is produced in the system, then a corresponding "molecular field" $Q_{lm}(\mathbf{R}_{lm})$ is excited which shifts the one-particle energies. Note in particular that the "polarizations" are functions only of the *total* distortion $\sum_{|p|}\delta n(\hat{\mathbf{p}})$ of the Fermi surface at a given point $\hat{\mathbf{p}}$, and are hence quite unaffected if particles are redistributed in energy at any point on the surface. In particular they are always zero in thermal equilibrium in the absence of external fields.

To sum up: the Landau theory of a Fermi liquid is completely equivalent to the theory of a free Fermi gas of mass m^*, subject to an infinite set of (generalized) molecular fields. In practice, one does not know the parameters F_l, Z_l of the molecular fields for $l \geq 2$, and in actual calculations they are frequently put equal to zero. It is evident that the "Fermi-liquid" terms (i.e., the molecular fields) will in general affect the response to external fields, but not the thermodynamics in zero field (since then there are no macroscopic polarizations). Thus, for instance, they have no effect on the specific heat.

Let us briefly consider a specific example, the spin susceptibility. If we apply a static external magnetic field $\mathbf{H}_{\mathrm{ext}}$, then, by symmetry arguments, the only polarization that can arise is the total spin polarization $\mathbf{S}$, and hence we can neglect all Landau terms except Z_0. Then we can proceed as in the Weiss theory of ferromagnetism: if the susceptibility of the "free quasiparticle gas" (that is, neglecting all Landau terms) is χ_0, we can put the actual induced spin polarization $\mathbf{S} = \gamma^{-1}\mathbf{M}$ equal to $\gamma^{-1}\chi_0$ times the *total* field, that is, the sum of the external and molecular fields. So we have the equations

$$\begin{aligned}\mathbf{S} &= \gamma^{-1}\chi_0\mathbf{H}_{\mathrm{total}}, \qquad \mathbf{H}_{\mathrm{total}} = \mathbf{H}_{\mathrm{ext}} + \mathbf{H}_{\mathrm{mol}},\\ \mathbf{H}_{\mathrm{mol}} &= -\gamma^{-1}\hbar^{-2}(dn/d\epsilon)^{-1}Z_0\mathbf{S}.\end{aligned} \tag{2.24}$$

Solving these for $\mathbf{S}$ in terms of $\mathbf{H}_{\mathrm{ext}}$, we obtain $\mathbf{S} =$

$\gamma^{-1}\chi\mathbf{H}_{\text{ext}}$, where the true susceptibility χ is given by

$$\chi = \frac{\chi_0}{1 + [\gamma^{-2}\hbar^{-2}(dn/d\epsilon)^{-1}Z_0]\chi_0} \tag{2.25}$$

or using Eq. (2.3) with the real density of states $dn/d\epsilon$

$$\chi = \frac{\chi_0}{1+\frac{1}{4}Z_0} = \frac{\frac{1}{4}\gamma^2\hbar^2(dn/d\epsilon)}{1+\frac{1}{4}Z_0}\,. \tag{2.26}$$

C. Dynamical responses

Very often, it is necessary to deal with situations in which physical quantities are not constant over the volume of the system, as assumed above, but vary in space and possibly also in time. The Landau theory is not competent to deal with arbitrary disturbances, but provided the variation is sufficiently slow (on a spatial scale large compared with a atomic spacing, and a time scale long compared to the inverse Fermi energy), it is intuitively plausible that we can proceed exactly as above, that is, describe the system as effectively a gas of quasiparticles of mass m^* subject to a set of molecular fields, which however are now varying in space and time with the polarizations which produce them. For instance, the field generated by a spin polarization density $\mathbf{S}(\mathbf{r}t)$ is now

$$\mathbf{H}_{\text{mol}}(\mathbf{r}t) = -\gamma^{-1}\hbar^{-2}(dn/d\epsilon)^{-1}Z_0\mathbf{S}(\mathbf{r}t), \tag{2.27}$$

the Hartree-like potential generated by a density fluctuation $\delta\rho(rt)$ is

$$V_H(\mathbf{r}t) = (dn/d\epsilon)^{-1}F_0\delta\rho(\mathbf{r}t), \tag{2.28}$$

and so on.

We can use these equations (and similar ones for the other molecular fields) to generate the space- and time-dependent susceptibilities (response functions) of the Fermi liquid from those of the ideal Fermi gas. Consider for instance the density response function χ_d. This is defined as follows [see Nozières and Pines (1966) for a much fuller discussion]: we imagine that we apply a *small* space- and time-dependent potential $\delta V_{\text{ext}}(\mathbf{r}t)$ to the system. The induced change in density $\delta\rho(\mathbf{r}'t')$ $(t' \geq t)$ is then linear in δV_{ext} and we define $\chi_d(\mathbf{r}'-\mathbf{r}, t'-t)$ by the equation[4]

$$\delta\rho(\mathbf{r}'t') = -\int d\mathbf{r}\,dt\chi_d(\mathbf{r}'-\mathbf{r}, t'-t)\delta V_{\text{ext}}(\mathbf{r}t). \tag{2.29}$$

Suppose now that the response function of a free gas of quasiparticles of mass m^* is $\chi_d^{(0)}(\mathbf{r}'-\mathbf{r}, t'-t)$. (We return to the explicit form of this function in a moment.) The situation is now somewhat more complicated than in the static case, because a space- and time-dependent perturbation will in general induce currents as well as density fluctuations, and these will then produce molecular vector potentials proportional to F_1 which in turn will affect the motion. It is possible in principle to take these effects into account, but for simplicity let us specialize to the case where all F_l other than F_0 are zero. In that case we can proceed by exact analogy with the static case, writing:

$$\delta\rho(\mathbf{r}'t') = -\int d\mathbf{r}\,dt\chi_d^{(0)}(\mathbf{r}'-\mathbf{r}, t'-t)\delta V_{\text{total}}(\mathbf{r}t), \tag{2.30}$$

$$\delta V_{\text{total}}(\mathbf{r}t) = \delta V_{\text{ext}}(\mathbf{r}t) + \delta V_H(\mathbf{r}t), \tag{2.31}$$

$$\delta V_H(\mathbf{r}t) = (dn/d\epsilon)^{-1}F_0\delta\rho(\mathbf{r}t) \equiv f_0\delta\rho(\mathbf{r}t). \tag{2.32}$$

This system of equations is easily solved by Fourier transformation. Let the Fourier transform of χ_d^0 be

$$\chi_d^0(\mathbf{q}\omega) \equiv \int d\mathbf{r} \int dt \exp[+i\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r}) - i\omega(t'-t)] \times \chi_d^0(\mathbf{r}'-\mathbf{r}, t'-t) \tag{2.33}$$

with a similar definition for $\chi_d(\mathbf{q}\omega)$. Then we find from Eqs. (2.29)–(2.32)

$$\chi_d(\mathbf{q}\omega) = \chi_d^0(q\omega)/[1+f_0\chi_d^0(q\omega)]. \tag{2.34}$$

The function $\chi_d^0(q\omega)$, the density response function of a free Fermi gas of mass m^* [or density of states $(dn/d\epsilon)$] can be straightforwardly calculated by any one of a number of techniques; it is the so-called Lindhard function (Lindhard, 1954). To zeroth order in q/k_F and ω/ϵ_F it has the form

$$\chi_d^0(q\omega) = (dn/d\epsilon)f(s), \qquad s \equiv \omega/qv_F \tag{2.35}$$

$$f(s) \equiv [1 - \tfrac{1}{2}s \ln |(1+s)/(1-s)|] + \tfrac{1}{2}i\pi s\theta(1-s). \tag{2.36}$$

The true density response function (density fluctuation propagator) is therefore from Eq. (2.14)

$$\chi_d(q\omega) = (dn/d\epsilon)\{f(s)/[1+F_0f(s)]\}. \tag{2.37}$$

We can calculate the dynamic spin susceptibility $\chi_{\text{sp}}(q\omega)$ (spin response function, spin fluctuation propagator) by an exactly similar technique. For the free gas $\chi_{\text{sp}}^0(q\omega)$ differs from $\chi_d^0(q\omega)$ only by a factor $\frac{1}{4}\gamma^2\hbar^2$; proceeding exactly as above we find

$$\chi_{\text{sp}}(q\omega) = \tfrac{1}{4}\gamma^2\hbar^2(dn/d\epsilon)\{f(s)/[1+\tfrac{1}{4}Z_0f(s)]\}. \tag{2.38}$$

In liquid ^{3}He the parameter F_0 is large and positive, while $\frac{1}{4}Z_0$ is negative and not far from -1. (See below.) This means that the expressions (2.37) and (2.38) are quite different not only from (2.35) but also from each other. In fact, since F_0 is very large, Eq. (2.37) can be well approximated by the form

$$\chi_d(q\omega) = K_S[s_0^2/(s_0^2-s^2)], \quad K_S \equiv (dn/d\epsilon)(1+F_0)^{-1}, \quad s_0^2 \equiv \tfrac{1}{3}(1+F_0), \tag{2.39}$$

where K_S is the static compressibility and s_0 is the speed of ordinary hydrodynamic sound, which in the limit $F_0 \to \infty$ is identical to that of "zero sound." Thus the density fluctuation spectrum of liquid ^{3}He, especially at high pressures, is very little different from that of an ideal elastic

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[4] The minus sign is a matter of convention: it ensures that the static limit of χ_d is positive and equal to the compressibility.

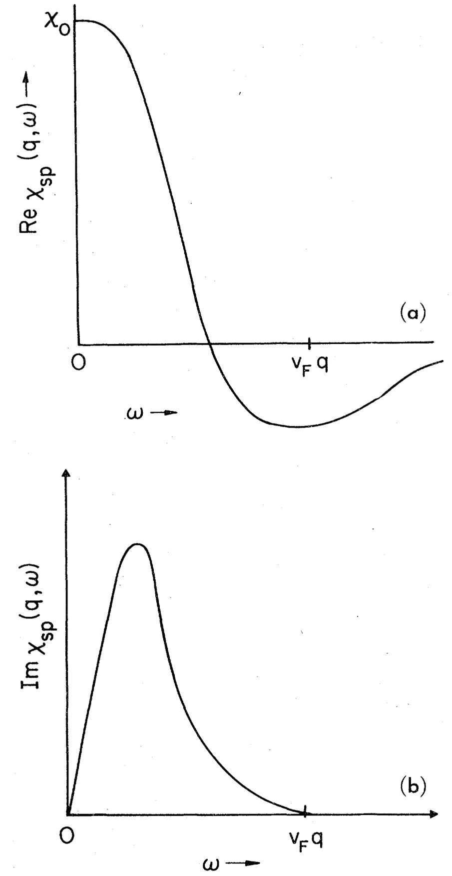

FIG. 1. (a) Real and (b) imaginary parts of the dynamic spin susceptibility $\chi_{sp}(q\omega)$ for $q \ll k_F$ and $\omega \ll \epsilon_F/\hbar$ (qualitative).

fluid. By contrast, the spin response function $\chi_{sp}(q\omega)$ has the form shown in Figs. 1(a) and (b), with an imaginary part which is strongly peaked at *low* frequencies [$\omega \sim (1+\frac{1}{4}Z_0)v_F q$]. Although this peak does not resemble a delta-function of frequency and hence does not represent a real propagating excitation, one sometimes thinks of it as representing a sort of quasielementary excitation—a so-called "paramagnon." Alternatively, since this strong peaking at low frequencies implies an anomalously long range in time for the functions $\chi_{sp}(\mathbf{r}'-\mathbf{r}, t'-t)$, one talks of "persistent spin fluctuations." Although the existence of those persistent spin fluctuations in no way invalidates the Landau Fermi-liquid description of liquid ³He in the low-temperature limit, they are believed to play an important role in the finite-temperature corrections to the Landau theory [see Pethick (1969)]. In addition, as we shall see later, they may have an important effect in the superfluid phases. For completeness we mention that if one calculates $\chi_{sp}{}^0(q\omega)$ to the next order in $(q/k_F)^2$ (but still zeroth order in ω/ϵ_F) and inserts it in (the spin analog of) Eq. (2.34), then for *small* ω/qv_F the result is

$$\chi_{sp}(q\omega) = \frac{1-\frac{1}{12}(q/k_F)^2}{1+\frac{1}{4}Z_0(1-\frac{1}{12}(q/k_F)^2)} \times \tfrac{1}{4}\gamma^2\hbar^2(dn/d\epsilon). \tag{2.40}$$

However, it is not entirely obvious that this formula should be taken too seriously, since once we go to higher order in q/k_F there are corrections to the Landau theory which may be as important as the correction to Eq. (2.38) incorporated in (2.40). At best (2.40) may give us some qualitative idea of the behavior of χ_{sp} at shorter wavelengths.

D. Comments

This concludes our brief and somewhat intuitive account of the Landau theory. What have we left out? Clearly one thing that has been left out entirely is any account of the mutual *scattering* of quasiparticles.

It is in fact possible to put these collisions in a semi-phenomenological way, and it is necessary to do so if one wishes to calculate the kinetic coefficients, for which collisions are all-important. As a result of collisions, a given quasiparticle has a finite lifetime: for a typical thermally excited quasiparticle ($\epsilon - \mu \sim k_BT$, where μ is the chemical potential or Fermi energy), the Pauli principle strongly restricts the possible collisions and, as a result of the usual phase-space considerations, the lifetime τ is proportional to T^{-2}. We may crudely estimate the constant from the transport lifetimes measured experimentally: there we find $\tau \sim 0.4 \times 10^{-12}T^{-2}$ sec (°K)². Thus the uncertainty in quasiparticle energy, $\hbar/\tau$, due to collisions is small compared to the Fermi energy (~a few °K) everywhere throughout the Fermi-liquid region, and for $T \leq 50$ mK it is even small compared to k_BT. Consequently, for almost all purposes except the calculation of the kinetic coefficients, the collisions of quasiparticles may be completely neglected. However, as we shall see in the next section, there is one special class of collision processes which plays a very special role.

Before leaving the subject of the normal phase of liquid ³He, I would like to comment briefly on the so-called "paramagnon model." This model (which has also been applied to strongly paramagnetic metals such as Pd) treats ³He as a system of particles with the bare mass m [and hence the free-gas density of states $(dn/d\epsilon)_0$ given in Eq. (2.2)] subject to a phenomenological repulsive interaction $I \equiv (dn/d\epsilon)_0{}^{-1}\bar{I}$ which operates only between antiparallel-spin particles. In the Hartree–Fock approximation the spin density fluctuation spectrum has precisely the form (2.38) but with $dn/d\epsilon$ replaced by $(dn/d\epsilon)_0$, v_F by v_{F_0} [given by Eq. (2.1)] and $Z_0/4$ by $-\bar{I}$. In particular the static spin susceptibility is

$$\chi_{sp} = \tfrac{1}{4}\gamma^2\hbar^2(dn/d\epsilon)_0(1-\bar{I})^{-1}. \tag{2.41}$$

If one wants to fit this expression to the experimentally observed susceptibility of ³He at high pressures one needs to take $(1-\bar{I})^{-1}$ as large as 20, and the peak of the spin density fluctuation spectrum is correspondingly pushed to very low frequencies compared to $v_{F_0}q$. One then considers the effects which these very low frequency paramagnons may have on the properties of the system.

Historically speaking the paramagnon model has had some importance in drawing attention to the many important effects associated with the strongly enhanced spin susceptibility of liquid ³He (and some metals). Unfortunately the impression is sometimes given in the literature that it is a *rival* description to the Fermi-liquid theory and even that the qualitative successes of the paramagnon model show that Fermi-liquid theory must be incorrect. This I do not believe to be true. One may treat the paramagnon model in either of two ways: either as an *intermediate* model, that is as a model which gives inputs to further calculations, the eventual output of which is (or may be) a Fermi-liquid description, or as a *final* model, that is as a particular choice of the parameters entering Fermi-liquid theory. In the first case there is clearly no contradiction, but it is not obvious *a priori* that there will be any particular relation between the input parameter $\bar{I}$ and (for instance) the output parameters m^* and Z_0. [There are in fact arguments to show that in the strongly paramagnetic limit the susceptibility will in fact be rigorously given by Eq. (2.41); if this is true, then a comparison with Eq. (2.26) gives the relation $(1-\bar{I}) = (m/m^*)(1+\frac{1}{4}Z_0)$.] In the second case, we simply have a particular choice of the Fermi-liquid parameters: in fact, the choice $m^* = m$, $-\frac{1}{4}Z_0 = \bar{I} \sim 0.95$. However, this choice is quite incompatible with a number of known properties of liquid ^{3}He (specific heat, compressibility, etc.) and must be rejected; at best, when interpreted in this second sense, the paramagnon model may give a crude qualitative guide to *some* of the magnetic properties. (We shall see, however, that any attempt to apply it quantitatively even to these fails disastrously as regards the superfluid phases.) Finally, one sometimes finds the quantity $\bar{I}$ used in the literature within the context of the Landau theory, simply as an alternative notation for $-Z_0/4$, and given its "Landau" value (~ 0.75). For a much more extended discussion of the relation between the paramagnon and Fermi-liquid theories, see Pethick (1969).

Let us conclude by reviewing the information we can obtain from experiment about liquid ^{3}He regarded as a normal Fermi liquid. First, there are a number of parameters which have little to do with the Fermi-liquid description [atomic ("bare") mass, gyromagnetic ratio, Fermi momentum, etc.]. Secondly, we can also obtain from experiment a number of the phenomenological parameters which enter the Landau description. The specific heat is unaffected by molecular-field effects, as argued above, and is given by Eq. (2.3): hence, a specific-heat measurement measures the quasiparticle density of states $dn/d\epsilon$, or equivalently [by Eq. (2.5)] the quasiparticle effective mass m^*. Also, several of the Landau parameters F_l, Z_l are accessible: Z_0 may be obtained (once we have $dn/d\epsilon$) from a measurement of the spin susceptibility χ, according to Eq. (2.26), and F_0 in a similar way from the compressibility [cf. Eq. (2.39)]. F_1 is related to m^* by the identity ("Landau effective-mass relation")

$$m^*/m = 1 + \tfrac{1}{3}F_1 \tag{2.42}$$

which will be proved below (Sec. VI) as a special case of the result for a superfluid system. Finally Z_1 is in principle obtainable from an experiment on spin diffusion in a high magnetic field (Leggett and Rice, 1968; Leggett, 1970; Corruccini *et al.*, 1971). Most of these quantities are tabulated in Wheatley, Table V.

III. THE COOPER INSTABILITY

As we saw at the end of the last section, the Fermi-liquid description of liquid ^{3}He does not actually exclude collisions between quasiparticles, but their only effect in the theory is to give a finite quasiparticle lifetime and finite kinetic coefficients. However, there is one special class of collisions which may have a much more profound effect, namely collisions between quasiparticles of equal and opposite momentum. As was first recognized by Cooper in the context of superconductivity, these collisions can lead to the formation of a collective bound state whose properties are profoundly different from those of the normal state. For a good discussion of the analogous points in superconductivity theory, see Rickayzen (1965), Sec. 4.2.

In this section we shall make a first, qualitative attempt to see why this particular class of collisions is especially important and to what kind of instability it gives rise, and to identify the parameters of the liquid which we need to know in order to treat the problem quantitatively. Then, having discussed the choice of these parameters in Sec. IV, we shall return to a more complete treatment of the problem in Sec. V. It should be emphasized that although the considerations of the present section do bring out most of the important physics involved, they are not to be taken seriously from a quantitative point of view.

A. Instability of a normal Fermi gas at zero temperature

Let us first consider, for orientation, the more or less trivial problem of two identical Fermi particles interacting through a potential $V(r)$. Schrödinger's equation is

$$-(\hbar^2/2m)(\nabla_1^2+\nabla_2^2)\Psi(\mathbf{r}_1,\mathbf{r}_2:\sigma_1\sigma_2) + [V(|\mathbf{r}_1-\mathbf{r}_2|)-E']\Psi(\mathbf{r}_1,\mathbf{r}_2:\sigma_1\sigma_2) = 0. \tag{3.1}$$

We can take the eigenfunctions of this equation to be a product of space and spin functions: let us, further, confine ourselves to the case of zero center-of-mass momentum, so that Ψ is not a function of the center-of-mass coordinate. Then the wave function is of the form

$$\Psi(\mathbf{r}_1\mathbf{r}_2:\sigma_1\sigma_2) = \varphi(\mathbf{r}_1-\mathbf{r}_2)\chi(\sigma_1\sigma_2). \tag{3.2}$$

Because of the antisymmetry requirement, we must either take $\varphi(\mathbf{r}_1-\mathbf{r}_2)$ to be a symmetric function of $\mathbf{r}_1$ and $\mathbf{r}_2$ (i.e., an even function of $\mathbf{r}_1-\mathbf{r}_2$) and $\chi(\sigma_1\sigma_2)$ an antisymmetric function of σ_1 and σ_2, or vice versa. The first case corresponds to even orbital angular momentum l and total spin zero (singlet spin function), the second to odd l and total spin one (triplet spin function). The energy depends on the wave function of relative motion $\varphi(\mathbf{r})$ ($\mathbf{r}\equiv\mathbf{r}_1-\mathbf{r}_2$), and in fact Eq. (3.1) becomes

$$[-(\hbar^2/2\mu)\nabla^2+V(r)]\varphi(\mathbf{r}) = E'\varphi(\mathbf{r}),$$

where $\mu\equiv m/2$ is the reduced mass. If we take Fourier transforms, putting

$$\varphi(\mathbf{r}) = \sum_k \varphi_k \exp(i\mathbf{k}\cdot\mathbf{r}),$$

$$V_{\mathbf{kk}'} \equiv \int d\mathbf{r}\exp[-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]V(r) \tag{3.3}$$

then we have

$$(2\mathcal{E}_k - E')\varphi_k = -\sum_{\mathbf{k}'} V_{\mathbf{kk}'}\varphi_{\mathbf{k}'}, \qquad \mathcal{E}_k \equiv \hbar^2k^2/2m. \tag{3.4}$$

Turning the sum over $\mathbf{k}'$ into an integral in the usual way (and remembering that we work in unit volume), we have

$$(2\mathcal{E}_k - E')\varphi_k = [-1/(2\pi)^3] \int d^3\mathbf{k}' V_{\mathbf{kk}'}\varphi_{\mathbf{k}'}. \tag{3.5}$$

The matrix element $V_{\mathbf{kk}'}$ of the potential is a function only of $|\mathbf{k} - \mathbf{k}'|^2 \equiv k^2 + k'^2 - 2kk'\cos\theta$, where θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. It can therefore be expanded in the form

$$V_{\mathbf{kk}'} = \sum_l (2l+1) V_l(k,k') P_l(\cos\theta),$$

$$V_l(k,k') \equiv \int (d\Omega/4\pi) V_{\mathbf{kk}'} P_l(\cos\theta), \tag{3.6}$$

where the $P_l(\cos\theta)$ are Legendre polynomials. We can now take the eigenfunctions of (3.5) to be functions of k times spherical harmonics corresponding to given angular momentum l and projection m

$$\varphi_{\mathbf{k}} = \psi_l(k) Y_{lm}(\hat{\mathbf{k}}) \tag{3.7}$$

(where $\hat{\mathbf{k}}$ indicates a unit vector along $\mathbf{k}$). The "radial" function $\psi_l(k)$ then satisfies the equation

$$(2\mathcal{E}_k - E')\psi_l(k) = [-1/(2\pi)^3] \int_0^\infty 4\pi k'^2 V_l(k,k')\psi_l(k')\, dk'. \tag{3.8}$$

If for given l this equation has one or more solutions for $E' < 0$, this indicates that for that l value there exists a bound state(s) in which the motion in coordinate space is bounded, i.e., such that $\varphi(\mathbf{r})$ tends to zero as r tends to infinity. This may or may not be the case, depending on the precise form of $V_l(k, k')$, or equivalently on the form of the coordinate space potential $V(r)$. If $V(r)$ is uniformly repulsive, then no bound state exists. Even if $V(r)$ is attractive, a bound state will not exist for any l if it is too weak. If bound states do exist, the most tightly bound always corresponds to $l = 0$ (see, e.g., de-Shalit and Talmi, 1963, p. 35).

This much for orientation. Now let us consider the following problem: we take as before two identical Fermi particles, but now require them to interact *in the presence of a filled Fermi sea* of $N - 2$ other particles, with some Fermi momentum $\hbar k_F$. That is, we insist that the two particles occupy only states with $k > k_F$. This is the so-called Cooper problem (Cooper, 1956). The interesting question is now not whether any states exist with energy less than zero, but whether any exist with energy less than twice the Fermi energy: or equivalently, whether we can do better than simply putting the last two particles into the lowest available pair of plane-wave states? From now on, therefore, we shall measure all energies from the Fermi energy $\epsilon_F = \hbar^2k_F^2/2m$, defining

$$\epsilon_k \equiv \mathcal{E}_k - \epsilon_F \equiv (\hbar^2/2m)(k^2 - k_F^2), \qquad E \equiv E' - 2\epsilon_F. \tag{3.9}$$

All calculations now go through exactly as above, except that we must restrict the sums over momentum to states such that $k > k_F$. Equation (3.8) is therefore replaced by

$$(2\epsilon_k - E)\psi_l(k) = -[(2\pi)^3]^{-1} \times \int_{k_F}^\infty 4\pi k'^2 V_l(k,k')\psi_l(k')\, dk' \qquad (k > k_F). \tag{3.10}$$

Whether or not Eq. (3.10) has a solution with negative E depends on the detailed behavior of $V_l(k, k')$. For the purposes of illustration let us assume the following particularly simple model potential which is a generalization of the one used by Bardeen *et al.* (1957) in the case of superconductivity

$$V_l(k,k') = V_l \quad \text{for} \quad k_F - \Delta k \le k, k' \le k_F + \Delta k$$
$$= 0 \text{ otherwise} \tag{3.11}$$

where we assume, moreover, the inequality $\Delta k \ll k_F$. Then to a good approximation we can rewrite the density of states factor as follows:

$$[(2\pi)^3]^{-1} \int 4\pi k'^2\, dk' \approx [(2\pi)^3]^{-1} 4\pi k_F^2 \int (dk'/d\epsilon_{k'})\, d\epsilon_{k'}$$
$$\approx [(2\pi)^3]^{-1} \cdot (4\pi k_F^2/\hbar v_F) \int d\epsilon_{k'} = \tfrac{1}{2}(dn/d\epsilon) \int d\epsilon_{k'}, \tag{3.12}$$

where we used Eqs. (2.1) and (2.2) [recall that $dn/d\epsilon$ is the density of states for *both* spins: one often finds in the literature the quantity $N(0)$, which is the density of states for *one* spin population and hence is equal to $\frac{1}{2}dn/d\epsilon$]. If we also write

$$\epsilon_c = \epsilon(k_F + \Delta k) \equiv (\hbar/m) k_F \Delta k \tag{3.13}$$

then Eq. (3.10) becomes

$$(2\epsilon_k - E)\psi_l(k) = -\tfrac{1}{2} V_l (dn/d\epsilon) \int_0^{\epsilon_c} \psi_l(k')\, d\epsilon_{k'}. \tag{3.14}$$

In contrast to (3.8), Eq. (3.14) always has a solution with $E < 0$ if $V_l < 0$. In fact, we see that $\psi_l(k)$ has the form

$$\psi_l(k) = A/(2\epsilon_k - E) \tag{3.15}$$

and the equation for the energy is obtained by substituting this in (3.14)

$$1 = -\tfrac{1}{2}(dn/d\epsilon) V_l \int_0^{\epsilon_c} \frac{d\epsilon_{k'}}{2\epsilon_{k'} - E}$$
$$= -\tfrac{1}{4}(dn/d\epsilon) V_l \ln\left(\frac{2\epsilon_c - E}{-E}\right). \tag{3.16}$$

In the so-called "weak coupling" limit, i.e., the limit $(dn/d\epsilon) V_l \ll 1$, the solution reduces to

$$E_{(l)} = -2\epsilon_c \exp\left(-\frac{4}{(dn/d\epsilon)\,|V_l|}\right) \quad (V_l < 0). \tag{3.17}$$

The binding energy is therefore a monotonically increasing function of $|V_l|$. Note that there is no special reason why

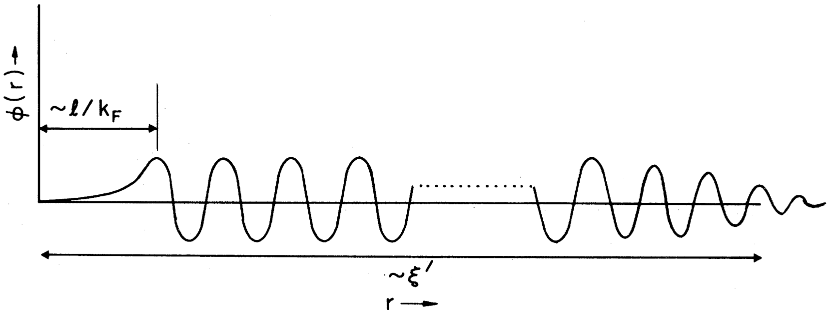

FIG. 2. Qualitative behavior of the Cooper-pair wave function for an $l\neq 0$ state. (The $1/r$ fall-off in the region $l/k_F \ll r \ll \xi_0$ is not shown).

$|V_l|$ should be largest for $l=0$, and hence the most stable solution may well correspond to finite l, in contrast to the case of Eq. (3.8).

What we have shown, then, is this: If the interaction between two particles with total momentum zero is attractive near the Fermi surface in any angular momentum state (that is, if any of the V_l's is negative), then at $T=0$ the filled Fermi sea is *unstable*; it is energetically advantageous to take a pair of particles out of the Fermi sea and let them form the bound state we have discussed. Clearly if it is energetically advantageous to do this with one pair, it is even more so to let a great many particles form bound states; however, because of the Fermi statistics they cannot do so quite independently, so that we cannot conclude very much about the actual many-body groundstate from the solution of the Cooper problem. We need in fact a completely new calculation, which will be given in Sec. V. Nevertheless, as we shall see, the pair wave function $\varphi(\mathbf{r})$ resulting from the Cooper problem is a surprisingly good guide, qualitatively speaking, to the properties of the real ground state wave function.

B. Properties of the wave function in the Cooper problem

It is interesting to investigate the nature of the relative motion in coordinate space. According to Eqs. (3.3), (3.7), and (3.15) the wave function for relative motion $\varphi(\mathbf{r})$ is given by

$$\varphi(\mathbf{r}) = \sum_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r}) Y_{lm}(\hat{\mathbf{k}})\psi_l(k). \tag{3.18}$$

Now, if we sum the quantity $\exp(i\mathbf{k}\cdot\mathbf{r})Y_{lm}(\hat{\mathbf{k}})$ over the direction of $\mathbf{k}$ for a given magnitude k of $\mathbf{k}$, we obtain just the wave function for free relative motion of two particles with total energy $\hbar^2k^2/m$, angular momentum l, and angular momentum projection m. (See, e.g., Landau and Lifshitz, 1965, p. 104.) This can be written in the form $Y_{lm}(\hat{\mathbf{r}})R_l(kr)$, where the radial wave function $R_l(kr)$ has the following properties: at short distances it is proportional to $(kr)^l$ (and hence vanishes at the origin if $l\neq 0$), then has its first maximum at $r\sim l/k$, and for $r\gg l/k$ is proportional to $[\sin(kr-\frac{1}{2}l\pi)]/kr$. Our wave function (3.18) is a linear superposition of such functions with the weighting factor $\psi_l(k)$ given by Eq. (3.15), that is,

$$\varphi(\mathbf{r}) = Y_{lm}(\hat{\mathbf{r}})\int_{k_F}^{k_F+\Delta k} \frac{A'}{2\epsilon_k+|E|} R_l(kr)\,dk \tag{3.19}$$

(where we included in the constant A' factors arising from the conversion of the sum over $\mathbf{k}'$ into an integral). The angular dependence of the wave function for relative motion is therefore exactly what we would expect for total angular momentum l and projection m. The radial dependence however is more interesting: we see that, crudely speaking, the range of k values involved is not of order Δk but of order $|E|/2\hbar v_F$. Let us define $\xi' \equiv 2\hbar v_F/|E|$: for weak coupling this length is very large compared to atomic dimensions. Then for $r\ll\xi'$ the contributions of all the components are approximately in phase, so that the radial part of $\varphi(\mathbf{r})$ can be replaced simply by $R_l(k_Fr)$, that is, it is just like the relative wave function of two free particles at the Fermi surface with angular momentum l. In particular, it is small for $r\ll l/k_F$, has its first maximum for $r\sim l/k_F$, and for $r\gg l/k_F$ (but still $r\ll\xi'$) approaches the asymptotic form $\sin(k_Fr-\frac{1}{2}l\pi)/k_Fr$. However, for $r\geq\xi'$ the different components of the radial wave function begin to interfere destructively and as a result the wave function falls off faster than r^{-1} as $r\rightarrow\infty$ (fast enough, in fact, for the state to be bound). Thus, the state we have constructed is indeed a bound state and the order of magnitude of the average separation of the particles is $\xi'=2\hbar v_F/|E|$, which tends to infinity exponentially as $V_l\rightarrow 0$. A sketch of the general behavior of the wave function for relative radial motion is shown in Fig. 2.

Although these results were obtained from the rather artificial model potential (3.11), use of a more realistic potential with a range in coordinate space of the order of atomic dimensions (rather than of order $(\Delta k)^{-1}$) does not change the conclusion qualitatively provided the attraction is weak enough; in particular the binding energy $|E|$ tends to zero exponentially with the strength of the attractive potential, while the average separation of the pair ξ' tends to infinity. (The reasons for this will become obvious in the Sec. V.) It should be particularly noted that although the attraction which is responsible for the formation of the bound state operates only when the relative separation is of the order of an atomic separation [or, in the case of the model potential (3.11), of order $(\Delta k)^{-1}$], the particles spend very little of their time within this region. In this respect the situation is very similar to that of a particle moving in one dimension and subject to a very weak attractive square-well potential; as is well known, in this case a bound state always exists, but the particle spends almost all of its time *outside* the well.

C. The transition temperature

If the normal state is unstable with respect to formation of pair bound states at $T=0$, it is presumably also unstable at low but finite temperatures. However, the instability does not persist in the high-temperature limit, since (apart from anything else) the use of Fermi statistics was essential to the proof, and at sufficiently high temperature it should be equally valid to use classical statistics. At what temperature, then, does the onset of instability occur?

The most satisfactory way to examine this question is to examine whether a small perturbation imposed on the normal-state distribution function will oscillate or will increase exponentially; if the latter, then the normal state is clearly unstable. Standard techniques for doing this have been developed within the equations-of-motion or Green's-function formalism (see, e.g., Thouless, 1960, Ambegaokar, 1969, for this technique in the theory of superconductivity; or, with special reference to ^{3}He, Ambegaokar, 1974). However, I have not found a simple way to present the essentials of the method without relying essentially on the second-quantization formalism, something which as a matter of policy is avoided in this review. Consequently, I will give here only a heuristic argument which, though it does give all the principal factors entering the criterion for the onset of instability, cannot in any way be taken seriously and in fact would (if believed) give a quite wrong numerical result. It is emphasized, therefore, that the ensuing discussion is intended *purely* to motivate the discussion of the effective interaction in the next section. A quantitatively correct derivation of the onset (critical) temperature is given in Sec. V.

What we may do is to try and generalize the "Cooper problem" of Subsection A in a natural way to finite temperature. This may be done as follows: At zero temperature the equilibrium normal state has all plane-wave states filled up to the Fermi energy, and all states above empty. At finite temperature, on the other hand, the equilibrium normal state has some plane-wave states below the Fermi surface empty and some above it full: in fact, the probability of a state with energy ϵ_k (measured, as always, with respect to the Fermi energy) being occupied is given by the usual Fermi function

$$n_k \equiv (\exp\beta\epsilon_k + 1)^{-1}, \qquad \beta \equiv 1/k_BT. \tag{3.20}$$

Now, the wave function of the bound state in the Cooper problem is a linear superposition of components in which one particle has momentum $\mathbf{k}$ and the other momentum $-\mathbf{k}$ (with appropriate spins depending on whether we look for a singlet or triplet solution) [cf. Eq. (3.3)]. If we are to form such a component, then the plane-wave states in question must be *both* empty. The probability of this is simply $(1 - n_k)^2$. We can therefore pose the analog of the Cooper problem at finite temperature in the following form: Consider two particles interacting in the presence of $N - 2$ other particles which "block" some of the available states, the probability of a given pair of states $(\mathbf{k}, -\mathbf{k})$ being unblocked being equal to $(1 - n_k)^2$. At what temperature does the energy of the resulting solution fall below the Fermi surface?

Clearly we can treat this problem exactly as in Subsection A, the only difference being that for a specific distribution of the other $N - 2$ particles the analogue of Eq. (3.4) is now

$$(2\epsilon_k - E)\varphi_{\mathbf{k}} = -\sum_{\mathbf{k}'}{}' V_{\mathbf{k}\mathbf{k}'}\varphi_{\mathbf{k}'}, \tag{3.21}$$

where the prime indicates that only those values of $\mathbf{k}'$ for which both the plane-wave states $\mathbf{k}'$ and $-\mathbf{k}'$ (with appropriate spins) are free, are included. Since the probability of both states being free is $(1 - n_{k'})^2 \equiv [1 - n(\epsilon_{k'})]^2$, when we average the solution with this factor the effect is evidently to multiply the effective density of states by a factor $[1 - n(\epsilon_{k'})]^2$. Equation (3.16) is therefore replaced by

$$1 = -\tfrac{1}{2}(dn/d\epsilon)V_l \int_{-\epsilon_c}^{\epsilon_c} \frac{[1 - n(\epsilon_{k'})]^2\, d\epsilon_{k'}}{2\epsilon_{k'} - E}, \tag{3.22}$$

where we notice that the integral now runs over energies below the Fermi surface as well as above, since there will also be some pairs of states below the Fermi surface which are unoccupied. Equation (3.22) reduces to (3.16) in the limit $T \to 0$.

We now ask for the temperature at which the energy falls below the Fermi surface (i.e., below which $E < 0$). Putting $E = 0$ in Eq. (3.22) and using the fact that $[1 - n(-\epsilon)] \equiv n(\epsilon)$, we obtain

$$\begin{aligned} 1 &= -\tfrac{1}{2}(dn/d\epsilon)V_l \int_0^{\epsilon_c} \frac{[1 - 2n(\epsilon_k)]}{2\epsilon_k}\, d\epsilon_k \\ &= -\tfrac{1}{2}(dn/d\epsilon)V_l \int_0^{\epsilon_c} \frac{\tanh\beta\epsilon_k/2}{2\epsilon_k}\, d\epsilon_k. \end{aligned}$$

While there is nothing wrong with the above calculation, there is no reason to suppose that anything particularly significant happens at the temperature defined by this equation. It is nevertheless amusing (though no doubt of no deep significance) that it does actually give the correct transition temperature, as obtained from the methods mentioned above, apart from an unwanted factor of $\frac{1}{2}$ on the right-hand side. In fact, the equation for the true transition temperature is

$$\begin{aligned} 1 &= -\tfrac{1}{2}(dn/d\epsilon)V_l \int_0^{\epsilon_c} \frac{\tanh\beta\epsilon_k/2}{\epsilon_k}\, d\epsilon_k \\ &= -\tfrac{1}{2}(dn/d\epsilon)V_k \ln(1.14\beta\epsilon_c). \end{aligned} \tag{3.23}$$

Inverting this, we find that the transition temperature T_c for pairs with relative angular momentum l is given by

$$k_BT_c = 1.14\epsilon_c \exp(-1/\lambda_l), \qquad \lambda_l \equiv \tfrac{1}{2}(dn/d\epsilon)\,|V_l|, \quad (V_l < 0). \tag{3.24}$$

If the quantity λ_l is small we see that the transition temperature is extremely sensitive to V_l and very much less sensitive to ϵ_c.

The qualitative conclusion of the considerations of this section, then, is that the parameters which are likely to play a role in determining the critical temperature of a degenerate Fermi system are (a) the density of states at the Fermi surface and (b) the pairing interaction $V(\mathbf{k}, \mathbf{k}')$, or more accurately its spherical harmonic decomposition $V_l(k, k')$ given by Eq. (3.6). To the extent that the simple model form (3.11) is valid, the latter is parametrized simply by the two quantities V_l and ϵ_c. In the next section we shall investigate what values these quantities are likely to have in the case of liquid ^{3}He.

IV. THE EFFECTIVE INTERACTION IN LIQUID ^{3}He

As we saw in Sec. II, liquid ^{3}He is a strongly interacting Fermi liquid and its low-lying states (in the normal phase) must be described in terms of Landau quasiparticles rather than real particles. Typically, we expect that the quasiparticle description should be a good one for excitation energies at least up to $\sim k_BT_L$, where T_L is the order of magnitude of the temperature below which the Landau theory seems to work well: in the case of liquid ^{3}He this is $\sim$100 mK. On the other hand, we know from experiment that the superfluid transition takes place only below 3 mK. Consequently, we would expect that it is possible to describe the superfluid (and hence, *a priori*, the instability which leads to it) in terms of (linear combinations of) Landau quasiparticle states; or, to put it intuitively, that it should be possible to visualize the Cooper pairing process as taking place between quasiparticles which can be regarded as fixed entities whose internal structure is insensitive to the superfluid transition. Moreover, the spatial extent of a quasiparticle is expected to be at most a few times an atomic dimension, whereas the smallest dimension occurring in the theory of Cooper pairing, namely $(\Delta k)^{-1}$, can be chosen to be much larger than this (this is in fact automatically assured by the condition $\Delta k \ll k_F$ which we assumed above). It should therefore be reasonable, for the purposes of considering pair formation, to assign to the quasiparticles spatial coordinates just like real particles. In other words, all the considerations of the last section (and all the theory we shall develop in the next few sections) go through unchanged provided only that particles are everywhere replaced by quasiparticles. The only difference this makes to the formulas derived is that the real mass m should be replaced everywhere by the quasiparticle effective mass m^*, or, equivalently, that the density of states $dn/d\epsilon$ should be taken to be the real value for the Fermi liquid. The molecular fields characteristic of the Fermi-liquid picture do not affect any of the conclusions of the last section, since at no stage did we consider a state involving any macroscopic polarizations. On the other hand, they play an extremely important role, as we shall see below, both in determining the effective interaction which we need as an input to the calculations of the last section and in characterizing the response of the superfluid state to an external field. [For a more extended discussion of most of the above points, see Leggett (1965)].

To apply the results of the last section, then, we need the matrix element $V(\mathbf{k}, \mathbf{k}')$ [or its angular momentum decomposition $V_l(k, k')$] for the scattering of a pair of *quasiparticles* from plane-wave states $\mathbf{k}, -\mathbf{k}$ to states $\mathbf{k}', -\mathbf{k}'$, where $\mathbf{k}$ and $\mathbf{k}'$ are near the Fermi surface. In general, $V(\mathbf{k}, \mathbf{k}')$ need not be simply a function of $|\mathbf{k} - \mathbf{k}'|$ as in the free-particle case, but since the total system is rotationally invariant it must be of the general form $V(k, k', \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')$. Also, the interaction may be spin dependent; but, again, invariance under spin rotation will constrain it to be of the form $A + B\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'$. Thus, finally, the most general form of the pairing interaction for two Landau quasiparticles is

$$V(\mathbf{k}, \mathbf{k}'\colon \boldsymbol{\sigma}, \boldsymbol{\sigma}') = V_1(k, k'\colon \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}') + V_2(k, k'\colon \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')\,\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'. \tag{4.1}$$

As we saw, when considering the Cooper instability one can consider spin singlet and spin triplet states separately, and associate them, respectively, with even and odd l; since $\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' = -\frac{3}{4}$ for a singlet state and $+\frac{1}{4}$ for a triplet, we have [from Eq. (3.6)]

$$\begin{aligned} V_l(k, k') &= \int (d\Omega/4\pi)\{V_1(k, k'\colon \cos\theta) \\ &\quad - \tfrac{3}{4}V_2(k, k'\colon \cos\theta\}P_l(\cos\theta) \qquad l = \text{even} \\ &= \int (d\Omega/4\pi)\{V_1(k, k'\colon \cos\theta) \\ &\quad + \tfrac{1}{4}V_2(k, k'\colon \cos\theta)\}P_l(\cos\theta) \qquad l = \text{odd}. \end{aligned} \tag{4.2}$$

The instability of the normal state will be determined by whichever of the V_l's is most attractive (negative) for k, k' near k_F. In particular if it turns out that the use of the model form (3.11) is justified, then the critical temperature T_c will be given by Eq. (3.26). Although the use of the model potential (3.11) at first sight looks somewhat artificial, we shall see below (end of Section V.E) that in fact in the context of Fermi-liquid theory it is quite natural.

The actual quantitative calculation of the pairing potential $V(\mathbf{k}, \mathbf{k}')$ from first principles, that is, from a knowledge of the interatomic forces, is an extremely formidable task,[5] and not surprisingly the resulting estimates of T_c have been scattered through many orders of magnitude. Rather than discuss them in detail, let us simply note one qualitative feature which is common to all. Although we are interested in pairing between quasiparticles rather than real particles, we should expect that some features of the "bare" interatomic potential might persist in $V(\mathbf{k}, \mathbf{k}')$. Now if we were to take $V(\mathbf{k}, \mathbf{k}')$ as simply the bare interatomic potential, its angular momentum decomposition $V_l(k, k')$ for k, k' near k_F would be just the effective potential felt by two free ^{3}He atoms with wave vector k_F in a state of relative angular momentum l. Since the relative wave function of such a pair is the function $R_l(k_Fr)$ discussed in the last section, it is intuitively plausible that we should have something like

$$V_l(k, k') \sim \int V(r)R_l^2(k_Fr)r^2\,dr. \tag{4.3}$$

This is actually correct: see Anderson and Morel, 1961, Eq. (3.8) (the constants arise from the normalization of the R_l's, etc.) To get an attractive value of V_l, therefore, we require that R_l should be small in the region of the repulsive hard-core potential ($r \leq 2.5$ Å) and, if possible, large in the region of the attractive van der Waals tail ($r \geq 3$ Å). Since R_l behaves as $(k_Fr)^l$ for small r and has its first maximum at $r \sim l/k_F$, it turns out that $l = 2$ and $l = 3$ are both good candidates. [A much more complete discussion of this approach, including the question of the renormalization of the potential via high-energy virtual transitions, is given by Anderson and Morel (1961).]

It is, however, possible to approach the problem of the quasiparticle pairing potential from a quite different point of view (see, e.g., Layzer and Fay 1971; Nakajima, 1973). Instead of trying to calculate it from first principles, one asks instead how much information about it can be obtained from the Fermi-liquid theory and the parameters entering it. Since we actually need the interaction $V_{\mathbf{kk}'}$ for all values

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[5] For references to some of these calculations, see, e.g., Østgaard (1969), Layzer and Fay (1974a).

of the momentum transfer $|\mathbf{k}-\mathbf{k}'|$ up to the maximum value $2k_F$, while (as we shall see) the Landau theory permits us to calculate $V_{\mathbf{kk}'}$ only for low momentum transfer, such an approach does not give a complete solution to the problem. However, it does bring out some extremely important qualitative features.[6]

What we shall do is to calculate the effective interaction $V_{\rm eff}(|\mathbf{r}-\mathbf{r}'|)$ between two quasiparticles at points $\mathbf{r}$ and $\mathbf{r}'$ and then take the pairing interaction $V_{\mathbf{kk}'}$ to be the Fourier transform of this quantity [cf. Eq. (3.3)]:

$$V_{\mathbf{kk}'} = \int d\mathbf{r}\, \exp[-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]V(\mathbf{r}). \tag{4.4}$$

In doing this, however, we should bear in mind that the "position" of a quasiparticle cannot be defined with accuracy greater than (say) an atomic spacing, and hence that the potential defined by Eq. (4.4) is meaningless for momentum transfers $|\mathbf{k}-\mathbf{k}'|$ of the order of k_F. For much smaller transfers, however, the expression (4.4) is meaningful. With this caution in mind, we note that according to Sec. II two quasiparticles of momentum $\mathbf{k}$, $\mathbf{k}'$ have a "direct" interaction via the Landau term in the Fermi-liquid Hamiltonian

$$V_{\rm direct}(r) = \{f_{\mathbf{kk}'} + \zeta_{\mathbf{kk}'}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\}\delta(\mathbf{r}) \qquad (\mathbf{k}\equiv\mathbf{p}/\hbar,\ \mathbf{k}'\equiv\mathbf{p}'/\hbar) \tag{4.5}$$

(where in evaluating $f_{\mathbf{kk}'}$, etc., we must remember that $\mathbf{k}$ and $\mathbf{k}'$ are practically parallel), so that according to Eq. (4.4)

$$V_{\mathbf{kk}',\rm direct} = \{f_{\mathbf{kk}'} + \zeta_{\mathbf{kk}'}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\} \equiv (dn/d\epsilon)^{-1}\{F_{\mathbf{kk}'} + Z_{\mathbf{kk}'}\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\}. \tag{4.6}$$

We emphasize again that this expression is to be taken seriously only when $|\mathbf{k}-\mathbf{k}'| \ll k_F$.

However, there is also an *indirect* interaction between the two particles, via the polarization of the liquid itself. To make the subsequent argument clearer, let us digress for a moment to the case of the attractive interaction between electrons in a metal due to exchange of virtual phonons. This can be viewed in the following way: Suppose we have an electron at point $\mathbf{r}$ and time t. This will attract the (positive) ion lattice and create a cloud of positive charge in the vicinity. The electron will then possibly move on, but because of the heavier ion mass the lattice will take sometime to revert to its original, unpolarized state. If now at sometime t' while the lattice is still partially polarized a second electron comes by at some point $\mathbf{r}'$, it will be attracted to the positive charge and its energy will be lowered. In this way is generated an effective attraction $V(\mathbf{r}-\mathbf{r}', t-t')$ between the two electrons at $(\mathbf{r}, t)$ and $(\mathbf{r}', t')$.

Just such a phenomenon can also take place in liquid ³He, the difference being that in this case the polarizable medium is not distinct from the atoms which are attracted. Consider for definiteness the case of spin polarization. A ³He atom at point $\mathbf{r}$ and time t will produce a molecular field which in turn produces a spin polarization of the neighboring liquid. This polarization persists for a fair time before dying out (cf. Sec. II). If now at time t' a second ³He atom comes by at point $\mathbf{r}'$, it will be either attracted or repelled (depending on its spin) by the liquid polarization. In this way a (spin-dependent) effective interaction is generated between the two ³He atoms, which is additional to the interaction (4.6).

Let us make this more quantitative. According to Eq. (2.27), the molecular field $\mathbf{H}_{\rm mol}(\mathbf{r}t)$ generated by a spin polarization $\mathbf{S}(\mathbf{r}t)$ is

$$\mathbf{H}_{\rm mol}(\mathbf{r}t) = -\gamma^{-1}\hbar^{-2}(dn/d\epsilon)^{-1}Z_0\mathbf{S}(\mathbf{r}t), \tag{4.7}$$

and hence for a single atom with $\mathbf{S}(\mathbf{r}t) = \hbar\boldsymbol{\sigma}$ at $\mathbf{r}$, t it is $-(\gamma\hbar)^{-1}(dn/d\epsilon)^{-1}Z_0\boldsymbol{\sigma} \equiv -(\gamma\hbar)^{-1}\zeta_0\boldsymbol{\sigma}$. Now by definition[7] of the dynamic spin susceptibility $\chi(\mathbf{r}'-\mathbf{r}, t'-t)$, the magnetization $\mathbf{M}(\mathbf{r}', t')$ produced at a nearby point $\mathbf{r}'$ at a subsequent time t' is

$$\begin{aligned}\mathbf{M}(\mathbf{r}', t') &= \gamma^2\hbar^2 \int \chi(\mathbf{r}'-\mathbf{r}, t'-t)H(\mathbf{r}t)\,d\mathbf{r}\,dt\\ &= -\chi(\mathbf{r}'-\mathbf{r}, t'-t)\gamma\hbar\zeta_0\boldsymbol{\sigma}.\end{aligned} \tag{4.8}$$

The molecular field at $\mathbf{r}'$, t' is $-\zeta_0/(\gamma\hbar)^2$ times this, and hence the change in energy of a second atom of spin $\boldsymbol{\sigma}'$ at $\mathbf{r}'$, t' is

$$\Delta E = -\gamma\hbar\boldsymbol{\sigma}'\cdot\mathbf{H}_{\rm mol}(\mathbf{r}'t') = (\gamma\hbar)^{-1}\cdot\zeta_0\boldsymbol{\sigma}'\cdot\mathbf{M}(\mathbf{r}'t'). \tag{4.9}$$

Combining this with Eq. (4.8), we find that the total decrease of energy when we have an atom of spin $\boldsymbol{\sigma}$ at $(\mathbf{r}, t)$ and a second of spin $\boldsymbol{\sigma}'$ at $(\mathbf{r}', t')$ $(t' > t)$ is

$$\Delta E = -\zeta_0^2\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\chi(\mathbf{r}'-\mathbf{r}, t'-t) \qquad \zeta_0 \equiv (dn/d\epsilon)^{-1}Z_0 \tag{4.10}$$

so that this represents an effective interaction between the particles. (In the "paramagnon" literature, $-4I$ corresponds to ζ_0.) For $t' < t$ the argument can obviously be reversed so that the effective interaction is just given by $\chi(\mathbf{r}-\mathbf{r}', t-t')$. So we can write in general

$$V_{\rm eff}(\mathbf{r}'-\mathbf{r}, t'-t) = -\zeta_0^2\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\cdot\tilde{\chi}(\mathbf{r}'-\mathbf{r}, t'-t), \tag{4.11}$$

where [bearing in mind that $\chi(t'-t) \equiv 0$ for $t' < t$, etc.]

$$\begin{aligned}\tilde{\chi}(\mathbf{r}'-\mathbf{r}, t'-t) \equiv{}& \tfrac{1}{2}[\chi(\mathbf{r}'-\mathbf{r}, t'-t)\\ &+ \chi(\mathbf{r}-\mathbf{r}', t-t')],\end{aligned} \tag{4.12}$$

where the factor of $\frac{1}{2}$ is added to avoid double-counting.[8] Because the coordinate-space potential (4.11) is time-dependent, the resulting pairing interaction $V_{\mathbf{kk}'}$ is a function of the energy transfer $\epsilon_{k'}-\epsilon_k \equiv \hbar\omega$ as well as of the

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[6] In the literature the ensuing considerations have very often been formulated within the language of the "paramagnon" theory (see Sec. II.D). However, I believe that to the extent that they are valid they can (and should) be reformulated in the language of the Fermi-liquid theory (cf. Nakajima, 1973).

[7] From now on we shall always include a factor of $(\gamma\hbar)^{-2}$ in the definition of χ (so that it is genuinely the spin rather than magnetization response function). We will also drop the subscript "sp" from now on.

[8] This is a slightly delicate point. To see it, note that $V_{\rm eff}$ is a variational derivative (cf. Sec. IX.C), i.e.,

$$\Delta E = \hbar^{-2}\int d\mathbf{r}\,d\mathbf{r}'\,dt\,dt'\, V_{\rm eff}(\mathbf{r}'-\mathbf{r}, t'-t)\int \delta S(\mathbf{r}t)\int \delta S(\mathbf{r}'t').$$

momentum transfer $|\mathbf{k}' - \mathbf{k}| \equiv \mathbf{q}$. In fact we have

$$\begin{aligned} V_{\mathbf{kk}',\text{indirect}} &= -\zeta_0^2 \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}' \iint \exp[-i(\mathbf{k}-\mathbf{k}')\\ &\quad \cdot(\mathbf{r}'-\mathbf{r})] \exp[-i\omega(t'-t)]\tilde{\chi}(\mathbf{r}'-\mathbf{r}, t'-t)\, d\mathbf{r}\, dt,\\ &\equiv -\zeta_0^2 \hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{\sigma}}' \tfrac{1}{2}[\chi(\mathbf{q}\omega) + \chi(-\mathbf{q}, -\omega)],\\ &= -\zeta_0^2 \hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{\sigma}}' \operatorname{Re}\chi(q\omega). \end{aligned} \tag{4.13}$$

Thus, apart from a factor, the indirect effective pairing interaction for the scattering $\mathbf{k} \to \mathbf{k}'$ is just given by the real part of the spin density fluctuation propagator (dynamic susceptibility) $\chi(\mathbf{q}\omega)$, where $\mathbf{q} \equiv \mathbf{k}' - \mathbf{k}$ and $\hbar\omega \equiv \epsilon_{k'} - \epsilon_k$. This quantity was discussed in Sec. II.

Several comments are worth making about the result (4.13). First we note that whatever the sign of the "coupling constant" ζ_0, the mechanism considered always produces an attraction between parallel-spin particles and a repulsion between antiparallel-spin ones. Thus it intrinsically favors spin triplet pairing and tends to suppress singlet pairing (Berk and Schrieffer, 1966; Layzer and Fay, 1971). Secondly, let us write out Eq. (4.13) explicitly using the definition $\zeta_0 = (dn/d\epsilon)^{-1} Z_0$ and Eq. (2.38). In this way we find

$$\begin{aligned} V_{kk',\text{indirect}} &= -\tfrac{1}{4} \hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{\sigma}}' Z_0^2 (dn/d\epsilon)^{-1}\\ &\quad \times \operatorname{Re}\{f(s)/[1+\tfrac{1}{4}Z_0 f(s)]\} \qquad (s \equiv \omega/v_F q), \end{aligned} \tag{4.14}$$

where $f(s)$ is the function defined in (2.36). In the limit $\omega/q \to 0$ the function $f(s)$ reduces to 1 and, adding Eq. (4.14) to the part of the direct interaction (4.5) associated with the molecular field parameter Z_0, we find for the total interaction associated with Z_0 (and hence with spin polarization)

$$V_{\mathbf{kk}'}{}^{\text{spin}} = (dn/d\epsilon)^{-1}[Z_0/(1+\tfrac{1}{4}Z_0)]\hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{\sigma}}'. \tag{4.15}$$

Since Z_0 is about -3 for liquid ³He, this expression is attractive and quite large. On the other hand, in the limit $q/\omega \to 0$ the function $f(s)$ tends to zero and we are left with the "direct" interaction $(dn/d\epsilon)^{-1} Z_0 \hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{\sigma}}'$ which, although still attractive, is considerably smaller.

Thirdly, we want to emphasize an important difference between the mechanism considered here and the mechanism of attraction between electrons in metals due to lattice polarization. In the latter case, the lattice constitutes essentially an independent system and its motion is very little affected by what is going on in the electron gas, provided only that the screening properties of the latter are not much changed. Consequently, when for instance the electron gas becomes superconducting the response of the lattice is practically unaffected. In the case of the spin polarization mechanism in liquid ³He, on the other hand, the medium which is polarized is identical to the atoms undergoing the indirect attraction: therefore, if as a result of the attraction the behavior of the atoms is changed—if for instance they form Cooper pairs—the response of the polarizable medium is automatically affected. This feature is of crucial importance to the Anderson–Brinkman (1973) theory of the stability of ³He–A, which we shall discuss below (Sec. IX).

Finally, just as one often thinks of the lattice polarization mechanism in metals as "exchange of virtual phonons," one may think of the process considered above as equivalent to "exchange of virtual paramagnons" or "spin fluctuation exchange"—a name it is frequently given in the literature.

Evidently, the spin fluctuation exchange mechanism in liquid ³He is only one of a whole class of similar processes: in fact, there will be one process for each different molecular field parameter. For instance, consider the contribution of the "molecular field" (Hartree field) associated with density fluctuations ("exchange of virtual zero sound quanta"). A density change $\delta\rho(\mathbf{r}t)$ gives rise, according to the arguments of Sec. II, to a Hartree-type potential $f_0\delta\rho(\mathbf{r}t)$ $[f_0 \equiv (dn/d\epsilon)^{-1}F_0]$, and by following through an argument exactly analogous to the one given above for the spin fluctuation case we conclude that this leads to a spin-independent indirect interaction of the form

$$V_{\mathbf{kk}'} = -f_0^2 \operatorname{Re}\chi_d(\mathbf{q}\omega), \tag{4.16}$$

where $\chi_d(\mathbf{q}\omega)$ is the density fluctuation propagator. As we saw in Sec. II, in the limit of strong repulsive interactions $(F_0 \to \infty)$ (which is well attained in liquid ³He, particularly at high pressures) this quantity may be well approximated by the form [cf. Eq. (2.39)]

$$\begin{aligned} \operatorname{Re}\chi_d(q\omega) &= (dn/d\epsilon)[1/(1+F_0)][c^2q^2/(c^2q^2-\omega^2)]\\ c^2 &= \tfrac{1}{3}v_F^2(1+F_0). \end{aligned} \tag{4.17}$$

Hence the total interaction associated with density fluctuations (adding the "direct" term f_0) is

$$\begin{aligned} V_{\mathbf{kk}'}{}^{\text{density}} &= (dn/d\epsilon)^{-1}\{F_0 - [F_0^2/(1+F_0)]\\ &\quad \times [c^2q^2/(c^2q^2-\omega^2)]\}. \end{aligned} \tag{4.18}$$

Although this interaction is repulsive in both the limits $\omega/q \to 0$ and $q/\omega \to 0$, it is strongly attractive over a wide range of intermediate frequencies. The same goes for other types of indirect attraction such as the one mediated by exchange of transverse current fluctuations. Consequently, if one is interested in calculating the actual value of the effective pairing interaction quantitively, it is by no means obvious that it is a good approximation to limit oneself to the exchange of spin fluctuations only. On the other hand, such a goal is probably none too hopeful anyway (see below). The special importance of the spin fluctuation exchange mechanism as distinct from other types of indirect interaction lies not so much in its absolute value as in the fact that it is rather specially sensitive to the superfluid transition when it occurs.

In principle, considerations of the above type drawn from the Fermi-liquid theory enable us to calculate the effective pairing interaction $V_{\mathbf{kk}'}$ for *small* momentum transfer $\mathbf{q} = \mathbf{k}' - \mathbf{k}$. Beyond this, however, they cannot help us, since as we have seen it makes no sense to try to use the Landau theory for large momentum transfers (at least in its simple form). On the other hand, the parameters V_l (or λ_l) which according to Eq. (3.26) determine the critical temperature involve integrals over the whole of the Fermi surface, that is, they involve momentum transfers up to $2k_F$. Consequently, if we wish to estimate the V_l quantitatively we must either make some plausible extrapolation of the small-q interaction to higher $\mathbf{q}$ [for instance by using the expression (4.14) but replacing $f(s)$ by the full "free-gas" susceptibility for $q \sim k_F$], or recalculate $V_{\mathbf{kk}'}$ from scratch. It is difficult to be very sure about the validity of the approximations involved in either technique, and there may therefore be something to be said for taking T_c (and hence, indirectly, V_l) as a phenomenological parameter to be taken from experiment and using all the above arguments merely to draw qualitative conclusions.

What, then, are the qualitative conclusions we can draw from the considerations advanced in this section? First, whatever its form and strength in detail, it is obvious that the spin-fluctuation exchange mechanism will always tend to favor spin triplet pairing and suppress spin singlet pairing, while for instance exchange of density fluctuations is independent of spin. Hence we should expect that $l = 1$ or $l = 3$ pairing is now favored, with $l = 2$ pairs suppressed. Secondly, if spin fluctuations in particular play a dominant role, this would to some extent account for the rather surprising pressure dependence of T_c observed experimentally. The point here is that according to Eq. (3.36) the transition temperature is exponentially sensitive to the parameter $\lambda_l \equiv -\frac{1}{2}(dn/d\epsilon)V_l$. Now we know that (from specific-heat measurements) $dn/d\epsilon$ changes by a factor of more than 2 between $p = 0$ and the melting curve; on the other hand, the fact that the observed T_c changes only by a factor of 3 over this pressure interval (Ahonen *et al.*, 1974), implies (unless we choose the cutoff energy ϵ_c to have a quite implausibly small value) that λ_l changes very little. Thus, the change of V_l with pressure must be just such as to cancel (very nearly) the change of the density of states $dn/d\epsilon$. This would be somewhat mysterious if $V_{\mathbf{kk}'}$ were some kind of "bare" interaction; however, we see from Eq. (4.15) that the static limit ($\omega/q \to 0$) of $V_{\mathbf{kk}'}{}^{\mathrm{spin}}$ at least is just such as to cancel the $dn/d\epsilon$, leaving only the much weaker pressure dependence of Z_0 (this quantity changes only by about 10% over the whole pressure range). Also we see from Eq. (4.18) that in the limit of large F_0 the static $V_{kk'}{}^{\mathrm{density}}$ is exactly $(dn/d\epsilon)^{-1}$, so that the resulting contribution to λ_l is exactly $-\frac{1}{2}$ and completely independent of pressure.

The third conclusion is one which has not been much emphasized in the literature (cf., however, Layzer and Fay, 1971; Nakajima, 1973): To the extent that we take spin-fluctuation exchange seriously as a major contribution to the effective interaction in ^{3}He, and are prepared to draw at least qualitative conclusions from the behavior of this mechanism in the low-q region, to that same extent we must be prepared for the interaction parameters λ_l for the different (odd) l to be close in value. The point is that if we assume that the interaction $V_{\mathbf{kk}'}$ is strongly peaked in the region of small momentum transfers, then since with our normalization the Legendre polynomials $P_l(\cos\theta)$ all tend to 1 for $\theta \to 0$, it follows from Eq. (3.6) that all the V_l (hence λ_l) for low l are close in value. This may have some fairly significant implications for quantitative calculations of the properties of the superfluid phases, although like most authors we shall ignore it in what follows (cf. Sec. V.C).

Finally, a remark on the energy dependence of $V_{\mathbf{kk}'}$. As we have seen for given momentum transfer q any indirect interaction coming from virtual polarization effects will be strongly energy-dependent; the scale of the energy variation will be $\sim\hbar cq$, where c is the typical velocity of the excitation which is exchanged. Thus, for density polarization (exchange of zero sound quanta) c is the speed of sound, while for spin polarization (exchange of paramagnons) c is the "paramagnon velocity," i.e., of order $v_F(1 + \frac{1}{4}Z_0)$. In any case c is of the order of, or larger than, the Fermi velocity v_F. Let us suppose now that the largest momentum transfer for which application of the Landau Fermi-liquid theory is justified is q_c; then when we integrate $V_{\mathbf{kk}'}$ over the Fermi surface to get $V_l(k, k')$, the typical scale of energy variation of $V_l(k, k')$ will be at least $\hbar v_F q_c$, which we may estimate as say $0.1\epsilon_F$. The point now is that for liquid ^{3}He the quantity $k_B T_c$ is less than $10^{-3}\epsilon_F$, so that the energy variation of the pairing potential is over a scale very large compared to $k_B T_c$. The same remark applies, *a fortiori*, to any contribution *not* associated with virtual polarization. Consequently, for $\epsilon_k, \epsilon_{k'} \leq k_B T_c$ we may approximate $V_l(k, k')$ by a constant [obtained by taking $\omega = 0$ everywhere in the expressions (4,14), (4,17), etc]. As we shall see below, this is in fact sufficient justification for replacing $V_l(k, k')$ by the model form (3.11).

V. THE WAVE FUNCTION OF THE SUPERFLUID STATE

In Sec. III we saw that if the interaction between quasiparticles near the Fermi surface is attractive in any angular momentum state, then below some critical temperature T_c the normal state is unstable with respect to formation of a bound pair with center of mass momentum zero. However, the trial wave function we constructed there was obviously inadequate in two respects: (1) It was not properly antisymmetrized—in fact we treated the last two particles on a quite different footing from the $N - 2$ filling the Fermi sea. (2) The binding energy we obtained was only of order 1 not of order N, since we correlated only two particles. We must therefore look for a trial wave function which (a) enables all N particles to enjoy the attractive potential and (b) is properly antisymmetrized.

A. Nature of the wave function

Let us therefore do the following (in the present subsection we stick to the case $T = 0$ and treat a nearly free gas rather than a Fermi liquid). Considering a given pair of particles 1, 2, we make up for them a wave function $\Psi(\mathbf{r}_1 - \mathbf{r}_2 : \sigma_1\sigma_2)$ which is correctly antisymmetrized with respect to the interchange $1 \rightleftarrows 2$ but is otherwise arbitrary. Then we construct the *same* wave function for the pair 3, 4, the pair 5, 6 etc., so that the total wave function of the system is

$$\Psi = \varphi(\mathbf{r}_1 - \mathbf{r}_2 : \sigma_1\sigma_2)\varphi(\mathbf{r}_3 - \mathbf{r}_4 : \sigma_3\sigma_4)\varphi(\mathbf{r}_5 - \mathbf{r}_6 : \sigma_5\sigma_6)\cdots. \tag{5.1}$$

Finally we antisymmetrize the wave function with respect to interchange of particles between pairs, and normalize it: that is,

$$\begin{aligned}\Psi = \mathcal{N}\{&\varphi(\mathbf{r}_1 - \mathbf{r}_2 : \sigma_1\sigma_2)\varphi(\mathbf{r}_3 - \mathbf{r}_4 : \sigma_3\sigma_4)\varphi\cdots\\ &-\varphi(\mathbf{r}_1 - \mathbf{r}_3 : \sigma_1\sigma_3)\varphi(\mathbf{r}_2 - \mathbf{r}_4 : \sigma_2\sigma_4)\cdots\}\\ \mathcal{N} = {}&\text{normalization factor.}\end{aligned} \tag{5.2}$$

In this way we produce a trial function for the system which has all the correct symmetry properties built in. Moreover, it is evident that if we can arrange the wave function of relative motion $\varphi(\mathbf{r}_1-\mathbf{r}_2:\sigma_1\sigma_2)$ so that particles 1 and 2 enjoy one another's attraction, then the same automatically follows for particles 3 and 4, etc.

We may expand φ in Fourier components:

$$\varphi(\mathbf{r}_1-\mathbf{r}_2:\sigma_1\sigma_2)=\sum_{\mathbf{k}}\exp(i\mathbf{k}\cdot\mathbf{r}_1-\mathbf{r}_2)\chi(\mathbf{k}:\sigma_1\sigma_2). \qquad (5.3)$$

Here, in contradistinction to the case of the single-pair wave function calculated in Sec. III, $\mathbf{k}$ can take values both below and above the Fermi surface. To satisfy the "internal" symmetry requirement, χ must be an even function of $\mathbf{k}$ if the spin function is a singlet, and an odd function if it is a triplet.

It seems intuitively plausible that for states *above* the Fermi surface the behavior of the χ which actually gives the ground state will be something like that calculated for the Cooper problem [i.e., the $\varphi_{\mathbf{k}}$ of Sec. III], and in particular that if the attraction is strongest in a state of angular momentum l, then the angular dependence of χ will correspond to this.

A trial wave function of the form (5.2) is indeed usually assumed to be the true ground state wave function of an anisotropic superfluid (and of a superconductor, which may be regarded as a special case of this). In a sense it seems to represent a Bose condensate of diatomic molecules of total momentum zero with a relative wave function $\varphi(\mathbf{r}_1-\mathbf{r}_2:\sigma_1\sigma_2)$. However, one should be extremely cautious in drawing quantitative conclusions from this picture. For instance, suppose that the orbital part of $\chi(\mathbf{k})$ has the angular dependence $\exp i\phi$ ($\phi=$ polar angle of $\mathbf{k}$). It is tempting to infer that each "diatomic molecule" has angular momentum $\hbar$, and therefore that the whole system has angular momentum $N\hbar/2$. But this is certainly incorrect, if for no other reason than that the *normal* ground state can actually be represented as a special case of Eq. (5.2), with $\chi(\mathbf{k})\sim e^{i\phi}$ or indeed any angular dependence we please (see below).

The wave function (5.2) is actually not the most general possible wave function corresponding to pair formation, since we have taken the center-of-mass momentum of the pairs equal to zero. More generally, we can write a system wave function of the type (5.2), but with $\varphi(\mathbf{r}_1-\mathbf{r}_2:\sigma_1\sigma_2)$ replaced by a more general function $\varphi(\mathbf{r}_1\mathbf{r}_2:\sigma_1\sigma_2)\equiv\varphi[\mathbf{r}_1-\mathbf{r}_2,\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2),\sigma_1\sigma_2]\equiv\varphi(\mathbf{r},\mathbf{R}:\sigma_1\sigma_2)[\mathbf{R}\equiv\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)]$. In particular a state in which the pairs are moving with center-of-mass momentum $\hbar\mathbf{K}$ is described by the form of φ

$$\varphi(\mathbf{r},\mathbf{R}:\sigma_1\sigma_2)=\exp i\mathbf{K}\cdot\mathbf{R}\varphi(\mathbf{r}:\sigma_1\sigma_2). \qquad (5.4)$$

However, for the moment we will assume that the center-of-mass is at rest and therefore that the pair wave function depends only on the relative coordinate, as in Eq. (5.3).

The wave function (5.2) is generally believed to represent very well the ground state of a superfluid Fermi gas. However, it is somewhat inconvenient to operate with in practice, since the calculation of any physical quantity will involve a great many terms of the "exchange" type, that is, terms coming from the cross terms in the square of the expression (5.2). Indeed, the mere calculation of the normalization constant is not an entirely trivial task.

To cope with this situation a number of alternative formal techniques have been introduced—the method of anomalous Green's functions, the method of canonical transformations, etc.[9] When used with appropriate caution, these methods all give the same results (at least to order $N^{-1/2}$) for the physically interesting quantities, namely the expectation values of one- and two-particle operators (and, more generally, n-particle operators where $n\ll N^{1/2}$). In this paper I shall use the method originally devised by Bardeen *et al.* (1957) to deal with the corresponding problem in superconductivity, since this needs considerably less formal groundwork than most other methods. In this and the next section I develop the theory for spin singlet pairing, and in Sec. VII generalize it to the slightly more complicated case of spin triplet pairing.

B. BCS method

Consider the case in which the pseudomolecular wave function (5.3) corresponds to spin singlet pairing ($S=0$). In that case, since there is only one spin singlet state, we can always write

$$\chi(\mathbf{k}:\sigma_1\sigma_2)=\chi(\mathbf{k})\sqrt{2}^{-1}(\uparrow\downarrow-\downarrow\uparrow), \qquad (5.5)$$

where we use an intuitive notation in which $\uparrow\downarrow$ indicates the eigenstate of the spin projections σ_1, σ_2 such that $\sigma_1=-\sigma_2=\frac{1}{2}\hbar$, etc. We must have $\chi(\mathbf{k})=\chi(-\mathbf{k})$ in order to preserve the antisymmetry. Bearing this in mind, we might as well rewrite Eq. (5.3) in the identically equivalent form

$$\begin{aligned}\varphi(\mathbf{r}_1-\mathbf{r}_2:\sigma_1\sigma_2)&=\sum_{\mathbf{k}}\chi(\mathbf{k})\sqrt{2}^{-1}[\exp(i\mathbf{k}\cdot\mathbf{r}_1)\\&\quad\times\exp(-i\mathbf{k}\cdot\mathbf{r}_2)\uparrow\downarrow-\exp(-i\mathbf{k}\cdot\mathbf{r}_1)\exp(i\mathbf{k}\cdot\mathbf{r}_2)\downarrow\uparrow]\\&\equiv\sum_{\mathbf{k}}\chi(\mathbf{k})\sqrt{2}^{-1}\{(\mathbf{k}\uparrow)_1(-\mathbf{k}\downarrow)_2-(-\mathbf{k}\downarrow)_1(\mathbf{k}\uparrow)_2\},\end{aligned} \qquad (5.6)$$

where the second form indicates explicitly that the wave function is an antisymmetrized linear combination of states in which particle 1 is in the plane-wave state with momentum $\mathbf{k}$ and spin up, and particle 2 in that with momentum $-\mathbf{k}$ and spin down. Consequently, when the system wave function is given by Eq. (5.1) and the pseudomolecular wave function is a spin singlet, the single-particle states $(\mathbf{k}\uparrow)$ and $(-\mathbf{k}\downarrow)$ are always occupied in pairs or not at all.

The basic idea of Bardeen *et al.* is now to write the system wave function not as an antisymmetrized product of wave functions referring to different *pairs of particles*, but as a product of wave functions referring to the state of occupation of different *pairs of single-particle states* $(\mathbf{k}\uparrow,-\mathbf{k}\downarrow)$. That is, one writes

$$\Psi=\prod_{\text{all }\mathbf{k}}\Phi_{\mathbf{k}}, \qquad (5.7)$$

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[9] For an account of some of these methods in the theory of superconductivity, see, e.g., Ambegaokar, 1969.

where $\Phi_{\mathbf{k}}$ describes the state of occupation of the pair of single-particle states $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$. Let us write the state of occupation in which both $\mathbf{k}\uparrow$ and $-\mathbf{k}\downarrow$ are empty as $|0, 0\rangle_{\mathbf{k}}$, and that in which both are full as $|1, 1\rangle_{\mathbf{k}}$. (States in which only one of the pair is occupied are not relevant here, although at finite temperatures we shall have to consider them.) So we can write the properly normalized form of $\Phi_{\mathbf{k}}$ as

$$\Phi_{\mathbf{k}} = u_{\mathbf{k}}\,|0, 0\rangle_{\mathbf{k}} + v_{\mathbf{k}}\,|1, 1\rangle_{\mathbf{k}}, \tag{5.8}$$

where $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are arbitrary complex parameters subject to the conditions

$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1, \qquad u_{\mathbf{k}} = u_{-\mathbf{k}}, \qquad v_{\mathbf{k}} = v_{-\mathbf{k}}. \tag{5.9}$$

The first condition ensures normalization, the second that the state considered is a spin singlet. [Recall that for Fermi particles the state $(-\mathbf{k}\uparrow, \mathbf{k}\downarrow)$ is conventionally defined to be *minus* the state $(\mathbf{k}\downarrow, -\mathbf{k}\uparrow)$.] Equations (5.7)–(5.9) define the most general "BCS-like" system wave function corresponding to singlet pairing at $T = 0$. Equations (5.7)–(5.8) may be written alternatively using second-quantization notation in the identically equivalent form

$$\Psi = \prod_{\text{all }\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} a_{\mathbf{k}\uparrow}^{+} a_{-\mathbf{k}\downarrow}^{+})\,|\text{vac}\rangle, \tag{5.7$'$}$$

where $|\text{vac}\rangle$ is the vacuum state. The normal ground state (filled Fermi sea) is a special case of this wave function, corresponding to the choice

$$\begin{aligned} &u_{\mathbf{k}} = \exp(i\phi_{\mathbf{k}}), \quad v_{\mathbf{k}} = 0 \qquad \text{for } |\mathbf{k}| > k_F \\ &u_{\mathbf{k}} = 0, \quad v_{\mathbf{k}} = \exp(i\phi_{\mathbf{k}}) \qquad \text{for } |\mathbf{k}| < k_F, \end{aligned} \tag{5.10}$$

where $\varphi_{\mathbf{k}}$ is an arbitrary real number depending on $\mathbf{k}$.

The system wave function described by Eqs. (5.7)–(5.9) has one very peculiar feature: apart from the special case (5.10) it corresponds to a superposition of states containing different numbers of particles. In what sense it can or should be "taken seriously" is perhaps a matter of debate: clearly if one were dealing with a truly isolated system it would be unrealistic, in that one has a presumably inviolable superselection rule forbidding superpositions corresponding to different numbers of conserved particles. However, in practice, in the case of superconductors, one is almost always dealing with an "open" system in which electrons can migrate in and out of the system via the current leads, etc., so that there is nothing particularly unrealistic in describing this particular piece of metal in a way which does not conserve electrons. Similarly, in the case of liquid ^{3}He, one may plausibly argue that in all practical situations the layer of solid ^{3}He which is believed to condense on the apparatus walls may play the role of a particle "reservoir." But in any case, if one feels unhappy with a particle-nonconserving wave function one may recover one corresponding to a fixed number of particles N by the following trick, which is due to Anderson (1958): one considers the family of BCS-like system wave functions $\Psi(\theta)$ given by

$$\Psi(\theta) = \prod_{\text{all }\mathbf{k}} \Phi_{\mathbf{k}}(\theta), \qquad \Phi_{\mathbf{k}}(\theta) = u_{\mathbf{k}}\,|0, 0\rangle_{\mathbf{k}} + v_{\mathbf{k}} \exp(i\theta)\,|1, 1\rangle_{\mathbf{k}} \tag{5.11}$$

which is, of course, automatically normalized provided Eq. (5.9) is satisfied. The members of this family differ only in the phase relation between components corresponding to different numbers of particles: since no physical operator has matrix elements between such components, this phase relation cannot affect any physical quantity and all members of the family are completely equivalent. If now we form the linear superposition

$$\Psi(N) = (2\pi)^{-1/2} \int_0^{2\pi} \Psi(\theta) \exp(-iN\theta/2)\, d\theta \tag{5.12}$$

it turns out that $\Psi(N)$ is precisely the N-particle function given by Eqs. (5.2), (5.3), and (5.5), with $\chi(\mathbf{k})$ proportional to $v_{\mathbf{k}}/u_{\mathbf{k}}$ (cf. Rickayzen, 1965, p. 142).[10]

From now on we shall use the particle-nonconserving BCS-like wave function (5.7). It must then be remembered that, according to the standard precepts of statistical mechanics, the equilibrium state at $T = 0$ is given by minimizing not the expectation value $\langle H\rangle$ for the Hamiltonian, but $\langle H\rangle - \mu\langle N\rangle$, where μ is the chemical potential. For an open system μ is fixed by the environment which acts as a particle reservoir, while for a closed system μ is fixed by the condition that $\langle N\rangle$ is set equal to the actual number $\bar{N}$ of particles (and $\mu = \partial\langle H\rangle/\partial\langle N\rangle$). The statistical fluctuations in the particle number are then of order $N^{1/2}$.

For reasons which will become clearer subsequently, it is convenient to express the parameters $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ entering (5.7) in terms of a single complex parameter $\Delta_{\mathbf{k}}$ in the following way:

$$\begin{aligned} v_{\mathbf{k}} &\equiv \Delta_{\mathbf{k}}/[\,|\Delta_{\mathbf{k}}|^2 + (E_{\mathbf{k}} + \epsilon_{\mathbf{k}})^2]^{1/2}, \\ u_{\mathbf{k}} &\equiv (E_{\mathbf{k}} + \epsilon_{\mathbf{k}})/[\,|\Delta_{\mathbf{k}}|^2 + (E_{\mathbf{k}} + \epsilon_{\mathbf{k}})^2]^{1/2}, \end{aligned} \tag{5.13}$$

where

$$E_{\mathbf{k}} \equiv +(|\Delta_{\mathbf{k}}|^2 + \epsilon_{\mathbf{k}}^2)^{1/2}, \qquad \Delta_{-\mathbf{k}} \equiv \Delta_{\mathbf{k}}. \tag{5.14}$$

The conditions (5.9) are then automatically satisfied. At this stage the function $\Delta_{\mathbf{k}} \equiv \Delta(\mathbf{k})$ simply forms a set of arbitrary complex parameters characterizing the wave function: its physical significance will become clearer below.

It should be emphasized that the *only* assumption we have made so far is that the wave function of the system has built into it a certain type (and only a certain type) of correlation property, namely the one described by Eq. (5.2). This property may, indeed, be taken as the defining characteristic of an "anisotropic superfluid" (of which, of course, the isotropic variety—e.g., a superconductor—is here regarded as a special case); it is often described by saying that *Cooper pairs* form in the system. We have *not*, so far, said anything about what determines the form of the parameters $u_{\mathbf{k}}$, $v_{\mathbf{k}}$ entering the wave function: this is the topic of the next subsection. It is essential to distinguish between those properties of an anisotropic superfluid which follow from the basic correlation properties of the wave

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[10] The choice $v_k = \exp(i\phi)$, where ϕ is the polar angle of $\mathbf{k}$, gives $\Psi(N)$ an apparent angular momentum. However, in the normal limit described by Eq. (5.10) this is clearly quite spurious, since the only effect in this case is to multiply the normal ground state wave function by a phase factor.

function, and those which follow from a particular form of the parameters entering it; this is a distinction we shall make repeatedly in what follows.

C. BCS solution and elementary excitations at $T = 0$

In this and subsequent subsections we will determine the optimum form of the wave function, that is of the function $\Delta_{\mathbf{k}}$, under the assumption that the kinetic energy $\epsilon_{\mathbf{k}}$ and pairing interaction $V_{\mathbf{kk}'}$ are some fixed functions, that the one-particle density of states is not too fast varying near the Fermi surface, and that the pairing interaction is "weak" and not too drastically energy-dependent. (These somewhat vague statements will acquire a rather more precise meaning below.) This collection of approximations we will hereafter label the "BCS" approximation.

Let us then consider the problem of finding the ground state at zero temperature. We choose a given form of the pairing energy $V_{\mathbf{kk}'}$ for spin singlet states and calculate the total energies of the state described by Eqs. (5.7)–(5.9), or rather the quantity $\langle H\rangle - \mu\langle N\rangle \equiv \langle K - \mu N\rangle + \langle V\rangle$, where K is the kinetic energy. Remembering that we measure all one-particle energies $\epsilon_{\mathbf{k}}$ from the chemical potential, we can write,

$$\langle K - \mu N\rangle = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \langle (n_{\mathbf{k}\downarrow} + n_{-\mathbf{k}\downarrow})\rangle. \tag{5.15}$$

But if the wave function for the pair state $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ has the form (5.8), then clearly

$$\langle n_{\mathbf{k}\uparrow} + n_{-\mathbf{k}\downarrow}\rangle = 2\,|\,v_{\mathbf{k}}\,|^2 \tag{5.16}$$

and so we have

$$\langle K - \mu N\rangle = \sum_{\mathbf{k}} 2\epsilon_{\mathbf{k}}\,|\,v_{\mathbf{k}}\,|^2. \tag{5.17}$$

Evidently, minimizing this term alone would lead to the conditions (5.10), i.e., to the normal ground state. However, we have to examine the potential terms as well.

To make the subsequent argument clearer, let us digress for a moment to a simple one-particle problem with a potential $V(\mathbf{r})$. The expectation value of the potential is simply $\int V(\mathbf{r})\,|\,\psi(\mathbf{r})\,|^2\,d\mathbf{r}$, and if we take the Fourier transforms $\psi_{\mathbf{k}}$ and $V_{\mathbf{k}}$ of the wave function and the potential, this can be written:

$$\langle V\rangle = \sum_{\mathbf{kk}'} V_{\mathbf{k}'-\mathbf{k}}\psi_{\mathbf{k}'}^*\psi_{\mathbf{k}}, \tag{5.18}$$

that is, it is a sum of terms referring to scattering from state $\mathbf{k}$ to state $\mathbf{k}'$, and each of these is a product of (a) the matrix element for the scattering process, (b) the probability amplitude that the system is in the initial state ($\psi_{\mathbf{k}}$), (c) the complex conjugate of the probability amplitude that it is in the final state ($\psi_{\mathbf{k}'}^*$). Note that since we are dealing with a single-particle problem, the statistics obeyed by the particle are quite irrelevant here.

Let us now go back to our BCS problem. We first notice that the only scattering processes we need to take into account are those in which a pair in $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ scatters into $(\mathbf{k}'\uparrow, -\mathbf{k}'\downarrow)$. [If one of the particles, say the one in $(\mathbf{k}\uparrow)$, scatters with a particle from another pair state $(\mathbf{k}''\sigma)$, we will produce a final state in which some particles are unpaired, that is, one which is not describable by (5.7)–(5.9).[11]] Now the initial state here is one in which not only is $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ known to be full, *but also* $(\mathbf{k}'\uparrow, -\mathbf{k}'\downarrow)$ *is known to be empty* (otherwise, the Pauli principle prevents the scattering). Hence the appropriate probability amplitude is not just $v_{\mathbf{k}}$ but $v_{\mathbf{k}}u_{\mathbf{k}'}$. Similarly the final state has $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ empty and $(\mathbf{k}'\uparrow, -\mathbf{k}'\downarrow)$ full, and the appropriate probability amplitude is $u_{\mathbf{k}}v_{\mathbf{k}'}$. We should now complex conjugate the latter, multiply by the matrix element $V_{\mathbf{kk}'}$ and sum over all possible values of $\mathbf{k}$ and $\mathbf{k}'$ [with the usual factor of $\frac{1}{2}$ which comes from the original expression for the potential energy $\frac{1}{2}\sum_{ij}V(\mathbf{r}_i - \mathbf{r}_j)$]. However, there is a slight complication: the potential may equally well scatter into $(\mathbf{k}'\downarrow, -\mathbf{k}'\uparrow) \equiv -(-\mathbf{k}'\uparrow, \mathbf{k}'\downarrow)$, and this results in the replacement of $V_{\mathbf{kk}'}$ by $V_{\mathbf{kk}'} + V_{\mathbf{k},-\mathbf{k}'}$. (Like most of the rather messy points connected with exchange effects in BCS theory, this one comes out a good deal more clearly in the notation of second quantization—see below.) The final expression is

$$\langle V\rangle = \sum_{\mathbf{kk}'} V_e(\mathbf{k}\cdot\mathbf{k}')u_{\mathbf{k}}^*v_{\mathbf{k}}u_{\mathbf{k}'}v_{\mathbf{k}'}^*, \tag{5.19}$$

where

$$V_e(\mathbf{k}, \mathbf{k}') \equiv \tfrac{1}{2}[V(\mathbf{k}, \mathbf{k}') + V(\mathbf{k}, -\mathbf{k}')] \tag{5.20}$$

is the even part of V.

If we write

$$F_{\mathbf{k}} \equiv u_{\mathbf{k}}^*v_{\mathbf{k}} \qquad (F_{\mathbf{k}} \equiv F_{-\mathbf{k}}) \tag{5.21}$$

then Eq. (5.19) becomes

$$\langle V\rangle = \sum_{\mathbf{kk}'} V_e(\mathbf{k}, \mathbf{k}')F_{\mathbf{k}}F_{\mathbf{k}'}^*. \tag{5.22}$$

Equation (5.19) may be alternatively derived in the second-quantized formalism. In this language we have

$$V = \tfrac{1}{2}\sum_{\mathbf{kk}'}\sum_{\alpha\beta} V_{\mathbf{kk}'}a_{\mathbf{k}'\alpha}{}^+a_{-\mathbf{k}'\beta}{}^+a_{-\mathbf{k}\beta}a_{\mathbf{k}\alpha} \tag{5.23}$$

and if we select the part corresponding to antiparallel spin pairing it is

$$V = \tfrac{1}{2}\sum_{\mathbf{kk}'} V_{\mathbf{kk}'}(a_{\mathbf{k}'\uparrow}{}^+a_{-\mathbf{k}'\downarrow}{}^+a_{-\mathbf{k}\downarrow}a_{\mathbf{k}\uparrow} + a_{\mathbf{k}'\downarrow}{}^+a_{-\mathbf{k}'\uparrow}{}^+a_{-\mathbf{k}\uparrow}a_{\mathbf{k}\downarrow}). \tag{5.24}$$

When we take the expectation value of the quantity $a_{\mathbf{k}'\uparrow}{}^+a_{-\mathbf{k}'\downarrow}{}^+a_{-\mathbf{k}\downarrow}a_{\mathbf{k}\uparrow}$, because the states of occupation of different pairs are uncorrelated it reduces to

$$\langle a_{\mathbf{k}'\uparrow}{}^+a_{-\mathbf{k}'\downarrow}{}^+\rangle\langle a_{-\mathbf{k}\downarrow}a_{\mathbf{k}\uparrow}\rangle = (u_{\mathbf{k}'}v_{\mathbf{k}'}^*)(u_{\mathbf{k}}^*v_{\mathbf{k}}) \equiv F_{\mathbf{k}}F_{\mathbf{k}'}^*. \tag{5.25}$$

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[11] This is why the presentation given here, which is explicitly based on a certain assumption about the nature of the superfluid-state *wave function*, is completely equivalent to the original derivation of BCS, which started from a Hamiltonian truncated so as to keep only the pair scattering terms discussed below.

By using the anticommutation relations of the Fermi operators, we can similarly write the second term as $F_{-\mathrm{k}}F_{-\mathrm{k}'}^*$. Consequently

$$\langle V\rangle = \tfrac{1}{2}\sum_{\mathrm{kk}'} V_{\mathrm{kk}'}\{F_{\mathrm{k}}F_{\mathrm{k}'}^* + F_{-\mathrm{k}}F_{-\mathrm{k}'}^*\} \tag{5.26}$$

which is identically equal to Eq. (5.22) because of the condition $F_{\mathrm{k}} = F_{-\mathrm{k}}$.

Putting together Eqs. (5.17) and (5.19), we have

$$\langle E\rangle - \mu\langle N\rangle = \sum_{\mathrm{k}} 2\epsilon_{\mathrm{k}} \mid v_{\mathrm{k}} \mid^2 + \sum_{\mathrm{kk}'} V_e(\mathbf{k}, \mathbf{k}')u_{\mathrm{k}}^* v_{\mathrm{k}} u_{\mathrm{k}'} v_{\mathrm{k}'}^*. \tag{5.27}$$

If we want to find the optimum wave function at $T = 0$, we must minimize this expression. Introducing Δ_{k} and E_{k} as defined in (5.13) and (5.14), we have

$$F_{\mathrm{k}} \equiv u_{\mathrm{k}}^* v_{\mathrm{k}} (= \langle a_{-\mathrm{k}\downarrow} a_{\mathrm{k}\uparrow}\rangle) = \Delta_{\mathrm{k}}/2E_{\mathrm{k}} \tag{5.28}$$

and

$$\mid v_{\mathrm{k}} \mid^2 = \tfrac{1}{2}(1 - \epsilon_{\mathrm{k}}/E_{\mathrm{k}}). \tag{5.29}$$

The total energy is

$$\begin{aligned} \langle E\rangle - \mu\langle N\rangle &= \sum_{\mathrm{k}} \epsilon_{\mathrm{k}}(1 - \epsilon_{\mathrm{k}}/E_{\mathrm{k}}) \\ &\quad + \sum_{\mathrm{kk}'} V_e(\mathbf{k}, \mathbf{k}')(\Delta_{\mathrm{k}}/2E_{\mathrm{k}})(\Delta_{\mathrm{k}'}^*/2E_{\mathrm{k}'}), \\ E_{\mathrm{k}} &\equiv +(\epsilon_{\mathrm{k}}^2 + \mid \Delta_{\mathrm{k}} \mid^2)^{1/2}, \end{aligned} \tag{5.30}$$

that is, it is a function of the set of complex variational parameters Δ_{k}.

To minimize the expression (5.30) it is convenient to use the fact that $|\Delta_{\mathrm{k}}|^2/E_{\mathrm{k}} \equiv 2\Delta_{\mathrm{k}}F_{\mathrm{k}}^*$, and $dE_{\mathrm{k}}/d\Delta_{\mathrm{k}} = 2F_{\mathrm{k}}^*$ (where the differentiation is to be taken so that Δ_{k} and Δ_{k}^* remain complex conjugates, *not* at constant Δ_{k}^*). We can then rewrite the first term of Eq. (5.30) as

$$\begin{aligned} \sum_{\mathrm{k}} \epsilon_{\mathrm{k}}(1 - \epsilon_{\mathrm{k}}/E_{\mathrm{k}}) &= \text{const} + \sum_{\mathrm{k}} (-E_{\mathrm{k}} + \mid \Delta_{\mathrm{k}} \mid^2/E_{\mathrm{k}}) \\ &= \text{const} + 2\sum_{\mathrm{k}} (F_{\mathrm{k}}^*\Delta_{\mathrm{k}} - \textstyle\int F_{\mathrm{k}}^*\, d\Delta_{\mathrm{k}}) \\ &= \text{const} + 2\textstyle\int \Delta_{\mathrm{k}}\, dF_{\mathrm{k}}^*. \end{aligned} \tag{5.31}$$

Then rewriting the second term of (5.32) explicitly in terms of F_k and differentiating with respect to F_{k}^*, we obtain

$$\Delta_{\mathrm{k}} = -\sum_{\mathrm{k}'} V_{\mathrm{kk}'}F_{\mathrm{k}'} \equiv -\sum_{\mathrm{k}'} V_{\mathrm{kk}'}(\Delta_{\mathrm{k}'}/2E_{\mathrm{k}'}). \tag{5.32}$$

This is the well-known $T = 0$ BCS gap equation. We defer detailed consideration of its properties until we have generalized it to the finite temperature case. Note however at once that it defines only an extremum of the energy, not necessarily an absolute minimum.

The simplest way of forming an elementary excitation at $T = 0$ is to change the state of occupation of a single pair state $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ while leaving all others unchanged. [There do exist also collective excitations involving a large number of pair states (Anderson, 1958), but we shall not consider these at this point.] We can either (a) break up the pair, that is, allow only *one* particle to occupy the pair state, or (b) form a pair-state wave function which is of the form (5.8) but with coefficients chosen so that it is orthogonal to the ground state. Let us call these two types of excitation "broken pair" (BP) and "excited pair" (EP) states, respectively, the ground state being referred to as the "ground pair" (GP) state.

There are two orthogonal broken pair states, which can be chosen to correspond to a single particle in $\mathbf{k}\uparrow$ or $-\mathbf{k}\downarrow$, respectively. The kinetic energy is just ϵ_{k}. We *lose* all the pairing energy associated with the ground pair state (we have in fact $F_{\mathrm{k}} \equiv 0$ for this state). Bearing in mind that F_{k} may occur in either of the sums in the expression for V, we have for the excitation energy of the excited pair state (energy relative to the ground pair state)

$$\begin{aligned} E_{\mathrm{BP}} - E_{\mathrm{GP}} &= \epsilon_{\mathrm{k}} - (\epsilon_{\mathrm{k}} - \epsilon_{\mathrm{k}}^2/E_{\mathrm{k}}) \\ &\quad - \mathrm{Re}(\Delta_{\mathrm{k}}/E_{\mathrm{k}}) \sum_{\mathrm{k}'} (V_{\mathrm{kk}'}\Delta_{\mathrm{k}'}^*/2E_{\mathrm{k}'}). \end{aligned} \tag{5.33}$$

With the help of the gap equation, Eq. (5.32), and the definition (5.14) this becomes simply

$$E_{\mathrm{BP}} - E_{\mathrm{GP}} = E_{\mathrm{k}}. \tag{5.34}$$

As to the excited pair state, one may easily show that the unique state (apart from a phase factor) which is normalized and orthogonal to the ground pair state is

$$\Phi_{\mathrm{k(EP)}} = \tilde{u}_{\mathrm{k}} \mid 0, 0\rangle_{\mathrm{k}} + \tilde{v}_{\mathrm{k}} \mid 1, 1\rangle_{\mathrm{k}}: \quad \tilde{u}_{\mathrm{k}} \equiv v_{\mathrm{k}}^*, \tilde{v}_{\mathrm{k}} \equiv -u_{\mathrm{k}}^*. \tag{5.35}$$

The kinetic energy is therefore $2\epsilon_{\mathrm{k}} \mid \tilde{v}_{\mathrm{k}} \mid^2 = 2\epsilon_{\mathrm{k}}(1 - \mid v_{\mathrm{k}} \mid^2) = \epsilon_{\mathrm{k}}(1 + \epsilon_{\mathrm{k}}/E_{\mathrm{k}})$ and the quantity $u_{\mathrm{k}}^* v_{\mathrm{k}} \equiv F_{\mathrm{k}}$ is replaced by $\tilde{u}_{\mathrm{k}}^*\tilde{v}_{\mathrm{k}} = -F_{\mathrm{k}}$. Consequently, the loss of (negative) potential energy is twice that for the BP states, and we find simply

$$E_{\mathrm{EP}} - E_{\mathrm{GP}} = 2E_{\mathrm{k}}. \tag{5.36}$$

So we have two elementary excitations with excitation energy E_{k}, and one with energy $2E_{\mathrm{k}}$. It is at first sight tempting to regard the EP state as in some sense equivalent to double excitation of the BP states, but this temptation should probably be resisted, for reasons we shall see in Sec. VI. We shall see below that Δ_{k} is usually independent of the magnitude of $\mathbf{k}$ to a good approximation and can therefore be taken to be a function only of $\mathbf{n}$, the direction of $\mathbf{k}$; since we have $E_{\mathrm{k}}^2 = \epsilon_{\mathrm{k}}^2 + \mid \Delta(\mathbf{n}) \mid^2$, it follows that $\mid \Delta(\mathbf{n}) \mid$ is the least possible excitation energy associated with the direction $\mathbf{n}$. $\Delta(\mathbf{n})$ is commonly referred to simply as the "energy gap," even though in general it may be complex.

D. BCS theory at finite temperature

The basic assumption of the BCS theory at finite temperatures is the obvious generalization of the zero-temperature one: the pair states $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ are assumed to be statistically independent and hence to be each described by its own density matrix or statistical ensemble. For any given pair $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ we have, as at zero temperature, the four basis states $\mid 0, 0\rangle_{\mathrm{k}}$, $\mid 1, 0\rangle_{\mathrm{k}}$, $\mid 0, 1\rangle_{\mathrm{k}}$, and $\mid 1, 1\rangle_{\mathrm{k}}$

(where $|1,0\rangle_{\mathbf{k}}$ is the state with $\mathbf{k}\uparrow$ occupied and $-\mathbf{k}\downarrow$ empty, etc.). Evidently the "broken-pair" states $|1,0\rangle_{\mathbf{k}}$ and $|0,1\rangle_{\mathbf{k}}$ feel nothing of the pairing interaction and have, as at $T=0$, an energy $\epsilon_{\mathbf{k}}$. From the states $|0,0\rangle_{\mathbf{k}}$ and $|1,1\rangle_{\mathbf{k}}$ we can form, as at $T=0$, two mutually orthogonal linear combinations. Let us write these as

$$\Phi_{\mathbf{k},\mathrm{GP}} = u_{\mathbf{k}}(T)\,|0,0\rangle_{\mathbf{k}} + v_{\mathbf{k}}(T)\,|1,1\rangle_{\mathbf{k}},$$
$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1 \tag{5.37a}$$

$$\Phi_{\mathbf{k},\mathrm{EP}} = v_{\mathbf{k}}(T)\,|0,0\rangle_{\mathbf{k}} - u_{\mathbf{k}}(T)\,|1,1\rangle_{\mathbf{k}} \tag{5.37b}$$

and define as before

$$v_{\mathbf{k}}(T) \equiv \Delta_{\mathbf{k}}(T)/[|\Delta_{\mathbf{k}}|^2 + (E_{\mathbf{k}}+\epsilon_{\mathbf{k}})^2]^{1/2},$$
$$u_{\mathbf{k}}(T) \equiv (E_{\mathbf{k}}+\epsilon_{\mathbf{k}})/[|\Delta_{\mathbf{k}}|^2 + (E_{\mathbf{k}}+\epsilon_{\mathbf{k}})^2]^{1/2}, \tag{5.38}$$

$$E_{\mathbf{k}}(T) \equiv +[\epsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}(T)|^2]^{1/2}. \tag{5.39}$$

Thus, just as at $T=0$, the complex parameter $\Delta_{\mathbf{k}}(T)$ enters as a parameter characterizing the trial pair wave function.

At this stage it is convenient to anticipate the result that in (stable and metastable) thermal equilibrium the energies of the broken pair and excited pair states are given relative to the ground pair states by $E_{\mathbf{k}}(T)$ and $2E_{\mathbf{k}}(T)$, respectively, just as in the zero temperature case. Using this, we can make the ansatz that the probabilities of occurrence of GP, BP, and EP states at temperature T are given by the Boltzmann distribution

$$P_{\mathrm{GP}} = n^{-1}, \qquad P_{\mathrm{BP}} = n^{-1}\exp[-\beta E_{\mathbf{k}}(T)],$$
$$P_{\mathrm{EP}} = n^{-1}\exp[-2\beta E_{\mathbf{k}}(T)], \tag{5.40}$$
$$n \equiv \{1 + 2\exp[-\beta E_{\mathbf{k}}(T)] + \exp[-2\beta E_{\mathbf{k}}(T)]\},$$
$$\beta \equiv 1/k_BT.$$

Up to this point, as in Subsection B, our description of the state of the system is quite general.

It is now straightforward to write down the free energy as a function of the parameters $\Delta_{\mathbf{k}}(T)$. The kinetic energy including the chemical-potential term is

$$\langle K-\mu N\rangle = \sum_{\mathbf{k}}\{P_{\mathrm{GP}}(\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}}^2/E_{\mathbf{k}}) + 2P_{\mathrm{BP}}\epsilon_{\mathbf{k}}$$
$$+ P_{\mathrm{EP}}(\epsilon_{\mathbf{k}} + \epsilon_{\mathbf{k}}^2/E_{\mathbf{k}})\}$$
$$= \sum_{\mathbf{k}}\epsilon_{\mathbf{k}}[1 - (\epsilon_{\mathbf{k}}/E_{\mathbf{k}})\tanh(\beta E_{\mathbf{k}}/2)]. \tag{5.41}$$

To calculate the expectation value of the potential energy let us as before consider a particular scattering process $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow) \to (\mathbf{k}'\uparrow, -\mathbf{k}'\downarrow)$. For any given quantum mechanical state of the system let $A_{\mathbf{k}1}$ be the probability amplitude that $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ is full and $A_{\mathbf{k}0}$ the amplitude that it is empty, and similarly for $\mathbf{k}'$. (At $T=0$, the quantities $A_{\mathbf{k}1}$ and $A_{\mathbf{k}0}$ are equivalent to $v_{\mathbf{k}}$ and $u_{\mathbf{k}}$, respectively). Then for any given quantum mechanical state the contribution from this scattering process is $V_e(\mathbf{k},\mathbf{k}')(A_{\mathbf{k}0}A_{\mathbf{k}'1})^* \times (A_{\mathbf{k}'0}A_{\mathbf{k}1}) = V_e(\mathbf{k},\mathbf{k}')A_{\mathbf{k}0}^*A_{\mathbf{k}1}A_{\mathbf{k}'0}A_{\mathbf{k}'1}^*$; since we have a thermal ensemble of quantum mechanical states, we must take the thermal average of this quantity. Because of the assumed independence of the different pair states, we can put $\langle A_{\mathbf{k}0}^*A_{\mathbf{k}1}A_{\mathbf{k}'0}A_{\mathbf{k}'1}^*\rangle = \langle A_{\mathbf{k}0}^*A_{\mathbf{k}1}\rangle\langle A_{\mathbf{k}'0}A_{\mathbf{k}'1}^*\rangle$. Hence the total potential energy is

$$\langle V\rangle = \sum_{\mathbf{kk}'}V_e(\mathbf{k},\mathbf{k}')\langle A_{\mathbf{k}0}^*A_{\mathbf{k}1}\rangle\langle A_{\mathbf{k}'0}A_{\mathbf{k}'1}^*\rangle$$
$$\equiv \sum_{kk'}V_e(\mathbf{kk}')F_{\mathbf{k}}(T)F_{\mathbf{k}'}^*(T), \tag{5.42}$$

where we have defined the generalization of Eq. (5.21) to finite temperatures:

$$F_{\mathbf{k}}(T) \equiv \langle A^*_{\mathbf{k}0}A_{\mathbf{k}1}\rangle. \tag{5.43}$$

Evidently, in second-quantized language we have [c.f. Eq. (5.28)]

$$F_{\mathbf{k}}(T) = \langle a_{-\mathbf{k}\downarrow}a_{\mathbf{k}\uparrow}\rangle, \tag{5.44}$$

where the brackets now indicate thermal as well as quantum mechanical averaging. It is now straightforward to calculate the value of $F_{\mathbf{k}}(T)$ in terms of $\Delta_{\mathbf{k}}(T)$. For the GP state $A_{k0}^*A_{\mathbf{k}1}$ is $u_{\mathbf{k}}^*v_{\mathbf{k}}$, for the EP state it is $-u_{\mathbf{k}}^*v_{\mathbf{k}}$, and for the BP states it is zero. Consequently, we have

$$F_{\mathbf{k}}(T) = u_{\mathbf{k}}^*v_{\mathbf{k}}(P_{\mathrm{GP}} - P_{\mathrm{EP}})$$
$$= [\Delta_{\mathbf{k}}(T)/2E_{\mathbf{k}}]\tanh(\beta E_{\mathbf{k}}/2). \tag{5.45}$$

Finally, the entropy for the pair state $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow)$ is calculated from the usual expression

$$S_{\mathbf{k}} = -k_B\sum_n P_n\ln P_n = -k_B(P_{\mathrm{GP}}\ln P_{\mathrm{GP}}$$
$$+ 2P_{\mathrm{BP}}\ln P_{\mathrm{BP}} + P_{\mathrm{EP}}\ln P_{\mathrm{EP}}),$$
$$= k_B\{-\beta E_{\mathbf{k}}[\tanh(\tfrac{1}{2}\beta E_{\mathbf{k}}) - 1]$$
$$+ 2\ln[1+\exp(-\beta E_{\mathbf{k}})]\}, \tag{5.46}$$

and the total entropy S is just the sum of (5.46) over $\mathbf{k}$.

Putting together Eqs. (5.41), (5.42), (5.45), and (5.46), we therefore finally have [combining a term from (5.41) with one from (5.46)]

$$F - \mu\langle N\rangle \equiv \langle K-\mu N\rangle - TS + \langle V\rangle,$$
$$= \sum_{\mathbf{k}}\{\epsilon_{\mathbf{k}} - E_{\mathbf{k}} + (|\Delta_{\mathbf{k}}|^2/E_{\mathbf{k}})\tanh(\tfrac{1}{2}\beta E_{\mathbf{k}})$$
$$- 2\beta^{-1}\ln[1+\exp(-\beta E_{\mathbf{k}})]\}$$
$$+ \sum_{\mathbf{kk}'}V_e(\mathbf{k},\mathbf{k}')\{[(\Delta_{\mathbf{k}}/2E_{\mathbf{k}})\tanh(\tfrac{1}{2}\beta E_{\mathbf{k}})]$$
$$\times[(\Delta_{\mathbf{k}'}^*/2E_{\mathbf{k}'})\tanh(\tfrac{1}{2}\beta E_{\mathbf{k}'})]\} \equiv f\{\Delta_{\mathbf{k}}\}. \tag{5.47}$$

Equation (5.49) is a general expression for the free energy in terms of $\Delta_{\mathbf{k}}(T)$, given our ansatz (5.42). Because $\Delta_{\mathbf{k}}$ is assumed to be an even function we can replace $V_e(\mathbf{k},\mathbf{k}')$ by $V(\mathbf{k},\mathbf{k}')$.

To find the form of $\Delta_{\mathbf{k}}(T)$ which minimizes the free energy it is convenient to note that according to Eq. (5.45) we have

$$(d/d\Delta_{\mathbf{k}})\{-E_{\mathbf{k}} - 2\beta^{-1}\ln[1+\exp(-\beta E_{\mathbf{k}})]\} = -2F_{\mathbf{k}}^*. \tag{5.48}$$

Consequently,

$$F - \mu\langle N\rangle = \sum_{\mathbf{k}} \{\text{const} + 2(\Delta_{\mathbf{k}} F_{\mathbf{k}}^* - \int F_{\mathbf{k}}^* \, d\Delta_{\mathbf{k}})\} + \langle V\rangle = \sum_{\mathbf{k}} \{\text{const} + 2 \int \Delta_{\mathbf{k}} \, dF_{\mathbf{k}}^*\} + \langle V\rangle. \qquad (5.49)$$

Differentiation with respect to $F_{\mathbf{k}}^*$ then gives the famous BCS gap equation

$$\Delta_{\mathbf{k}}(T) = -\sum_{\mathbf{k}'} V(\mathbf{k}, \mathbf{k}')[\Delta_{\mathbf{k}'}(T)/2E_{\mathbf{k}'}(T)] \times \tanh[\beta E_{\mathbf{k}'}(T)/2]. \qquad (5.50)$$

Before examining some properties of this equation, it should be strongly emphasized that in general it defines only an *extremum* of the free energy. In the case of superconductivity this does not usually lead to confusion, since for the model forms of $V(\mathbf{k}, \mathbf{k}')$ usually chosen there is only one extremum (other than the trivial one $\Delta_{\mathbf{k}} = 0$), which is therefore automatically an absolute minimum. On the other hand, in the case of $l \neq 0$ pairing the gap equation has in general many solutions, and some of them are not even local minima but are saddle points. This is why we have insisted on finding an explicit form for the free energy itself as a function of $\Delta_{\mathbf{k}}$ [Eq. (5.47)]: we can then immediately compare the free energies of different solutions to Eq. (5.50) and determine the lowest.

The properties of the solutions to Eq. (5.50) are extensively discussed by Anderson and Morel (1961). We simply quote some of the results:

(1) To the extent that $V(\mathbf{k}, \mathbf{k}')$ is independent of the magnitude of $\mathbf{k}$, $\mathbf{k}'$, then so is the solution $\Delta_{\mathbf{k}}(T)$. In particular if $V(\mathbf{k}, \mathbf{k}')$ has the model form (3.11), then we find

$$\begin{aligned} \Delta_{\mathbf{k}}(T) &= \Delta(\mathbf{n}: T) \qquad &&\text{for } k_F - \Delta k \leq k \leq k_F + \Delta k \\ &= 0 &&\text{otherwise.} \end{aligned} \qquad (5.51)$$

Moreover if $V(\mathbf{k}, \mathbf{k}')$ or more precisely its Legendre decomposition $V_l(k, k')$ is slowly varying over a range of k, k' large compared to (say) $k_B T_c/v_F$ (i.e., over an energy range large compared to $k_B T_c$) then $\Delta_{\mathbf{k}}(T)$ is also slowly varying over a similar range, and hence can be set equal to a constant $\Delta(\mathbf{n}: T)$ for $\epsilon_{\mathbf{k}} \leq k_B T_c$ to a good approximation. This condition should be very well fulfilled for liquid ³He (see the end of the last section). Since the behavior of $V_l(k, k')$ and $\Delta_{\mathbf{k}}$ for $\epsilon_{\mathbf{k}} \gg k_B T_c$ is of importance mainly in fixing the critical temperature (cf. below), it is then quite adequate for most purposes to replace $V_l(k, k')$ by the model form (3.11), with any choice of ϵ_c and V_l such that (a) the critical temperature T_c fits the experimental value and (b) the weak coupling condition $\epsilon_c \gg k_B T_c$ is preserved. (See however below, Sec. IX.) From now on we shall usually assume, for simplicity, the model form (3.11) (see also the end of Subsection D below).

(2) If in Eq. (3.11) all V_l are positive, no solution to the gap equation exists. If some are negative, then as in Eq. (3.26) we define for $V_l < 0$ the quantity $\lambda_l = -\frac{1}{2}(dn/d\epsilon) V_l$, and suppose that the largest λ_l corresponds to $l = l_0$. Then the gap equation has a solution only for $T < T_c$ where

$$k_B T_c = 1.14\epsilon_c \exp - 1/\lambda_{l_0}, \qquad \lambda_{l_0} \equiv -\tfrac{1}{2}(dn/d\epsilon) V_{l_0}. \qquad (5.52)$$

This is, not surprisingly, the same formula as was mentioned in Sec. III for the temperature at which the normal phase became unstable.

(3) Provided that the next largest value of λ_l is not too close to λ_{l_0} (precisely, provided $\exp(-1/\lambda_l) \ll \exp(-1/\lambda_{l_0})$ for $l \neq l_0$), then for $T < T_c$ the gap function $\Delta(\mathbf{n}: T)$ is *approximately* a combination of spherical harmonics corresponding to angular momentum l_0:

$$\Delta(\mathbf{n}: T) \approx \Delta(T) f_\Delta(\mathbf{n}), \qquad f_\Delta(\mathbf{n}) = \sum_m a_{l_0 m}^{(\Delta)} Y_{l_0 m}(\mathbf{n}). \qquad (5.53)$$

In this expression the form of $f_\Delta(\mathbf{n})$, i.e., the coefficients $a_{l_0 m}^{(\Delta)}$, may in principle be functions of temperature; however, in the simplest cases at least it is found that they are not. In using Eq. (5.53) we shall choose $\Delta(T)$ so that $f_\Delta(\mathbf{n})$ is normalized:

$$\int (d\Omega/4\pi) \, |f_\Delta(\mathbf{n})|^2 = 1. \qquad (5.54)$$

(4) Provided Eq. (5.53) is satisfied, the gap $\Delta(T)$ is of the general form

$$\Delta(T) = k_B T_c \varphi(T/T_c)$$

and the function $f_\Delta(\mathbf{n})$, insofar as it has any temperature dependence at all, is a function only of T/T_c. Thus, once T_c is determined all other parameters of the theory can be expressed in terms of it. It is for this reason that, in BCS theory, one often treats T_c as a phenomenological parameter to be determined from experiment rather than calculated.

(5) For any given l_0 there exist in general *several* solutions to Eq. (5.50): these may include in general saddle points and free energy maxima as well as local free energy minima. In case of doubt the most stable minimum must be found by going back to the explicit expression for the free energy, Eq. (5.47). It is of course entirely conceivable that a form of $f(\mathbf{n})$ which is the absolute minimum at one temperature ceases to be so as the temperature is varied.

(6) In general any function of the form (5.53) with $l_0 \neq 0$ will have nodes either at points or along lines on the Fermi surface. In other words, the energy gap for excitations will be zero at these points (or along these lines). However, it should be pointed out that Eq. (5.53) is only an approximation and that mixing in of spherical harmonics with $l \neq l_0$ will inevitably occur (unless all V_l are rigorously zero for $l \neq l_0$) even if the actual magnitude of the admixture is very small; it may well be that one effect of this admixture is to eliminate the nodes of $\Delta(\mathbf{n})$. (I know of no general proof that this is so, but pilot calculations in specific cases seem to point in this direction.) Even if this is so, however, there will still be points at which the gap is very small.

(7) The general form of the "overall magnitude of the gap" $\Delta(T)$ is much as in the simple s-wave BCS theory;

it increases as $(T_c - T)^{1/2}$ for T near T_c and tends to a constant value $\Delta(0)$ in the limit $T \to 0$. We have in fact $\Delta(0) = ak_BT_c$ [cf. (4) above] where $a \leq 1.75$.

We finally observe in passing that provided the gap equation is satisfied the excitation energy of the "broken pair" and "excited pair" states can be calculated analogously to the $T = 0$ case, and we find for given $\mathbf{k}$

$$E_{\mathrm{BP}} - E_{\mathrm{GP}} = E_{\mathbf{k}}(T), \qquad E_{\mathrm{EP}} - E_{\mathrm{GP}} = 2E_{\mathbf{k}}(T). \tag{5.55}$$

Therefore the ansatz (5.40) used in deriving the gap equation is self-consistent.

E. The Ginzburg–Landau region

Many of the above considerations can be confirmed and made more quantitative if we consider specifically the region of temperature near T_c. In the case of superconductivity this is the region of validity of the celebrated theory of Ginzburg and Landau (1950) and it is therefore often known as the "Ginzburg–Landau" region.

Let us *assume* that in this region it is legitimate to treat the gap $\Delta_{\mathbf{k}}$ as small and to expand the free energy in powers of it: the results will show that this assumption is self-consistent. For the moment we will use the model potential (3.11). Moreover, although we shall treat $\Delta_{\mathbf{k}}$ as an arbitrary parameter we assume it is a function only of $\mathbf{n}$; and we assume also the "weak-coupling" condition $\beta\epsilon_c \gg 1$ for all T of interest.

We define the (dimensionless) quantity

$$\begin{aligned}\Psi(\mathbf{n}:T) &\equiv \sum_{|k|} F_{\mathbf{k}} \equiv \tfrac{1}{2}(dn/d\epsilon) \int_{-\epsilon_c}^{\epsilon_c} d\epsilon_k F_{\mathbf{k}} \\ &= \tfrac{1}{2}(dn/d\epsilon)\Delta(\mathbf{n}) \int_{-\epsilon_c}^{\epsilon_c} \frac{\tanh\frac{1}{2}\beta E_k}{2E_k}\, d\epsilon_k \\ &\{E_{\mathbf{k}} \equiv +[\epsilon_k^2 + |\Delta(\mathbf{n}, T)|^2]^{1/2}\}.\end{aligned} \tag{5.56}$$

Evidently, we may expand $\Psi(\mathbf{n}, T)$ in powers of $\Delta(\mathbf{n}, T)$:

$$\Psi(\mathbf{n}, T) = A(T)\Delta(\mathbf{n}) + B(T)\,|\Delta(\mathbf{n})|^2\Delta(\mathbf{n}) + \mathcal{O}(\Delta^5). \tag{5.57}$$

The precise form of $A(T)$ and $B(T)$ need not concern us for the moment, but we notice that both are smooth functions of T in the region of interest and that $A(T)$ is positive and increases with decreasing T, while $B(T)$ is negative. We now notice that according to Eqs. (5.47), (5.49), and (5.56) we have

$$\begin{aligned}F = F_0(T) &+ 2\int (d\Omega/4\pi)\Delta(\mathbf{n})\, d\Psi^*(\mathbf{n}) \\ &+ \int\!\!\int (d\Omega/4\pi)(d\Omega'/4\pi)\, V(\mathbf{n}, \mathbf{n}')\Psi(\mathbf{n})\Psi^*(\mathbf{n}').\end{aligned} \tag{5.58}$$

Using Eqs. (5.57) and (5.58) we may expand the free energy in powers of either $\Delta(\mathbf{n})$ or $\Psi(\mathbf{n})$. In the case of a BCS superconductor the expansion in Δ is conventional, but for an anisotropic superfluid there are advantages in using $\Psi(\mathbf{n})$ as the expansion parameter, particularly if one wants to deal with the possible admixture of spherical harmonics of the "wrong" l. Inverting Eq. (5.57) to obtain Δ in terms of Ψ and inserting in Eq. (5.58) we find

$$\begin{aligned}F\{\Psi(\mathbf{n}):T\} = F_0(T) &+ A^{-1}(T) \int (d\Omega/4\pi)\, |\Psi(\mathbf{n})|^2 \\ &+ \int\!\!\int (d\Omega/4\pi)(d\Omega'/4\pi)\, V(\mathbf{n}, \mathbf{n}')\Psi(\mathbf{n})\Psi^*(\mathbf{n}') \\ &+ \tfrac{1}{2}\{|B(T)|/[A(T)]^4\} \int (d\Omega/4\pi)\, |\Psi(\mathbf{n})|^4 + \mathcal{O}(\Psi^6).\end{aligned} \tag{5.59}$$

At this stage we find the explicit form of $A(T)$: expanding Eq. (5.56) in powers of Δ and comparing with Eq. (5.57), we get

$$A(T) = \tfrac{1}{2}(dn/d\epsilon) \int_{-\epsilon_c}^{\epsilon_c} \frac{\tanh\frac{1}{2}\beta\epsilon}{2\epsilon}\, d\epsilon = \tfrac{1}{2}(dn/d\epsilon) \ln(\zeta\beta\epsilon_c), \tag{5.60}$$

$$\zeta \equiv \tfrac{1}{2} \exp - \int_0^\infty \ln z\, \operatorname{sech}^2 z\, dz = 1.14 \tag{5.61}$$

(where in the intermediate steps we integrated by parts and used the condition $\beta\epsilon_c \gg 1$).

Let us write

$$\Psi(\mathbf{n}) = \sum_{lm} a_{lm} Y_{lm}(\mathbf{n}), \tag{5.62}$$

where Y_{lm} is a normalized spherical harmonic. Then using the definition (4.2) of V_l and the addition theorem for spherical harmonics, we find that the quadratic terms are

$$F_{(2)} = \sum_{lm} \{A^{-1}(T) + V_l\}\, |a_{lm}|^2. \tag{5.63}$$

Moreover the quartic (and subsequent) terms are always positive. We see therefore that at sufficiently high T [such that $A^{-1}(T) + V_l > 0$ for all l] the minimum of the free energy always occurs at $a_{lm} \equiv 0$, that is, for the normal state. The critical temperature below which the free energy is lowered by a nonzero choice of $\Psi(\mathbf{n})$ is given by

$$A^{-1}(T_c) + V_{l_0} = 0, \tag{5.64}$$

where l_0 is the angular momentum for which V_{l_0} is most attractive (negative). This just gives back Eq. (5.52).

Below T_c we see that the coefficient of $|a_{l_0m}|^2$ in (5.63) is negative, while the coefficients of $|a_{lm}|^2$ for $l \neq l_0$ are still positive (unless V_l is very close to V_{l_0}). To a first approximation, therefore, it is sensible to look for a form of $\Psi(\mathbf{n})$ which involves only the $l = l_0$ spherical harmonics:

$$\Psi(\mathbf{n}) = \Psi(T)f(\mathbf{n}), \qquad f(\mathbf{n}) = \sum_m a_{l_0m} Y_{l_0m}(\mathbf{n}), \tag{5.65}$$

where $f(\mathbf{n})$ is normalized as in Eq. (5.54). [Note that since the relation (5.57) between $\Psi(\mathbf{n})$ and $\Delta(\mathbf{n})$ is nonlinear, the function $f(\mathbf{n})$ defined in (5.65) is only approximately equal to the function $f_\Delta(\mathbf{n})$ defined in (5.53)—but see below.]

With the ansatz (5.65) for $\Psi(\mathbf{n})$, the quadratic terms in the free energy take the simple form

$$\begin{aligned} F_{(2)} &= [A^{-1}(T) - A^{-1}(T_c)] \int (d\Omega/4\pi)\, |\Psi(\mathbf{n})|^2 \\ &\approx -A^{-2}(T_c)(dA/dT)_{T_c}(T - T_c) \\ &\quad \times \int (d\Omega/4\pi)\, |\Psi(\mathbf{n})|^2. \end{aligned} \tag{5.66}$$

Adding the quartic terms from Eq. (5.59), and taking into account that $A(T)$ and $B(T)$ are smoothly varying near T_c, we find the final form of the free energy up to terms of order $|\Psi|^4$:

$$\begin{aligned} F\{\Psi(\mathbf{n}), T\} &= \alpha[(T - T_c)/T_c] \int (d\Omega/4\pi)\, |\Psi(\mathbf{n})|^2 \\ &\quad + \tfrac{1}{2}\beta \int (d\Omega/4\pi)\, |\Psi(\mathbf{n})|^4, \end{aligned} \tag{5.67}$$

$$\alpha \equiv -A^{-2}(T_c)(dA/dT)_{T_c} T_c > 0, \tag{5.68}$$

$$\beta \equiv -A^{-4}(T_c) B(T_c) > 0. \tag{5.69}$$

This is exactly of the form of the Ginzburg–Landau expansion of the free energy for a superconductor, except that in that case the "order parameter" $\Psi(\mathbf{n})$ is simply a constant Ψ, and the integral over the Fermi surface is therefore omitted. Explicit evaluation of the constants occurring in Eqs. (5.68) and (5.69) gives [directly from Eq. (5.60)]

$$-T_c(dA/dT)_{T_c} = \tfrac{1}{2}(dn/d\epsilon)[\equiv N(0)] \tag{5.70}$$

and [from a comparison of Eq. (5.57) with an expansion of Eq. (5.56)]

$$\begin{aligned} B(T_c) &= +\tfrac{1}{2}(dn/d\epsilon) \int_{-\epsilon_c}^{\epsilon_c} \frac{d}{d(\epsilon^2)} \cdot \frac{\tanh\frac{1}{2}\beta_c\epsilon}{2\epsilon}\, d\epsilon \\ &\approx \tfrac{1}{2}(dn/d\epsilon)(k_BT_c)^{-2} \cdot \frac{1}{8} \int_0^\infty \frac{1}{z}\frac{d}{dz}\left(\frac{\tanh z}{z}\right) dz, \\ &= -\tfrac{1}{2}(dn/d\epsilon)(\pi k_B T_c)^{-2}\, \tfrac{7}{8}\zeta(3), \end{aligned} \tag{5.71}$$

where $\zeta(3)$ is the Riemann zeta-function.

It should be observed that the particular definition of the "order parameter" $\Psi(\mathbf{n})$ we have used is not unique: we can always multiply $\Psi(\mathbf{n})$ by an arbitrary (temperature-dependent) constant, provided we scale the parameters α and β appropriately. A particularly useful choice is to multiply $\Psi(\mathbf{n})$ by the factor $A^{-1}(T_c)$: the resulting quantity, to lowest order in $T_c - T$, is just the gap function $\Delta(\mathbf{n})$ [cf. Eq. (5.57)].

In this way we can rewrite the expression for the free energy in the GL region in terms of $\Delta(\mathbf{n})$:

$$\begin{aligned} F\{\Delta(\mathbf{n}), T\} &= \tfrac{1}{2}(dn/d\epsilon)\{-(1 - T/T_c) \int (d\Omega/4\pi) \\ &\quad \times |\Delta(\mathbf{n})|^2 + \tfrac{1}{2}\cdot\tfrac{7}{8}\zeta(3)(\pi k_B T_c)^{-2} \int (d\Omega/4\pi)\, |\Delta(\mathbf{n})|^4\}, \end{aligned} \tag{5.72}$$

which is the generalization of the well-known expansion for a superconductor. [This expression can equally well be obtained directly from Eq. (5.58) by expanding in Δ rather than Ψ.] Notice that if we define a dimensionless gap and temperature Δ/k_BT_c and T/T_c, then T_c falls out of the problem entirely: consequently, for pairing with given l, $\Delta(\mathbf{n})/k_BT_c$ is a universal function of T/T_c [cf. point (4) in Sec. V.D above].

As we just saw, to lowest order in $T_c - T$ the gap $\Delta(\mathbf{n})$ is simply a constant times $\Psi(\mathbf{n})$. This is actually a special case of a much more general situation. Generally speaking, to the extent that it is legitimate to approximate $\Psi(\mathbf{n})$ by Eq. (5.65), it is also legitimate to treat $\Psi(\mathbf{n})$ and $\Delta(\mathbf{n})$ as strictly proportional: i.e., to put $f(\mathbf{n}) \equiv f_\Delta(\mathbf{n})$ [cf. Eqs. (5.65) and (5.53)]; the error in both cases is of order (at most) $(\ln\beta_c\epsilon_c)^{-1}$ (see below). In other words, to the extent that $\Psi(\mathbf{n})$ and $\Delta(\mathbf{n})$ are given by a combination of spherical harmonics of a single l value, the angular dependence must be the same in the two cases. In common with almost all work on anisotropic superfluids, we shall in the rest of this paper make this approximation, whether or not we are in the GL region.

Let us substitute the form (5.65) in the free energy (5.67) and define

$$\kappa \equiv \int (d\Omega/4\pi)\, |f(\mathbf{n})|^4. \tag{5.73}$$

Then using the normalization condition on $f(\mathbf{n})$ we find

$$F\{\Psi : T\} = \alpha[(T - T_c)/T_c]\Psi^2 + \tfrac{1}{2}\kappa\beta\Psi^4. \tag{5.74}$$

The equilibrium value of Ψ is therefore

$$\begin{aligned} &\Psi(T) = 0, \quad T > T_c \\ &\Psi(T) = (\alpha/\kappa\beta)^{1/2}[(T_c - T)/T_c]^{1/2}, \quad T < T_c. \end{aligned} \tag{5.75}$$

Alternatively, using the definition (5.53) and the form of free energy (5.72) and putting $f_\Delta(\mathbf{n}) \equiv f(\mathbf{n})$, we find for $\Delta(T)$

$$\begin{aligned} &\Delta(T) = 0, \quad T > T_c \\ &\Delta(T) = 3.06\kappa^{-1/2}(k_BT_c)(1 - T/T_c)^{1/2}, \quad T < T_c \\ &(3.06 = [8\pi^2/7\zeta(3)]^{1/2}). \end{aligned} \tag{5.76}$$

which is the same formula as for a BCS superconductor except for the factor $\kappa^{-1/2}$.

From Eqs. (5.74) and (5.75) we can also calculate the free energy as a function of T. We find for $T < T_c$ [adding also the "normal" term F_0 which was not written out in Eq. (5.74)]

$$F(T) = F_0(T) - \tfrac{1}{2}(\alpha^2/\kappa\beta)[(T_c - T)/T_c]^2. \tag{5.77}$$

From this it follows that among possible forms of the order parameter $\Psi(\mathbf{n})$ satisfying Eq. (5.65), the most stable is the one which corresponds to the minimum value of κ [Eq. (5.73)], that is, the one which maximizes $\Psi(T)$ and $\Delta(T)$.

We would like to remark in passing that the ansatz (5.65) for the order parameter $\Psi(\mathbf{n})$ is not exact in general. If we go back to Eq. (5.59), we can easily convince ourselves that even if all V_l except V_{l_0} are zero or positive, it is nevertheless energetically advantageous for $\Psi(\mathbf{n})$ to contain some admixture of $l \neq l_0$ spherical harmonics, since the fourth-order term can usually be reduced by doing so. To illustrate this, let us consider the case when all V_l except V_{l_0} are zero, take Ψ to be the overall amplitude of the main part of $\Psi(\mathbf{n})$ (i.e., the l_0 spherical harmonic), and let x be the admixture of $l \neq l_0$ harmonics. If x is finite, it will give a positive contribution of order $A^{-1}(T_o)x^2$ to the quadratic terms in Eq. (5.62): on the other hand, in general the lowest-order contribution to the quartic terms will be of order $\beta x \Psi^3$, which can always be made negative by a suitable choice of the sign of x. Consequently, we find $|x| \sim \beta A(T_o)|\Psi|^3$ and hence [using Eq. (5.75)] the ratio of $|x|$ to $|\Psi|$ is of order

$$|x|/|\Psi| \sim [d(\ln A)/d(\ln T)]_{T_c} \cdot [(T_c - T)/T_c]$$
$$\sim [\ln(\beta_c \epsilon_c)]^{-1}[(T_o - T)/T_o]. \tag{5.78}$$

The admixture is therefore very small in the weak-coupling limit. However, it will be strongly enhanced if some other V_l is negative and close in value to V_{l_0}.

In contrast to $\Psi(\mathbf{n})$, the gap $\Delta(\mathbf{n})$ can in principle contain only spherical harmonics of one l value l_0, if all V_l except V_{l_0} are zero. In general the relative mixing in of spherical harmonics with $l \neq l_0$ is of order (some constant times) $|\lambda_{l'}|$, where $\lambda_{l'}$ is the coupling constant for the spherical harmonic of the interaction $V_{l'}$ (other than V_{l_0}) with the largest magnitude (whether it is repulsive or attractive). This point is discussed in detail in Appendix A of Anderson and Morel (1961) for the zero-temperature case, and the considerations are obviously similar for finite temperature; one sees that the relative mixing is proportional to $(T_c - T)$ near T_o. If we suppose that $|\lambda_{l'}|$ is of the same order of magnitude as $\lambda_l \sim (\ln \beta_c \epsilon_o)^{-1}$, we see that the mixing in of "wrong" spherical harmonics in $\Delta(\mathbf{n})$ will usually be of the same order as that for $\Psi(\mathbf{n})$: hence, as stated above, if one is going to neglect it in $\Delta(\mathbf{n})$ it is generally consistent to do so also in $\Psi(\mathbf{n})$, i.e., to make the two quantities strictly proportional. It should be emphasized that it is *always* consistent to do so in the GL region (unless the λ_l's are extremely close).

Finally I would like to sketch a brief justification, in the framework of the methods of this section, of the statement that, provided the interaction is slowly varying over an energy range large compared to k_BT_c, then it is legitimate for most purposes to replace it by the model form (3.11), with the parameters V_l and ϵ_o ($\gg k_BT_c$) chosen so as to reproduce the experimentally observed critical temperature. First we go back to the gap equation (5.50) and note that at the critical temperature itself, when $\Delta_\mathbf{k}$ tends to zero, it becomes

$$\Delta_\mathbf{k} = -\sum_{\mathbf{k}'} V(\mathbf{k}, \mathbf{k}')(\Delta_{\mathbf{k}'}/2\epsilon_{\mathbf{k}'}) \tanh \beta\epsilon_{\mathbf{k}'}/2. \tag{5.79}$$

This is a *linear* equation, and since $V(\mathbf{k}, \mathbf{k}')$ has the form (4.1), its solution must be of the form

$$\Delta_\mathbf{k} = \Delta(\mathbf{n})\psi_l(k), \tag{5.80}$$

where $\Delta(\mathbf{n})$ corresponds to a combination of spherical harmonics of a given l value, and $\psi_l(k)$ obeys the integral equation (cf. Sec. III)

$$\psi_l(k) = -\tfrac{1}{2}(dn/d\epsilon) \int V_l(k, k')\psi_l(k')$$
$$\times (\tanh \beta\epsilon_{k'}/2/2\epsilon_{k'})\, d\epsilon_{k'}. \tag{5.81}$$

We normalize ψ_l so that $\psi_l(k) \to 1$ as $\epsilon_k \to 0$. We do not in general know the solution of Eq. (5.81) without a detailed knowledge of $V_l(k, k')$. However, if $V_l(k, k')$ is effectively constant over an energy range ϵ_0, then evidently $\psi_l(k)$ will similarly be constant over this range.

We will now *assume* that (5.80) is valid at all temperatures below T_c [with $\psi_l(k)$ independent of T] and show that this is consistent. We go back to Eq. (5.49), writing out $\langle V \rangle$ explicitly:

$$F = \text{const.} + 2\sum_\mathbf{k} \int \Delta_\mathbf{k}\, dF_\mathbf{k}^* + \sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} F_\mathbf{k} F_{\mathbf{k}'}^*,$$
$$F_\mathbf{k} \equiv (\Delta_\mathbf{k}/2E_\mathbf{k}) \tanh(\tfrac{1}{2}\beta E_\mathbf{k}). \tag{5.82}$$

This equation is of course quite generally valid, independently of the choice of potential. We define as before the quantity $\Psi(\mathbf{n})$:

$$\Psi(\mathbf{n}) \equiv \sum_{|k|} F_\mathbf{k} \equiv \tfrac{1}{2}(dn/d\epsilon) \int F_\mathbf{k}\, d\epsilon_k. \tag{5.83}$$

The crucial observation, now, is that provided $\psi_l(k)$ is effectively constant and equal to 1 over an energy range $\epsilon_0 \gg k_BT_c$ (and hence, *a fortiori*, $\gg \Delta$), we can write to a good approximation *either*

$$F_\mathbf{k} = \{\Delta(\mathbf{n})/2[\epsilon_k^2 + |\Delta(\mathbf{n})|^2]^{1/2}\} \tanh \tfrac{1}{2}\beta[\epsilon_k^2$$
$$+ |\Delta(\mathbf{n})|^2]^{1/2} \qquad (\epsilon_\mathbf{k} \ll \epsilon_0) \tag{5.84}$$

or simply

$$F_k = \Delta(\mathbf{n})\psi_l(k)/2\epsilon_k \qquad (\epsilon_\mathbf{k} \gg k_BT_c) \tag{5.85}$$

(or, of course, in the region $k_BT_c \ll \epsilon_k \ll \epsilon_0$, either of these expressions). When these are substituted into $\Psi(\mathbf{n})$ and it is expanded in terms of $\Delta(\mathbf{n})$ as in Eq. (5.57), we see that while $A(T_c)$ evidently depends on the form of $\psi_l(k)$, neither dA/dT nor $B(T)$ does so (and, more generally, no higher coefficients will do so). Moreover, when we substitute Eqs. (5.80) and (5.84) into (5.82) and re-express the result in terms of $\Delta(\mathbf{n})$ and then $\Psi(\mathbf{n})$, it is only the *quadratic* terms which will be affected by $\psi_l(k)$, not the higher ones [and the temperature dependence of the quadratic terms is independent of $\psi_l(k)$]. Consequently, the only effect of $\psi_l(k)$ is on the transition temperature itself: in fact, putting the quadratic term in $\Psi(\mathbf{n})$ equal to zero precisely gives back Eq. (5.81) for the transition temperature. Equation (5.67) still follows, the only difference being that $A(T_c)$ is in general not given by Eq. (5.60) but by a more general expression involving $\psi_l(k)$. However, when we rewrite the free energy in terms of $\Delta(\mathbf{n})$ rather than $\Psi(\mathbf{n})$ we find as before that $A(T_c)$ drops out of the problem. We conclude therefore that once given T_c, the gap $\Delta(\mathbf{n})$ and hence all thermodynamic properties (see next section) are independent of the high-energy behavior of $V_{\mathbf{k}\mathbf{k}'}$, so that we may take any convenient form which gives the right T_c—e.g., the model form (3.11). An alternative derivation of the corresponding result for the zero-temperature case is given by Anderson and Morel (1961).

One caution is, however, necessary here: although the thermodynamic properties do not depend on the cutoff in Eq. (3.11), it is less obvious that this is true for certain expectation values—e.g., that of the dipole energy (see Sec. X below). It may therefore be necessary for certain purposes to choose an unambiguous value of the cutoff energy ϵ_c [Eq. (3.13)]. If so, probably the most sensible choice is to impose the cutoff at the point where the inverse lifetime of a (normal) quasiparticle becomes of the same order as its energy [for a detailed justification of this procedure, see Morel and Nozières (1962)]. In the case of liquid ^{3}He the resulting value of ϵ_0, converted into temperature units, would be of the order of 1°K.

F. Summary

The major results of this section which we shall need subsequently are:

(1) At zero temperature the wave function of an anisotropic superfluid with spin singlet pairing may plausibly be taken in the form described by Eqs. (5.7)–(5.9). Similarly, at finite temperatures the state of the system is described by Eqs. (5.37)–(5.40). The state of the system in thermal equilibrium is completely described by the set of variational parameters $\Delta_{\mathbf{k}}(T)$.

(2) In the approximation we have adopted [which is analogous to that made by Bardeen *et al.* (1957) for an isotropic superconductor] the parameters $\Delta_{\mathbf{k}}(T)$ satisfy the BCS-like gap equation (5.50). The solutions to this equation have the properties (1)–(4) noted in Sec. V.C.

(3) Near the critical temperature T_c, to the extent that we neglect the mixing in of spherical harmonics with l other than the "dominant" value l_0, the free energy can be written in the form (5.67). The most stable solution is one which minimizes the quantity κ defined in Eq. (5.73).

In what follows we shall often want to distinguish between result (1) which is a very general ansatz concerning the correlation properties of the wave function (or statistical description), and results (2) and (3), which are much more detailed statements about the actual values of the variational parameters $\Delta_{\mathbf{k}}(T)$ appearing in the statistical description [or of quantities like $\Psi(\mathbf{n})$ which give equivalent information]. Recent advances in our understanding of the new phases of ^{3}He have involved the (partial) abandonment of (2) and (3) but not of (1). In the next section, therefore, we shall usually concentrate on expressing physical quantities in terms of the parameters $\Delta_{\mathbf{k}}(T)$ rather than explicitly in terms of temperature. The latter we shall only be able to do after we have investigated (in Sec. IX) the reasons why the results (2) and (3) probably do not apply to real liquid ^{3}He, and have obtained the corrections to the behavior of $\Delta_{\mathbf{k}}$ as a function of T. It should be said at once, however, that the main effect of these corrections is to alter, possibly, the relative stability of different solutions of the gap equation for the same l value: apart from this they do not affect the results of the present section qualitatively. As remarked above, we will call the collection of approximations leading from (1) to (2) and (3) "the BCS approximation."

To conclude, a remark on the appropriateness of the theory of this section to a real Fermi liquid such as ^{3}He. We shall see below (Sec. VI.D) that the quantities Q_{lm} and $\mathbf{R}_{lm}$ of Sec. II (the spherical harmonics of the "net" deformation of the Fermi surface), which are zero in the normal state in equilibrium, remain zero in equilibrium in the superfluid state. Consequently, the pairing process itself produces no molecular fields and the only modification necessary to the above theory is the replacement of the real mass m by the quasiparticle effective mass m^* [or equivalently, the interpretation of $(dn/d\epsilon)$ as the true density of states for the interacting system] and of the interatomic pairing potential $V(\mathbf{k}, \mathbf{k}')$ by an effective pairing potential between quasiparticles. Thus the above theory is more or less ready-made for application to the ^{3}He problem. However, we must remember that when in the next section we calculate responses to external fields, it will be essential to take the molecular fields into account.

VI. THERMODYNAMIC AND CORRELATION PROPERTIES OF THE ANISOTROPIC SUPERFLUID

In the present section we shall derive expressions for some properties of the anisotropic superfluid which are accessible to experimental measurement, and also for some which while not so directly accessible are of considerable theoretical interest. We deal here as in the last section with the case of spin singlet pairing: the results are, with one exception, trivially generalized to the case of triplet pairing which we shall discuss in the next section. For general reference we recall here the principal result of Sec. V.B: at temperature T the possible states of a pair of plane-wave states $\mathbf{k}\uparrow$, $-\mathbf{k}\downarrow$ are a "ground pair" (GP) state which is a linear combination of $|0, 0\rangle$ and $|1, 1\rangle$, an "excited pair" (EP) state which is also a linear combination but orthogonal to the GP state, and two "broken pair" (BP) states in which $\mathbf{k}\uparrow$ and $-\mathbf{k}\downarrow$ are, respectively, alone occupied. The energy of the BP and EP states relative to GP are $E_{\mathbf{k}}(T)$ and $2E_{\mathbf{k}}(T)$, respectively, where $E_{\mathbf{k}} \equiv +[\epsilon_k^2 + |\Delta_{\mathbf{k}}|^2(T)]^{1/2}$.

A. Specific heat

The specific heat per unit volume is most easily calculated from the expression (5.46) for the entropy.[12] In carrying out the differentiation with respect to T we must remember that $E_{\mathbf{k}}$ is itself a function of T. We find

$$C_v \equiv T(dS/dT) = \sum_{\mathbf{k}} \{k_B \tfrac{1}{2}\beta^2[E_{\mathbf{k}} + \beta(dE_{\mathbf{k}}/d\beta)] \times (E_{\mathbf{k}} \operatorname{sech}^2\beta E_k/2)\}. \qquad (6.1)$$

Above T_c, where $E_{\mathbf{k}} \equiv \epsilon_k \equiv$ const., this expression reduces to the well-known expression for the specific heat of a normal Fermi liquid, and is numerically given by Eq. (2.3). At and below T_c the general behavior is much like that of the specific heat of a BCS superconductor, with C_v jumping discontinuously at T_c and then falling sharply at lower temperatures, owing to the exponential fall-off of the function $\operatorname{sech}^2\beta E_k/2$. However, there is a significant difference in the behavior in the limit $T \rightarrow 0$: in the BCS case the gap Δ is constant and, since $E_k \geq \Delta$, the specific heat therefore falls off exponentially in the low-temperature limit. In the case of an anisotropic superfluid, however, the gap $\Delta(\mathbf{n})$ may in general have nodes and the low-temperature specific heat will then be proportional to a power of T. Evidently, if all the nodes are point nodes and the gap $\Delta(\mathbf{n})$ tends to

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[12] The alternative derivation of Anderson and Morel (1961) appears to contain some errors which, however, cancel out in the final result.

zero as the nth power of distance from a particular node on the Fermi surface, then the contribution to the specific heat is proportional to $T^{(2/n)+1}$. The result for the case $n = 1$ is worth quoting explicitly: if we consider the ith node and choose the axes momentarily so that it corresponds to the pole, then by a suitable choice of the zero of the polar angle φ we can always write $\Delta(\mathbf{n})$ near the pole in the form

$$|\Delta(\mathbf{n})|^2 \equiv |\Delta(\theta, \varphi)|^2 \cong \Delta_0^2(a_i^2 \cos^2\varphi + b_i^2 \sin^2\varphi)\theta^2, \tag{6.2}$$

where Δ_0 is the (real) overall magnitude of the gap, at zero temperature, normalized as in Eq. (5.53). The resulting contribution to Eq. (6.1) can be evaluated by taking new variables $x \equiv \Delta_0 a_i \theta \cos\varphi$, $y \equiv \Delta_0 b_i \theta \sin\varphi$, $z = \epsilon$, and then transforming to spherical polar coordinates. If all the nodes correspond to $n = 1$ (as for instance is the case for the most popular model of ³He-A—see Sec. IX), then the total specific heat in the limit $T \to 0$ is

$$C_v(T \to 0) = [24C' k_B^4 (dn/d\epsilon)\Delta_0^{-2} \sum_i a_i^{-1} b_i^{-1}] T^3,$$

$$C' \equiv \sum_{n=1}^{\infty} (-1)^{n+1} n^{-4} = \tfrac{7}{8}\zeta(4), \tag{6.3}$$

where ζ is the Riemann zeta-function.

It is helpful to express this as a fraction of the normal state specific heat at T_c:

$$C_v(T \to 0)/C_n(T_c) = \{[63\zeta(4)/\pi^2](k_B T_c/\Delta_0)^2 \times \sum_i a_i^{-1} b_i^{-1}\} (T/T_c)^3. \tag{6.4}$$

This result does not depend on the BCS approximation, although the actual value of the ratio $(k_B T_c/\Delta_0)$ may do so.

The above results should probably be taken with a pinch of salt, since, as we pointed out in the last section, it is quite possible that the true gap has no nodes, owing to the mixing-in of spherical harmonics with $l \neq l_0$. If so, then the specific heat will be exponential in the low-temperature limit. {Actually not quite, since collective excitations [see Sec. XII] will contribute a power-law term, whose coefficient is, however, very small.} Nevertheless, if the mixing is weak, one may expect that the nodes turn into low minima of order $\Delta_m \ll k_B T_c$, and one may then hope that in the temperature region $\Delta_m \ll k_B T \ll k_B T_c$ the above results may be approximately valid.

In the opposite limit $(T \to T_c)$ we see that the term in Eq. (6.1) proportional to E_k^2 will give just the normal state specific heat. The specific heat *jump* ΔC_v defined as the difference between C_v just below and just above T_c, is therefore given by

$$\Delta C_v = \tfrac{1}{2} k_B \beta_c^3 \sum_{\mathbf{k}} [E_{\mathbf{k}}(dE_{\mathbf{k}}/d\beta)] \operatorname{sech}^2(\tfrac{1}{2}\beta_c E_{\mathbf{k}})$$

$$= \tfrac{1}{2}(dn/d\epsilon)[-(d\Delta^2/dT)]_{T_c} \tag{6.5}$$

[where we used Eqs. (5.39), (5.53), and (5.54)]. In Sec. IX we shall see that quite generally (independent of the BCS approximation) it turns out that in the limit $T \to T_c$ we have

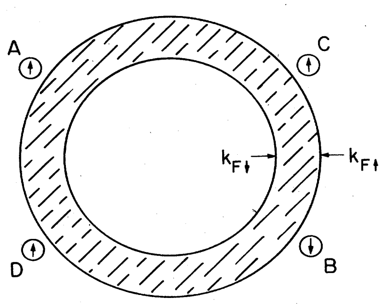

FIG. 3. Effect of a magnetic field on the formation of Cooper pairs. Particles A and B cannot form a pair, but particles C and D can (see Sec. VII.A).

$$\Delta^2(T) = c(k_B T_c)^2(1 - T/T_c), \tag{6.6}$$

where c is some positive numerical constant. Substituting this in Eq. (6.5) we find

$$\Delta C_v/C_n(T_c) = 3c/2\pi^2. \tag{6.7}$$

In the BCS approximation the constant c is 9.3 κ^{-1} [cf. Eq. (5.76)] and hence

$$[\Delta C_v/C_n(T_c)]_{\mathrm{BCS}} = 1.42\kappa^{-1} \leq 1.42. \tag{6.8}$$

B. Spin susceptibility

The spin susceptibility of an anisotropic superfluid with spin singlet pairing is strongly reduced in low magnetic fields. To see this qualitatively, we notice that the wave function of a Fermi superfluid involves the pairing of particles over the Fermi surface with opposite spin and opposite momentum. Now imagine that while the system is in the normal phase we apply to it a weak magnetic field; the effect of this is to split the "up-spin" and "down-spin" Fermi surfaces as shown in Fig. 3. If now we try to form Cooper pairs with opposite spin and momentum, then, crudely speaking, the up-spins in the shaded region are excluded from the pairing and so the energy of condensation is reduced. The system then has the choice between maintaining its original polarization and thus losing condensation energy, or reducing the polarization so as to allow more pairing. It turns out that at low fields it is energetically advantageous to do the second, so the susceptibility is reduced.

Let us now consider the effect quantitatively. We first neglect Fermi liquid effects. If a weak magnetic field H is applied to the system, the energies of the single-particle states $\mathbf{k}\uparrow$, $\mathbf{k}\downarrow$ are shifted:

$$\epsilon_{k\uparrow} = \epsilon_k - \tfrac{1}{2}\mu H, \quad \epsilon_{k\downarrow} = \epsilon_k + \tfrac{1}{2}\mu H \qquad (\mu \equiv \gamma\hbar). \tag{6.9}$$

To first order in H the energies of the GP and EP states are not affected, since they are linear combinations of $|0, 0\rangle$ and $|1, 1\rangle$. The energy of the BP state $|1, 0\rangle$ is, however, shifted down by $\tfrac{1}{2}\mu H$, and its energy relative to the GP state therefore becomes $E_{\mathbf{k}} - \frac{1}{2}\mu H$. Similarly, the energy (relative to GP) of $|0, 1\rangle$ becomes $E_{\mathbf{k}} + \frac{1}{2}\mu H$. So the probability of occupation of $|1, 0\rangle$ and $|0, 1\rangle$ is now

$$P(1,0) = n^{-1}\exp[-\beta(E_{\mathbf{k}} - \tfrac{1}{2}\mu H)],$$
$$P(0,1) = n^{-1}\exp[-\beta(E_{\mathbf{k}} + \tfrac{1}{2}\mu H)] \tag{6.10}$$

where the normalization factor n is unaffected to first order in H and given as in Eq. (5.42) by

$$n = [1 + 2\exp(-\beta E_{\mathbf{k}}) + \exp(-2\beta E_{\mathbf{k}})]. \tag{6.11}$$

Since $|1, 0\rangle$ and $|0, 1\rangle$ contribute, respectively, $\frac{1}{2}\mu$ and $-\frac{1}{2}\mu$ to the total magnetization while GP and EP contribute nothing, we have

$$M = \tfrac{1}{2}\mu \sum_{\mathbf{k}} [P(1,0) - P(0,1)] \approx \tfrac{1}{4}\mu^2 H \sum_{\mathbf{k}} \tfrac{1}{2}\beta\,\mathrm{sech}^2\tfrac{1}{2}\beta E_{\mathbf{k}},$$
$$= \tfrac{1}{4}\mu^2 H (dn/d\epsilon) \int (d\Omega/4\pi) \int_0^\infty d\epsilon_k \tfrac{1}{2}\beta\,\mathrm{sech}^2\tfrac{1}{2}\beta E_{\mathbf{k}}. \tag{6.12}$$

Thus, the susceptibility χ_0 obtained neglecting Fermi liquid effects is

$$\chi_0 = \tfrac{1}{4}\mu^2(dn/d\epsilon)Y(T) \equiv \tfrac{1}{4}\gamma^2\hbar^2(dn/d\epsilon)Y(T), \tag{6.13}$$

where

$$Y(T) \equiv \int (d\Omega/4\pi)\, Y(\mathbf{n}\colon T),$$
$$Y(\mathbf{n}, T) \equiv \int_0^\infty d\epsilon_k \tfrac{1}{2}\beta\,\mathrm{sech}^2\tfrac{1}{2}\beta E_{\mathbf{k}}. \tag{6.14}$$

The function $Y(T)$ is the so-called Yosida function, suitably generalized to the case of an anisotropic gap. In the normal case ($\Delta_{\mathbf{k}} \equiv 0$, $E_{\mathbf{k}} \equiv \epsilon_k$) is is of course equal to 1 and we recover the formula (2.3). In the GL region (near T_c) we can expand $Y(\mathbf{n}, T)$ in powers of $\Delta(\mathbf{n}) \equiv \Delta f(\mathbf{n})$:

$$Y(\mathbf{n}, T) = 1 - \tfrac{7}{4}\zeta(3)(\Delta^2/\pi^2 k_B{}^2 T_c{}^2)\,|f(\mathbf{n})|^2 + \cdots \tag{6.15}$$

so that

$$1 - Y(T) \approx \tfrac{7}{4}\zeta(3)[\Delta^2(T)/\pi^2 k_B{}^2 T_c{}^2]. \tag{6.16}$$

If furthermore we take $\Delta(T)$ to have its BCS value (5.76), we get

$$1 - Y(T)_{\mathrm{BCS}} \approx 2\kappa^{-1}(1 - T/T_c). \tag{6.17}$$

At lower temperatures the Yosida function will depend [as in Eq. (6.14)] on the specific form of the gap function, and also on whether the BCS approximation is valid: however its general form is always qualitatively like that illustrated in Fig. 4 [cf. Anderson and Morel (1961), Fig. 3]. The Yosida function is in some sense a measure of the fraction of the particles near the Fermi surface which are free to redistribute themselves, or the "effective density of states near the Fermi surface"; it should not be confused with the "density of the normal component" or "normal fraction" ρ_n/ρ discussed below.

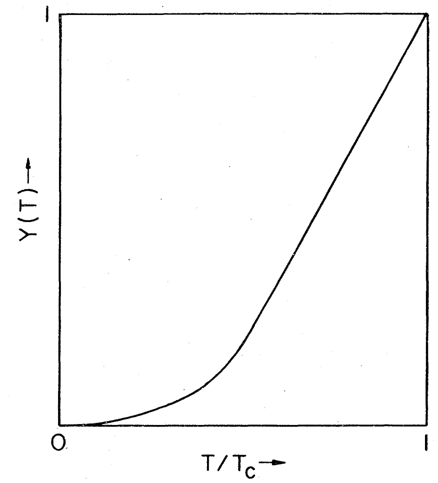

FIG. 4. General form of the Yosida function $Y(T)$.

We now consider Fermi liquid corrections to formula (6.13). It is evident from the derivation above that the induced magnetization associated with different bits of the Fermi surface is in general different, i.e., the polarization is anisotropic, so that strictly speaking we should consider all possible molecular field parameters Z_l. However, since $\Delta_{\mathbf{k}}$ and hence $E_{\mathbf{k}}$ is an even function of $\mathbf{n}$, only those fields associated with even l in Eqs. (2.20)–(2.23) come into play. Since we do not know Z_l for $l > 1$, we shall simply neglect all the Z_l except Z_0. It may be shown [see Leggett (1965), Sec. 4], that this result is exact to lowest order in $(1 - T/T_c)$ and also (for singlet pairing only) as $T \to 0$, where it reduces to Eq. (6.13): it seems unlikely that the error will be very great even in the intermediate region.[13] We can then simply use directly the techniques of Section II to obtain Eq. (2.25), where, however, χ_0 is now the value calculated for the *superfluid* phase neglecting Fermi liquid corrections. Substituting Eq. (6.13) for χ_0, we get

$$\chi(T) = \tfrac{1}{4}\gamma^2\hbar^2(dn/d\epsilon)[Y(T)/1 + \tfrac{1}{4}Z_0 Y(T)]. \tag{6.18}$$

This formula reduces to Eq. (2.25) in the limit $T \to T_c$ and to Eq. (6.13) in the limit $T \to 0$; this is quite natural since all enhancement effects vanish as the unenhanced susceptibility tends to zero. For $Z_0 \sim -3$, the value for real liquid ³He, the difference between Eqs. (6.18) and (6.13) is very pronounced; in Fig. 5 we show the approximate form of $\chi(T)$ from Eq. (6.18) as a fraction of the normal state susceptibility χ_n.

C. Normal density

The so-called normal density of a superfluid may be defined as follows. We imagine that the system is placed, for instance, in a tube and the walls are then moved with velocity $\mathbf{v}$ and the system allowed to come to equilibrium.

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[13] The expression obtained if Z_2 (but not higher Z_l's) is finite has been written down explicitly by Wölfle [1974 (Eq. 26)].

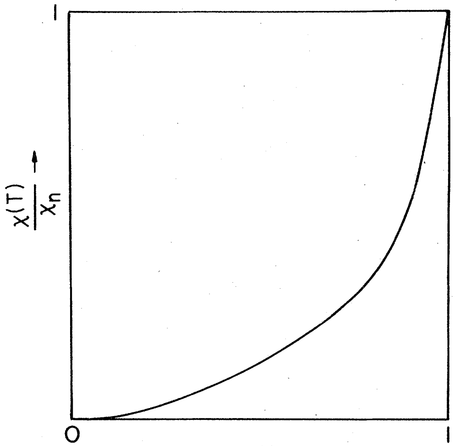

FIG. 5. Magnetic susceptibility of a superfluid with singlet pairing corrected for Fermi-liquid effects [Eq. (6.18)].

It is then *assumed* that the Cooper pairs do not change their wave function when this happens, and the normal density ρ_n is defined as the ratio of the equilibrium momentum of the system to the velocity $\mathbf{v}$: i.e., for the moment at least, by the formula

$$\mathbf{P} = \rho_n \mathbf{v}. \tag{6.19}$$

Whether or not this definition of the normal density is applicable to a specific situation depends, of course, on whether the assumption that the Cooper pairs do not change their wave function, i.e., remain at rest, is justified or not. In many physical situations of interest strong arguments in favor of this assumption can be given. For instance, if the tube is bent round to form an annulus then the condition of single valuedness on the pair wave function $\varphi(\mathbf{r}_1\mathbf{r}_2\sigma_1\sigma_2)$ implies that the center-of-mass angular momentum associated with the wave function is quantized and hence the flow velocity is quantized also; in that case, it usually happens that the original wave function corresponding to zero flow velocity is highly metastable and the Cooper pairs do indeed stay at rest. This, however, is a somewhat delicate subject (the problem is common to superconductors and liquid He II as well as Fermi superfluids), and we refer the reader to the literature (e.g., Langer and Fisher, 1967).

We can also define the *superfluid density* ρ_s as the ratio of the momentum to the velocity of the Cooper pairs when the rest of the liquid is at rest. Such a situation is realized, for instance, in "fourth sound" experiments carried out in narrow tubes where the nonsuperfluid (normal) component (that is, crudely speaking, the particles in broken pair states) are forced by collisions with the stationary walls to remain in an equilibrium distribution appropriate to zero net flow. Evidently, from Galilean invariance we have the result

$$\rho_s = \rho - \rho_n, \tag{6.20}$$

where ρ is the total mass density of the liquid. In defining the superfluid density it is understood that the Cooper pairs are moving *uniformly*, that is, that the pair wave function is of the form (we omit spin variables)

$$\varphi(\mathbf{r}_1\mathbf{r}_2) = \exp i\mathbf{K}\cdot\mathbf{R}\varphi_0(\mathbf{r}_1 - \mathbf{r}_2), \qquad R \equiv (\mathbf{r}_1 + \mathbf{r}_2)/2, \tag{6.21}$$

where φ_0 is the equilibrium wave function at rest and the velocity $\mathbf{v}_s$ of the pairs is defined by

$$\mathbf{v}_s = \hbar\mathbf{K}/2m. \tag{6.22}$$

Under these conditions, if the rest of the liquid is also moving with some velocity $\mathbf{v}_n$, we can combine Eq. (6.19) and the definition of ρ_s to give

$$\mathbf{P} = \rho_s\mathbf{v}_s + \rho_n\mathbf{v}_n \tag{6.23}$$

which is one of the basic equations of superfluid hydrodynamics. It should be emphasized, however, that Eq. (6.23) is only valid when the motion of the Cooper pairs is (at least locally) of the simple type described by Eq. (6.21) (cf. Sec. X.D below).

To calculate the normal density we proceed from its definition as follows. If the walls are moving uniformly with velocity $\mathbf{v}$, then the equilibrium state of the system is obtained by minimizing not the original free energy F but $F - \mathbf{v}\cdot\mathbf{P}$ (i.e., by minimizing the free energy in the form of the moving walls). This is equivalent to replacing the kinetic energy ϵ_k of a particle in the plane-wave state $\mathbf{k}$ by $\epsilon_k - \mathbf{v}\cdot\mathbf{p}$, $\mathbf{p} \equiv \hbar\mathbf{k}$. The calculation is now very similar to that for the spin susceptibility. The GP and EP states have their energy unshifted, at least to order $\mathbf{v}$. (Here we implicitly use the assumption that the pairing is still between states $\mathbf{k}$ and $-\mathbf{k}$ and not, say, between $\mathbf{k} + \mathbf{K}/2$ and $-\mathbf{k} + \mathbf{K}/2$; this is evidently equivalent to the assumption that the Cooper pairs remain at rest.) The energy of $|1\,0\rangle$ is shifted down by $-\hbar\mathbf{v}\cdot\mathbf{k}$, and that of $|0\,1\rangle$ up by the same amount. Omitting most of the intermediate steps, which are exactly analogous to those for the susceptibility, we find

$$\mathbf{P} = \sum_{\mathbf{k}} \hbar\mathbf{k}\{P(1,0) - P(0,1)\} = \sum_{k} \hbar^2\mathbf{k}\mathbf{v}\cdot\mathbf{k}\tfrac{1}{2}\beta\,\mathrm{sech}^2\tfrac{1}{2}\beta E_k. \tag{6.24}$$

Evidently, for an anisotropic gap (hence $E_\mathbf{k}$ anisotropic) the momentum $\mathbf{P}$ is in general not parallel to the velocity $\mathbf{v}$. The defining equation (6.15) must therefore be replaced by

$$\mathbf{P}_i = \sum \rho_{ij}{}^{n0} v_j, \tag{6.25}$$

where the *tensor* superfluid density $\rho_{ij}{}^{n0}$ is given by (the superscript zero indicates that Fermi liquid effects have not yet been taken into account):

$$\rho_{ij}{}^{n0} = \sum_{\mathbf{k}} \hbar^2 k_i k_j \tfrac{1}{2}\beta\,\mathrm{sech}^2\tfrac{1}{2}\beta E_\mathbf{k} = \tfrac{1}{3}\hbar^2 k_F{}^2(dn/d\epsilon)Y_{ij}(T)$$

$$= Nm^* Y_{ij}(T), \tag{6.26}$$

where [cf. the definition (6.14)]

$$Y_{ij}(T) \equiv 3\int (d\Omega/4\pi)\, n_i n_j \int_0^\infty d\epsilon_k \tfrac{1}{2}\beta\, \mathrm{sech}^2 \tfrac{1}{2}\beta E_{\mathbf{k}}$$

$$\equiv 3\int (d\Omega/4\pi)\, n_i n_j Y(\mathbf{n}: T) \tag{6.27}$$

and we used Eq. (2.5) to obtain the last form of Eq. (6.26). For an isotropic system we have $Y_{ij}(T) = \delta_{ij} Y(T)$: more generally we evidently have (regarding Y_{ij} as components of a matrix Y)

$$\mathrm{Tr}\hat{Y}(T) = 3Y(T). \tag{6.28}$$

Evidently the principal axes of $\hat{Y}$ will be determined by the orientation of the gap function $\Delta(\mathbf{n})$. The factors which determine this orientation will be discussed in Sec. X.

The result (6.26) is actually inconsistent, since for $E_{\mathbf{k}} = |\epsilon_{\mathbf{k}}|$ (normal state) the function $Y_{ij}(T)$ is equal to δ_{ij} and hence the normal density is Nm^* rather than the real mass density Nm. This inconsistency is removed when we consider the Fermi liquid corrections. Evidently a uniform motion of all or part of the liquid induces no spin polarization, nor any change of the total density, so the only Landau parameters to come in are the F_l with $l \geq 1$. We will only keep F_1; the arguments showing that this is exact in the limit $T \to T_c$ and probably a good approximation at all T are parallel to those developed in the last section for dropping Z_l with $l > 0$.

To take account of the effect of the molecular field associated with F_1 we use a method parallel to that used in Sec. II to deal with Z_0. According to the argument above, imposing a velocity $\mathbf{v}$ on the walls of the system has the same effect on the Hamiltonian as imposing an external potential $\mathbf{A}_{\mathrm{ext}}$. Replacing $\mathbf{v}$ by $\mathbf{A}_{\mathrm{ext}}$ in the definition of the tensor $\hat{\rho}_n$ and using Eq. (2.19), we can therefore write in analogy with Eq. (2.24)

$$\mathbf{P} = \hat{\rho}_{n0}\mathbf{A}_{\mathrm{total}}, \qquad \mathbf{A}_{\mathrm{total}} = \mathbf{A}_{\mathrm{ext}} + \mathbf{A}_{\mathrm{mol}},$$

$$\mathbf{A}_{\mathrm{mol}} = -(F_1/p_F^2)(dn/d\epsilon)^{-1}\mathbf{P}. \tag{6.29}$$

Solving these equations for $\mathbf{P}$ in terms of $\mathbf{A}_{\mathrm{ext}}$, we find

$$\mathbf{P} = \hat{\rho}_n \mathbf{A}_{\mathrm{ext}}, \qquad \hat{\rho}_n \equiv \frac{\hat{\rho}_{n0}}{1 + F_1 p_F^{-2}(dn/d\epsilon)^{-1}\hat{\rho}_{n0}}$$

$$= \frac{Nm^*\hat{Y}(T)}{1 + \frac{1}{3}F_1\hat{Y}(T)}. \tag{6.30}$$

This result is consistent in the normal limit $[\hat{Y}(T) = \hat{1}]$ if and only if the parameters m^* and F_1 are related by the "Landau effective-mass relation" (2.42), namely

$$m^*/m = 1 + \tfrac{1}{3}F_1 \tag{6.31}$$

which is a well-known result of the theory of a normal Fermi liquid (Landau, 1956).

Near T_c where $\hat{Y}(T)$ is only weakly different from $\hat{1}$, it follows from Eq. (6.30) that the superfluid density tensor has the form

$$\hat{\rho}_s(T)/\rho = [1 - \hat{Y}(T)]/(1 + \tfrac{1}{3}F_1). \tag{6.32}$$

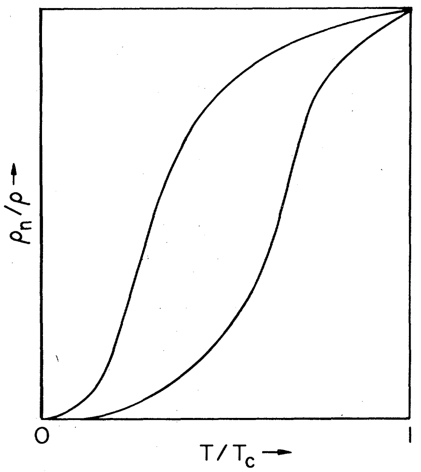

FIG. 6. Temperature dependence of two different eigenvalues of the normal density tensor (qualitative).

Since F_1 is large for liquid ³He (~6–15), depending on pressure, the effect of the Fermi liquid corrections is to depress $\hat{\rho}_s/\rho$ considerably below its value for a weakly interacting gas. Near T_c we can obtain a more explicit expression for the anisotropy of $\hat{\rho}_s(T)$ by expanding $1 - \hat{Y}(T)$ in powers of $|\Delta(\mathbf{n})|^2$:

$$\hat{\rho}_{ij}{}^s(T)/\rho = \{[1 - Y(T)]/(1 + \tfrac{1}{3}F_1)\} \int (d\Omega/4\pi) \times 3 n_i n_j |f(\mathbf{n})|^2. \tag{6.33}$$

Near T_c, therefore, the absolute magnitude of $\hat{\rho}_s$ (or $\hat{\rho}_n$) is affected by the Fermi liquid terms but the anisotropy is not. On the other hand, at lower T the anisotropy can be very much enhanced by the Fermi liquid effects: in Fig. 6 we sketch qualitatively the temperature dependence of two different eigenvalues of the normal density tensor where $F_1 \sim 15$. We see that in one temperature region the liquid can be effectively normal in one direction and almost completely superfluid in another!

D. Spatial correlations: angular momentum

Let us first briefly consider how the superfluid condensation changes the *equilibrium* expectation value of "one-particle" quantities,[14] that is, those of the form

$$M^{(1)} = \sum_{i=1}^{N} f(\mathbf{r}_i, \mathbf{p}_i, \boldsymbol{\sigma}_i). \tag{6.34}$$

Since the system remains homogeneous on condensation (in the absence of boundary effects, etc.) the expectation value cannot depend on $\mathbf{r}_i$. It is also intuitively obvious that in the case of a superfluid with singlet pairing there can be no dependence on $\boldsymbol{\sigma}_i$, since the singlet spin function is invariant under spin rotation and any function of $\boldsymbol{\sigma}_i$ for a particular spin $\frac{1}{2}$ can be written $\alpha + \boldsymbol{\beta}\cdot\boldsymbol{\sigma}_i$, where $\boldsymbol{\beta}$ must

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[14] Strictly speaking, one-quasiparticle quantities: $\mathbf{r}_i$, etc. should strictly refer to the quasiparticles. This does not affect the qualitative conclusions. "Equilibrium" here means "in thermal equilibrium in the absence of external fields."

transform like a spin vector. In view of this the expectation value of the quantity M can always be written in the form

$$\langle M^{(1)}\rangle = \sum_{\mathbf{k}} f_{\mathbf{k}}\langle n_{\mathbf{k}}\rangle. \tag{6.35}$$

For instance, if $M^{(1)}$ is the total particle number then $f_{\mathbf{k}} = 1$, while if $M^{(1)}$ is the total momentum then $f_{\mathbf{k}} = \hbar\mathbf{k}$.

The expectation value of $\langle n_{\mathbf{k}}\rangle$ is simply Eq. (5.41) divided by $\epsilon_{\mathbf{k}}$:

$$\langle n_{\mathbf{k}}\rangle = [1 - (\epsilon_{\mathbf{k}}/E_{\mathbf{k}}) \tanh\tfrac{1}{2}\beta E_{\mathbf{k}}]. \tag{6.36}$$

Hence the difference of $\langle M\rangle$ from its normal state value $\langle M\rangle_n$ (which can be obtained simply by letting $E_{\mathbf{k}} \to |\epsilon_{\mathbf{k}}|$) is

$$\langle M^{(1)}\rangle - \langle M_n^{(1)}\rangle = \sum_{\mathbf{k}} f_{\mathbf{k}}[\tanh\tfrac{1}{2}\beta\epsilon_{\mathbf{k}} - (\epsilon_{\mathbf{k}}/E_{\mathbf{k}}) \tanh\tfrac{1}{2}\beta E_{\mathbf{k}}]. \tag{6.37}$$

If we transform the sum in (6.37) into an integral in the standard way and neglect the small variation in the density of states as a function of ϵ (see below) we find

$$\langle M^{(1)}\rangle - \langle M_n^{(1)}\rangle = \tfrac{1}{2}(dn/d\epsilon) \int (d\Omega/4\pi) \int d\epsilon_{\mathbf{k}} f(\mathbf{n}, \epsilon_{\mathbf{k}}) \times [\tanh\tfrac{1}{2}\beta\epsilon_{\mathbf{k}} - (\epsilon_{\mathbf{k}}/E_{\mathbf{k}}) \tanh\tfrac{1}{2}\beta E_{\mathbf{k}}]. \tag{6.38}$$

Hence if $f(\mathbf{n}, \epsilon_{\mathbf{k}})$ is an even function of $\epsilon_{\mathbf{k}}$ (in particular, a constant), then $\langle M^{(1)}\rangle = \langle M_n^{(1)}\rangle$ identically. If we take into account the fact that $dn/d\epsilon$ is actually a function of ϵ, then since the quantity in brackets is appreciable only for $|\epsilon_{\mathbf{k}}| \leq \Delta(T)$, we find a correction of order $[f\, d/d\epsilon(dn/d\epsilon) \cdot \Delta^2] \sim Nf(\Delta/\epsilon_F)^2$, where f is a typical value of $f(\mathbf{n}, \epsilon_{\mathbf{k}})$. Since $(\Delta/\epsilon_F)^2 \sim 10^{-7}$, this correction is generally negligible for most purposes (though we shall see that it is important for a detailed understanding of the A transition).

An important special case of this result is that all the spherical harmonics of the deformation of the "net" Fermi surface, Q_{lm} and $\mathbf{R}_{lm}$, (Sec. II.B) are zero in the *equilibrium* superfluid state (i.e., in the absence of external fields) just as in the equilibrium normal state. Consequently the generalized molecular fields are also always zero, which is why we may legitimately treat the condensation of Landau quasiparticles simply by replacing m by m^* and $V(\mathbf{k}, \mathbf{k}')$ by the quasiparticle pairing interaction in the method of Sec. V (cf. Sec. V.F).

Quite generally, we see that only a very few special one-particle operators [such as $K - \mu N$, Eq. (5.43)] will have their expectation values appreciably affected by the superfluid transition.

The situation is very different when we come to *two-particle* operators, that is, those of the form

$$M^{(2)} = \tfrac{1}{2}\sum_{ij} f(\mathbf{r}_i, \mathbf{r}_j, \mathbf{p}_i, \mathbf{p}_j, \boldsymbol{\sigma}_i, \boldsymbol{\sigma}_j). \tag{6.39}$$

We might already guess, from the form of the pseudo-molecular system wave function (5.2), that such quantities might be quite radically affected by the superfluid condensation. Let us first consider the case in which $M^{(2)}$ has no momentum or spin dependence:

$$M^{(2)} = \tfrac{1}{2}\sum_{ij} g(\mathbf{r}_i - \mathbf{r}_j), \qquad g(\mathbf{r}) = \sum_{\mathbf{k}} g(\mathbf{k}) \exp(i\mathbf{k}\cdot\mathbf{r}). \tag{6.40}$$

The contributions to $\langle M^{(2)}\rangle$ which arise specifically from pair formation may be calculated in exact analogy to the calculation of $\langle V\rangle$ in Sec. V. (In both cases there are other contributions, but the most important ones are of the "Hartree–Fock" type and depend on $\langle n_{\mathbf{k}}\rangle$, so by the argument above they are essentially the same as in the normal phase.) We find

$$\langle M^{(2)}\rangle_{(\text{pairs})} = \sum_{\mathbf{k}\mathbf{k}'} g(\mathbf{k} - \mathbf{k}') F_{\mathbf{k}} F_{\mathbf{k}'}^*, \tag{6.41}$$

where $F_{\mathbf{k}}$ is given by Eq. (5.45). At this stage it is helpful to define the Fourier transform of $F_{\mathbf{k}}$:

$$F(\mathbf{r}) \equiv \sum_{\mathbf{k}} F_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r}). \tag{6.42}$$

Then we find

$$\langle M^{(2)}\rangle_{\text{pairs}} = \int d\mathbf{r} g(\mathbf{r}) \,|F(\mathbf{r})|^2. \tag{6.43}$$

Consequently, $F(\mathbf{r})$ has the physical significance of a sort of wave function for the Cooper pairs. One easily checks from Eq. (5.44) that in second-quantized language $F(\mathbf{r})$ is given by

$$F(\mathbf{r}) = \langle \psi_\downarrow(\mathbf{R} - \mathbf{r}/2)\psi_\uparrow(\mathbf{R} + \mathbf{r}/2)\rangle, \tag{6.44}$$

where $\psi_\uparrow(\mathbf{x})$ creates a particle of spin-up at $\mathbf{x}$, etc. From Eq. (5.45) the explicit form of $F(\mathbf{r})$ is

$$F(\mathbf{r}) = \sum_{\mathbf{k}} (\Delta_{\mathbf{k}}/2E_{\mathbf{k}}) \tanh\tfrac{1}{2}\beta E_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r}). \tag{6.45}$$

This function has properties very similar to those of the wave function of the Cooper problem, discussed in Sec. III.

In fact, if $\Delta_{\mathbf{k}} = \Delta(T)f(\mathbf{n})$ as in Eq. (5.53), then we find *approximately* (exactly in the limit $T \to T_c$)

$$F(\mathbf{r}) = f(\hat{\mathbf{r}})\tilde{F}(r). \tag{6.46}$$

At not too large distances $\tilde{F}(r)$ has the form of the radial function of two free particles at the Fermi surface with relative angular momentum l. However, for $r \gg \xi_0 \equiv \hbar v_F/\pi k_B T_c$ it falls off exponentially,[15] giving a bound state:

$$\tilde{F}(r) \sim \exp(-r/\xi'), \qquad r \gg \xi_0\colon \xi' \sim \xi_0 \equiv \hbar v_F/\pi k_B T_c. \tag{6.47}$$

Thus, the size of a Cooper pair is of order ξ_0. For liquid ³He the quantity ξ_0 is of the order of a few hundred Å.[16] Notice,

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[15] Whether or not the fall-off is strictly exponential depends on the way we treat the cut-off. Cf. Rickayzen (1965), Sec. 4.8.

[16] Definitions of ξ_0 differing from Eq. (6.47) by various numerical factors may be found in the literature.

incidentally, that the situation is qualitatively somewhat different at zero temperature: in that case the pair wave function has a range $\hbar v_F/|\Delta(\mathbf{n})|$ which is strongly anisotropic (see Anderson and Morel, 1961, Appendix C). The result (6.47) is qualitatively valid if $k_B T$ is not too small compared to Δ; the more general case will not be discussed here.

The maximum value of $F(\mathbf{r})$ is of order $\Psi(T)$ (cf. Eqs. (5.56) and (5.65), which in turn is of order $(\Delta/\epsilon_F)\ln\beta_c\epsilon_c$. Thus the change in $\langle M^{(2)}\rangle$ due to the formation of pairs, although of order $(\Delta/\epsilon_F)^2$, is multiplied by the factor $(\ln\beta_c\epsilon_c)^2$, which may be fairly large (for ³He it is probably at least 40). Certainly, in the limit of extreme weak coupling $\beta_c\epsilon_c \rightarrow \infty$) the "direct" effect of Cooper pair formation which we have just calculated far outweighs the indirect effects which arise, for instance, from the change of $\langle n_\mathbf{k}\rangle$ and are of order $(\Delta/\epsilon_F)^2$. Exactly what error is introduced in liquid ³He by omitting these "indirect" effects is not a simple question, but it is usually hoped that it is small (cf. Sec. IX, below).

The integral

$$\int d\mathbf{r}\,|\mathbf{F}(\mathbf{r})|^2 = \sum_\mathbf{k} (|\Delta_\mathbf{k}|^2/4E_\mathbf{k}^2)\tanh^2\tfrac{1}{2}\beta E_\mathbf{k} \equiv n_c(T) \tag{6.48}$$

is evidently of order $N\Delta/\epsilon_F$ (for $k_BT \lesssim \Delta$) rather than $N(\Delta/\epsilon_F)^2$. Following Anderson and Morel (1961)[17] we may regard it as the "number of condensed pairs": the reason for this identification will become clear shortly.

We now consider more general two-particle operators of the form

$$M^{(2)} = \tfrac{1}{2}\sum_{ij} g(\mathbf{r}_i - \mathbf{r}_j, \mathbf{p}_i - \mathbf{p}_j). \tag{6.49}$$

The expression for the contribution to $M^{(2)}$ from pair formation may be worked out either by analogy with the calculation of $\langle V\rangle$ in Sec. V (where we remember that the relative momentum of a state $|\mathbf{k}\uparrow, -\mathbf{k}\downarrow\rangle$ is $2\mathbf{k}$) or by second-quantizing $M^{(2)}$ and factorizing the resultant average similarity to Eq. (5.25). After a certain amount of calculation we find the intuitively plausible result

$$\langle M^{(2)}\rangle_{\text{pairs}} = \int F^*(\mathbf{r})g(\mathbf{r}, -i\hbar\nabla)F(\mathbf{r})\,d\mathbf{r}. \tag{6.50}$$

A particularly interesting case of Eq. (6.50) is the *relative* orbital angular momentum

$$\mathbf{L}_{\text{rel}} \equiv -\tfrac{1}{2}i\hbar\sum_{ij}(\mathbf{r}_i - \mathbf{r}_j)\times(\mathbf{p}_i - \mathbf{p}_j). \tag{6.51}$$

According to Eq. (6.50) we find

$$\langle\mathbf{L}_{\text{rel}}\rangle_{\text{pairs}} = -i\hbar\int F^*(\mathbf{r})\cdot(\mathbf{r}\times\nabla)F(\mathbf{r})\,d\mathbf{r} \tag{6.52}$$

and after a little algebra this becomes

$$\langle\mathbf{L}_{\text{rel}}\rangle_{\text{pairs}} = -i\hbar n_c(T)\int(d\Omega/4\pi)f^*(\mathbf{n})(\mathbf{n}\times\partial/\partial\mathbf{n})f(\mathbf{n}), \tag{6.53}$$

where $f(\mathbf{n})$ is the function defined by Eq. (5.65) which expresses the anisotropy of the order parameter (or, approximately, of the gap). If for instance $f(\mathbf{n}) = h(\theta)e^{im\phi}$, then the relative orbital angular momentum is $m\hbar n_c(T)$. In general the relative angular momentum is of order $\hbar n_c$, i.e., $\hbar N(\Delta/\epsilon_F)$. [The above results are derived and discussed in more detail by Anderson and Morel (1961), whose $I(\mathbf{r})$ is identical to our $F(\mathbf{r})$ apart from a factor of 2.]

However, it cannot be too strongly emphasized that this does *not* necessarily imply that the *total* orbital angular momentum of the system is of order $N(\Delta/\epsilon_F)$, even if the shape of the system is such that boundary effects do not make $\langle L_{\text{rel}}\rangle$ cancel out over the system as a whole (cf. Sec. X). In fact, the total angular momentum is

$$\mathbf{L} = N^{-1}\mathbf{L}_{\text{rel}} + \mathbf{L}_{\text{c.o.m.}}, \qquad \mathbf{L}_{\text{c.o.m.}} = \mathbf{R}\times\mathbf{P}, \tag{6.54}$$

where $\mathbf{R}$ and $\mathbf{P}$ are respectively the center-of-mass coordinate and total momentum operator of the system. The explicit contribution of the "pair relative angular momentum" (6.51) to the total angular momentum of the system is therefore of order 1, not of order N; and to the best of my knowledge there exists no rigorous argument relating $\mathbf{L}_{\text{c.o.m.}}$ to $\mathbf{L}_{\text{rel}}$ (that of Anderson and Morel (1961) is at best suggestive). In fact this rather fundamental question—whether the total angular momentum of an anisotropic superfluid is finite and, if so, what order it is in Δ/ϵ_F—is at the moment highly controversial; a definitive answer may well require the solution of the $l \neq 0$ pairing problem with "realistic" (nonperiodic) boundary conditions, which to the author's knowledge has not been achieved to date and seems to be a formidable task.

VII. SPIN TRIPLET PAIRING

In the last two sections we developed the theory of an anisotropic superfluid for the case of spin singlet pairing, that is, when the "pseudomolecular" wave function has the form (5.5). In this section we shall consider the rather more complicated case where the pseudomolecular wave function has the form of a triplet wave function in spin space. (The question of mixed singlet and triplet pairing is considered briefly in Sec. VII.C.)

Two particles in a state of total spin 1 can occupy any of three substates corresponding to the three values of spin projection S_z: in the intuitive notation used in Eq. (5.5), these are

$$|\uparrow\uparrow\rangle, S_z = +1: \quad (\sqrt{2})^{-1}|\uparrow\downarrow + \downarrow\uparrow\rangle, S_z = 0: \quad |\downarrow\downarrow\rangle, S_z = -1. \tag{7.1}$$

Consequently, the pseudomolecular wave function $\varphi(\mathbf{r}_1 - \mathbf{r}_2: \sigma_1\sigma_2)$ [Eq. (5.1)] [or its Fourier transform $\chi(\mathbf{k}: \sigma_1\sigma_2)$, Eq. (5.3)] can be written for spin triplet pairing in the general form (the normalization of $\varphi_{\uparrow\downarrow}$ is chosen with an eye to subsequent convenience)

$$\varphi(\mathbf{r}_1 - \mathbf{r}_2: \sigma_1\sigma_2) = \varphi_{\uparrow\uparrow}(\mathbf{r}_1 - \mathbf{r}_2)|\uparrow\uparrow\rangle + \varphi_{\uparrow\downarrow}(\mathbf{r}_1 - \mathbf{r}_2)|\uparrow\downarrow + \downarrow\uparrow\rangle + \varphi_{\downarrow\downarrow}(\mathbf{r}_1 - \mathbf{r}_2)|\downarrow\downarrow\rangle. \tag{7.2}$$

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[17] Our definition of n_c differs from that of these authors by a factor of $\frac{1}{4}$: cf. below.

The Pauli principle forces the function φ (and hence its components $\varphi_{\uparrow\uparrow}$, etc.) to be *odd* functions of $\mathbf{r}_1 - \mathbf{r}_2$. Now, it may happen that by a suitable choice of axes we can ensure that $\varphi_{\uparrow\downarrow}$ is zero everywhere in space, that is, that the spin state of the Cooper pair is always a linear superposition of $S_z = +1$ and $S_z = -1$ states. A state of the system satisfying this condition is called an "equal-spin-pairing" or ESP state, and its treatment is particularly simple. We shall discuss ESP states in the first subsection, and then go on in Subsection VII.B to treat the more general case.

A. ESP states

Provided the pairing interaction conserves spin (which is the case for liquid ^{3}He to the extent that we neglect the dipole forces), the treatment of ESP states is an almost trivial extension of the calculation of Sec. V for the singlet case. At first sight at least, the "up" and "down" spin particles form completely independent and noninteracting systems, and we can solve the pairing problem for these two systems separately. Let us consider for definiteness the "up" spins, and proceed by analogy with the BCS method developed in Sec. V. In analogy with Eqs. (5.7) and (5.8), we can write a "wave function for up-spin states" in the form

$$\Psi_\uparrow = \prod_{\text{all }\mathbf{k}} \Phi_{\mathbf{k}\uparrow}, \qquad \Phi_{\mathbf{k}\uparrow} = u_{\mathbf{k}\uparrow} \mid 0, 0\rangle_{\mathbf{k}\uparrow} + v_{\mathbf{k}\uparrow} \mid 1, 1\rangle_{\mathbf{k}\uparrow},$$
$$\mid u_{\mathbf{k}\uparrow} \mid^2 + \mid v_{\mathbf{k}\uparrow} \mid^2 = 1, \tag{7.3}$$

where however the state $\mid 1, 1\rangle_{\mathbf{k}\uparrow}$ is now the state in which the plane-wave states $\mathbf{k}\uparrow$ and $-\mathbf{k}\uparrow$ (not $-\mathbf{k}\downarrow$!) are both occupied and $\mid 0, 0\rangle_{\mathbf{k}\uparrow}$ that in which they are both empty. Because of the Fermi statistics we must now have

$$u_{\mathbf{k}\uparrow} = u_{-\mathbf{k}\uparrow}, \qquad v_{\mathbf{k}\uparrow} = -v_{-\mathbf{k}\uparrow}. \tag{7.4}$$

In second-quantized notation Eq. (7.3) reads

$$\Psi_\uparrow = \prod_{\text{all }\mathbf{k}} (u_{\mathbf{k}\uparrow} + v_{\mathbf{k}\uparrow} a_{\mathbf{k}\uparrow}{}^+ a_{-\mathbf{k}\uparrow}{}^+) \mid \text{vac}\rangle. \tag{7.3$'$}$$

The treatment now proceeds in exact analogy to that of Sec. V, the only difference being that to avoid counting the pair of states ($\mathbf{k}\uparrow$, $-\mathbf{k}\uparrow$) twice we must divide both Eqs. (5.15) and (5.19) (and similar expression at finite T) by a factor of two. We introduce the quantities $\Delta_{\mathbf{k}\uparrow}(T)$, $E_{\mathbf{k}\uparrow}$, $F_{\mathbf{k}\uparrow}$ and $\Psi_\uparrow(\mathbf{n}: T)$ [not to be confused with the total wave function occurring in Eq. (7.3)] by the relations analogous to Eqs. (5.38) (5.39) (5.44)–(5.45) and (5.56), respectively:

$$v_{\mathbf{k}\uparrow}(T) = \Delta_{\mathbf{k}\uparrow}(T)/[\mid \Delta_{\mathbf{k}\uparrow} \mid^2 + (E_{\mathbf{k}\uparrow} + \epsilon_{\mathbf{k}})^2]^{1/2},$$
$$u_{\mathbf{k}\uparrow}(T) \equiv (E_{\mathbf{k}\uparrow} + \epsilon_{\mathbf{k}})/[\mid \Delta_{\mathbf{k}\uparrow} \mid^2 + (E_{\mathbf{k}\uparrow} + \epsilon_{\mathbf{k}})^2]^{1/2}, \tag{7.5}$$

$$E_{\mathbf{k}\uparrow}(T) \equiv +(\epsilon_{\mathbf{k}}^2 + \mid \Delta_{\mathbf{k}\uparrow} \mid^2)^{1/2}, \tag{7.6}$$

$$F_{\mathbf{k}\uparrow} \equiv \langle a_{-\mathbf{k}\uparrow} a_{\mathbf{k}\uparrow} \rangle = [\Delta_{\mathbf{k}\uparrow}(T)/2E_{\mathbf{k}\uparrow}] \tanh \beta E_{\mathbf{k}\uparrow}/2, \tag{7.7}$$

$$\Psi_\uparrow(\mathbf{n}: T) \equiv \sum_{|k|} F_{\mathbf{k}\uparrow} = \tfrac{1}{2}(dn/d\epsilon)\Delta_\uparrow(\mathbf{n})$$
$$\times \int_{-\epsilon_c}^{\epsilon_c} \frac{\tanh\frac{1}{2}\beta E_{\mathbf{k}\uparrow}}{2E_{\mathbf{k}\uparrow}}\, d\epsilon_{\mathbf{k}}. \tag{7.8}$$

The free energy associated with up-spin states is therefore $\frac{1}{2}\sum_{\mathbf{k}} f\{\Delta_{\mathbf{k}\uparrow}\}$, where $f\{\Delta_{\mathbf{k}}\}$ is the functional defined in (5.45). The down-spin states can be handled in an exactly similar way, with definitions exactly analogous to (7.3)–(7.8), and we therefore find for the total free energy

$$F\{\Delta_{\mathbf{k}\uparrow}, \Delta_{\mathbf{k}\downarrow}\} = \tfrac{1}{2} \sum_{\mathbf{k}} [f\{\Delta_{\mathbf{k}\uparrow}\} + f\{\Delta_{\mathbf{k}\downarrow}\}]. \tag{7.9}$$

This leads to two *independent* gap equations for the "up" and "down" gaps $\Delta_{\mathbf{k}\uparrow}$, $\Delta_{\mathbf{k}\downarrow}$:

$$\Delta_{\mathbf{k}\uparrow}(T) = -\sum_{\mathbf{k}'} V(\mathbf{k}, \mathbf{k}')[\Delta_{\mathbf{k}'\uparrow}(T)/2E_{\mathbf{k}'\uparrow}(T)]$$
$$\times \tanh\tfrac{1}{2}\beta E_{\mathbf{k}'\uparrow}(T) \tag{7.10}$$

and a similar equation for $\Delta_{\mathbf{k}\downarrow}$. Note that because $\Delta_{\mathbf{k}\uparrow}$, $\Delta_{\mathbf{k}\downarrow}$ are odd functions of $\mathbf{k}$, only the odd part of the potential $V(\mathbf{k}, \mathbf{k}')$ enters into Eq. (7.10). Finally in the Ginzburg–Landau region we find for the free energy the expression

$$F\{\Psi_\uparrow(\mathbf{n}), \Psi_\downarrow(\mathbf{n}), T\} = \alpha[(T - T_c)/T_c] \int (d\Omega/4\pi)$$
$$\times \tfrac{1}{2}(\mid \Psi_\uparrow(\mathbf{n}) \mid^2 + \mid \Psi_\downarrow(\mathbf{n}) \mid^2)$$
$$+ \tfrac{1}{2}\beta \int (d\Omega/4\pi)\{\tfrac{1}{2}(\mid \Psi_\uparrow(\mathbf{n}) \mid^4 + \mid \Psi_\downarrow(\mathbf{n}) \mid^4)\}. \tag{7.11}$$

As to the BCS-like wave function, it is just the product $\Psi = \Psi_\uparrow\Psi_\downarrow$ (at $T = 0$): at finite T the statistical matrix is similarly factorized.

In the treatment we have given so far, there is nothing to correlate $\Delta_{\mathbf{k}\uparrow}$ and $\Delta_{\mathbf{k}\downarrow}$: all we know is that both are solutions of an equation of the form (7.10). We could, for instance (in the case of $l = 1$), choose $\Delta_{\mathbf{k}\uparrow} \sim n_x + in_y$, $\Delta_{\mathbf{k}\downarrow} \sim n_y + in_z$. In this case we find [cf. Eq. (7.6)] that $E_{\mathbf{k}\uparrow} \neq E_{\mathbf{k}\downarrow}$, i.e., for the same wave vector $\mathbf{k}$ the excitation energies of the BP states of up-spin (i.e., where $\mathbf{k}\uparrow$ is occupied and $-\mathbf{k}\uparrow$ empty or vice versa) are not equal to those of the corresponding down-spin states (and similarly of course for the EP states). If, however, it happens that the condition $E_{\mathbf{k}\uparrow} = E_{\mathbf{k}\downarrow}$ is satisfied for all $\mathbf{k}$ (i.e., $\Delta_\uparrow(\mathbf{n}) = a(\mathbf{n})\Delta_\downarrow(\mathbf{n})$, where $a(\mathbf{n})$ is a complex number of modulus unity), then we call the state in question a *unitary* (ESP) state. A more general definition of unitarity will be given in the next subsection.

Evidently, almost all the results of Sec. VI carry through quite unchanged to the spin triplet (ESP) case, provided we make the replacement in all formulas

$$\sum_{\mathbf{k}} g(E_{\mathbf{k}}) \rightarrow \sum_{\mathbf{k}} \tfrac{1}{2}\{g(E_{\mathbf{k}\uparrow}) + g(E_{\mathbf{k}\downarrow})\}, \tag{7.12}$$

where $g(E_{\mathbf{k}})$ is any function occurring in the formulas of Sec. VI (or of course a similar replacement for functions of $\Delta_{\mathbf{k}}$, etc.). There is, however, one important exception, namely the spin susceptibility. Let us consider the case in which the pairs form in an ESP state "with respect to the direction of the magnetic field," i.e., they have $S_z = \pm 1$ with respect to the field direction. If we repeat the intuitive argument of Sec. VI.B which showed that the susceptibility of a singlet paired superfluid must be reduced, we find that it now does not work: we now wish to pair a particle in the state $\mathbf{k}\uparrow$ with one in $-\mathbf{k}\uparrow$ (not $-\mathbf{k}\downarrow$) and this is still possible even when the system is polarized by the magnetic field (cf. Fig. 3). Consequently, the polarization by the field in no way inhibits superfluid condensation and, conversely, formation of the superfluid state does not prevent the system polarizing just as in the normal phase. Thus the spin susceptibility of a triplet superfluid in a ESP state with respect to the field direction is almost identical[18] to that of the normal phase. It turns out, as one might expect from symmetry considerations, that in bulk samples the pairs do indeed form along the field axis (see Sec. X below), so that the conclusion is that the susceptibility as measured experimentally (by either static or NMR techniques) will indeed be the same for an ESP state as for the normal phase (cf. also Sec. VII.D below).

Looking at the results of this subsection, one might well conclude that in an ESP-type triplet superfluid the spin-up and spin-down particles form two completely independent and noninteracting systems. It is then natural to ask: Does the *relative phase* of $\Delta_{k\uparrow}$ and $\Delta_{k\downarrow}$ [or $\Psi_\uparrow(\mathbf{n})$ and $\Psi_\downarrow(\mathbf{n})$, etc.] have any physical meaning? The answer to this question depends crucially on how one intends, eventually, to go back from the number (and spin) nonconserving BCS-type wave function

$$\Psi = \Psi_\uparrow \Psi_\downarrow \equiv \prod_{\text{all } \mathbf{k}} (u_{\mathbf{k}\uparrow} + v_{\mathbf{k}\uparrow} a_{\mathbf{k}\uparrow}^{+} a_{-\mathbf{k}\uparrow}^{+}) \times (u_{\mathbf{k}\downarrow} + v_{\mathbf{k}\downarrow} a_{\mathbf{k}\downarrow}^{+} a_{-\mathbf{k}\downarrow}^{+}) \mid \text{vac}\rangle \tag{7.13}$$

to the number-conserving wave function (5.2). There are two different obvious possibilities for the generalization of Eq. (5.12): If we define $\Psi_\uparrow(\theta_\uparrow)$ and $\Psi_\downarrow(\theta_\downarrow)$ analogously to Eq. (5.11), then we can write either (a)

$$\Psi(N) = \Psi(N_\uparrow)\Psi(N_\downarrow),$$
$$\Psi(N_\uparrow) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \Psi_\uparrow(\theta_\uparrow) \exp(iN_\uparrow \theta_\uparrow/2)\, d\theta_\uparrow, \quad \text{etc.}$$
$$(N_\uparrow + N_\downarrow = N) \tag{7.14a}$$

or (b)

$$\Psi(N) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \Psi_\uparrow(\theta)\Psi_\downarrow(\theta) \exp(iN\theta/2)\, d\theta,$$
$$\theta_\uparrow \equiv \theta_\downarrow \equiv \theta). \tag{7.14b}$$

These two alternative wave functions correspond to quite different hypotheses about the nature of the N-particle state. In (a), we have simply $N_\uparrow$ particles condensed into one pseudomolecular wave function, with spin $S_z = 1$, and the remaining $N_\downarrow$ particles condensed into a quite independent pseudomolecular wave function with $S_z = -1$. Evidently, in such a state the relative phase of the two pseudomolecular functions cannot be meaningful—and indeed we see that since both $\theta_\uparrow$ and $\theta_\downarrow$ are separately averaged over in Eq. (7.14a), the difference of phase between $\Delta_{k\uparrow}$ and $\Delta_{k\downarrow}$ has likewise been averaged over and retains no physical meaning. In state (b), on the other hand, we have all N particles condensed into a *single* pseudomolecular wave function which is a *linear combination* of $S_z = 1$ and $S_z = -1$ components—as indeed was assumed in Eq. (7.2). Clearly the relative phase of the two components does in general have a physical meaning—and we see that in Eq. (7.14b) the *relative* phase of $\Delta_{k\uparrow}$ and $\Delta_{k\downarrow}$ has *not* been averaged over (although the absolute phases have) and hence retains a meaning. For reasons which will become obvious in Sec. X, the moment that we introduce even a minute spin-nonconserving force into the problem it is essential to choose Eq. (7.14b) rather than (7.14a).

B. General triplet states

In general it is not possible to write an arbitrary wave function of the form (7.2) as an ESP state with respect to any set of spin axes. In this subsection we consider the more general (non-ESP) case, referring the reader to the classic paper of Balian and Werthamer (1963) for a more thorough and rigorous treatment.

As a first step, let us consider how we should describe an ESP state when we transform the spin axes so that the "pseudomolecular" wave function (7.2) no longer has $\varphi_{\uparrow\downarrow} \equiv 0$ with respect to the new axes. In particular, let us consider a particular type of ESP state such that originally $\varphi_{\uparrow\uparrow}(\mathbf{r}_1 - \mathbf{r}_2) = \exp(i\chi)\varphi_{\downarrow\downarrow}(\mathbf{r}_1 - \mathbf{r}_2)$, where χ is a constant phase. By a suitable choice of new axes we can always ensure that now $\varphi_{\uparrow\uparrow} = \varphi_{\downarrow\downarrow} = 0$, i.e., in the new axes

$$\varphi(\mathbf{r}_1 - \mathbf{r}_2 : \sigma_1\sigma_2) = \varphi_{\uparrow\downarrow}(\mathbf{r}_1 - \mathbf{r}_2) \mid \uparrow\downarrow + \downarrow\uparrow\rangle. \tag{7.15}$$

This function is very similar to (the Fourier transform of) Eq. (5.5), and the corresponding BCS-like wave function obviously involves pairing of the single-particle plane-wave states $\mid \mathbf{k}\uparrow\rangle$ and $\mid -\mathbf{k}\downarrow\rangle$, as in Eq. (5.7). In fact, the subsequent treatment goes through in *exact* analogy with that of Sec. V (including all factors of 2), with the sole exception that $v_{\mathbf{k}}$, $\Delta_{\mathbf{k}}$, $\Psi(\mathbf{n})$, etc. are *odd* functions of $\mathbf{k}$ and only the odd part of $V(\mathbf{n}, \mathbf{n}')$ therefore contributes to the gap equation (5.52). Call these functions $\Delta_{\mathbf{k}0}$, $\Psi_0(\mathbf{n})$, etc.

Evidently we need a description of the system which gives both the ESP results and those of the last paragraph as special cases, but is valid for an arbitrary choice of axes. Such a description may be obtained as follows: We treat the quantities $\Delta_{\mathbf{k}\uparrow}$, $\Delta_{\mathbf{k}\downarrow}$, $\Delta_{\mathbf{k}0}$ as components of a symmetric 2×2 matrix $\hat{\Delta}$ with elements $\Delta_{\mathbf{k}\uparrow\uparrow}$, $\Delta_{\mathbf{k}\downarrow\uparrow} \equiv \Delta_{\mathbf{k}\uparrow\downarrow}$, $\Delta_{\mathbf{k}\downarrow\downarrow}$:

$$\Delta_{\mathbf{k}\uparrow} \rightarrow \Delta_{\mathbf{k}\uparrow\uparrow}, \qquad \Delta_{\mathbf{k}\downarrow} \rightarrow \Delta_{\mathbf{k}\downarrow\downarrow}, \qquad \Delta_{\mathbf{k}0} \rightarrow \Delta_{\mathbf{k}\uparrow\downarrow} \equiv \Delta_{\mathbf{k}\downarrow\uparrow}. \tag{7.16}$$

It is assumed that the components of $\hat{\Delta}$ transform into one another under transformation of spin axes in a way to be specified below. Moreover, the "excitation energy" $E_{\mathbf{k}}$ should be replaced by a matrix defined by

$$\hat{E}_{\mathbf{k}} \equiv +(\epsilon_{\mathbf{k}}^2 + \hat{\Delta}_{\mathbf{k}}\hat{\Delta}_{\mathbf{k}}^{\dagger})^{1/2}. \tag{7.17}$$

Note however in the unitary case ($\mid \Delta_{\mathbf{k}\uparrow} \mid = \mid \Delta_{\mathbf{k}\downarrow} \mid$ in ESP axes) $\Delta_{\mathbf{k}}\Delta_{\mathbf{k}}^{\dagger} \propto \hat{1}$ and hence $E_{\mathbf{k}}$ is just a number (cf. below). Similarly the quantities $F_{\mathbf{k}}$, $F(\mathbf{r})$, and $\Psi(\mathbf{n})$ are replaced

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[18] There is in fact a very small change ($< 0.5\%$) due to the fact that in a field the Fermi surfaces are shifted and hence the parameters which determine the condensation energy may change very slightly. See Sec. XIII.

by 2 × 2 matrices:

$$\hat{F}_{\mathrm{k}} \equiv (\hat{\Delta}_{\mathrm{k}}/2\hat{E}_{\mathrm{k}}) \tanh\tfrac{1}{2}\beta\hat{E}_{\mathrm{k}}, \qquad \hat{\Psi}(\mathbf{n}) \equiv \sum_{|\mathbf{k}|} \hat{F}_k \tag{7.18}$$

$$\hat{F}(\mathbf{r}) = \sum_{\mathrm{k}} F_{\mathrm{k}} \exp(i\mathbf{k}\cdot\mathbf{r})$$

[for the order of $\hat{\Delta}$ and $\hat{E}$, cf., Balian and Werthamer (1963)]. Now we obtain all scalar quantities such as the free energy by the simple replacement

$$\sum_{\mathrm{k}} f\{\Delta_{\mathrm{k}}\} \rightarrow \tfrac{1}{2}\,\mathrm{Tr}\hat{f}\{\hat{\Delta}_{\mathrm{k}}\}, \tag{7.19}$$

where $\mathrm{Tr}\hat{f}$ indicates the trace of the *matrix* function $\hat{f}\{\hat{\Delta}_{\mathrm{k}}\}$. The gap equation retains the form (5.50), but $\hat{\Delta}_{\mathrm{k}}$ and $\hat{E}_{\mathrm{k}}$ are now matrices, so it is in effect three independent equations for the three independent elements of $\hat{\Delta}$. Notice in particular, for future reference, that the potential energy now has the form

$$\langle V\rangle = \int (d\Omega/4\pi) \int (d\Omega'/4\pi)\, V(\mathbf{n},\mathbf{n}')\tfrac{1}{2}\,\mathrm{Tr}\{\hat{\Psi}(\mathbf{n})\hat{\Psi}^{\dagger}(\mathbf{n}')\}. \tag{7.20}$$

This formalism is guaranteed, by construction, to give the correct results for the particular choices of spin axes leading either to ESP or to "$S_z = 0$ only" pairing. It is also, of course, necessary that quantities like the free energy are invariant under rotation of axes. This will be so if and only if the matrix $\hat{\Delta}_{\mathrm{k}}\cdot\hat{\Delta}_{\mathrm{k}}^{\dagger}$ undergoes a *unitary* transformation when the spin axes are rotated. (The easiest way to see this is to notice that the description of the "double-broken-pair" state in which both $\mathbf{k}\uparrow$ and $\mathbf{k}\downarrow$ are occupied, but not $-\mathbf{k}\uparrow$ or $-\mathbf{k}\downarrow$, is invariant under rotation of axes, and hence its energy E must be an invariant. Since the energy was $E_{\mathrm{k}\uparrow} + E_{\mathrm{k}\downarrow}$ in the ESP frame, by our prescription it is $\mathrm{Tr}\hat{E}_{\mathrm{k}}$: if this is to be invariant, $\hat{E}_{\mathrm{k}}$ must undergo a unitary transformation when the axes are rotated.) In the unitary case this condition is of course trivially satisfied, since then $\Delta_k\cdot\Delta_k^{\dagger}$ is just a number.

To give a physical meaning to this formalism, one must know how $\hat{\Delta}_{\mathrm{k}}$ itself transforms under rotation. Formally speaking a large number of choices is possible, consistent with the above requirements. However, by far the simplest choice is the prescription: *The elements of $\hat{F}_{\mathrm{k}}$ transform like the amplitudes of the three magnetic substates.* That is, if we write a general triplet spin wave function in the form[19]

$$\chi = \chi_{\uparrow\uparrow}\,|\uparrow\uparrow\rangle + \chi_{\uparrow\downarrow}\,|\uparrow\downarrow + \downarrow\uparrow\rangle + \chi_{\downarrow\downarrow}\,|\downarrow\downarrow\rangle \tag{7.21}$$

then under rotation of the spin axes $F_{\mathrm{k}\uparrow\uparrow}$ transforms like $\chi_{\uparrow\uparrow}$, etc. Note that the preservation of the normalization condition then automatically guarantees that $\hat{F}_{\mathrm{k}}\cdot\hat{F}_{\mathrm{k}}^{\dagger}$ undergoes a unitary transformation, and hence so does $\Delta_{\mathrm{k}}\cdot\Delta_{\mathrm{k}}^{\dagger}$. If the state is unitary, the elements of $\hat{\Delta}_{\mathrm{k}}$ of course transform like those of $\hat{F}_{\mathrm{k}}$. The above choice is compatible with the microscopic definition, in second-quantized language

$$F_{\mathrm{k}\alpha\beta} = \langle a_{-\mathrm{k}\alpha}a_{\mathrm{k}\beta}\rangle, \tag{7.22}$$

which of course reduces to Eq. (7.7) in the appropriate special case. The fact that the elements of $\hat{F}$ transform like the magnetic substates, plus the considerations of Subsection VI.D, make it clear that the elements of $\hat{F}_{\mathrm{k}}$ have the physical significance of the amplitudes of the various S_z components in the Cooper pair wave function. In the unitary case they can also be regarded as representing directly the corresponding components of the pseudomolecular wave function (7.2).

So far we have been formally considering an ESP state which is transformed out of the original axes. But, if we consider any one particular value of $\mathbf{k}$, it is always possible to choose the axes to make it "ESP" (since any function of the form (7.21) can be reduced to $S_z = \pm 1$ components only by a suitable choice of axes). Moreover, the spin-conserving potential scatters $S_z = +1$ pairs into $S_z = +1$ states only, and so on. Consequently the description developed above is quite general, and can be applied even when the "right" axes are different for different $\mathbf{k}$ (non-ESP states). In fact, one can see quite directly that the form of the terms in the free energy coming from $K - \mu N - TS$ is right; since each $\mathbf{k}$-value contributes independently, it is possible to choose the spin axes appropriately for each $\mathbf{k}$ (or $\mathbf{n}$), write the contribution to F in the form obtained by using Eq. (7.12), and then transform the axes for each $\mathbf{k}$ so that the final frame of reference is uniform for all $\mathbf{k}$. As to the potential terms, one can check that these are given correctly by Eq. (7.20) [with the definitions (7.18) and (7.22)] by writing $\langle V\rangle$ out explicitly:

$$\begin{aligned}\langle V\rangle &= \sum_{\mathrm{k}\mathrm{k}'\alpha\beta} V(\mathbf{k},\mathbf{k}')F_{\mathrm{k}\alpha\beta}F_{\mathrm{k}'\beta\alpha}{}^{*}\\ &\equiv \sum_{\mathrm{k}\mathrm{k}'\alpha\beta} V(\mathbf{k},\mathbf{k}')\langle a_{\mathrm{k}'\alpha}{}^{+}a_{-\mathrm{k}'\beta}{}^{+}\rangle\langle a_{-\mathrm{k}\beta}a_{\mathrm{k}\alpha}\rangle\end{aligned} \tag{7.23}$$

which is just a factorization of the second-quantized version of the spin-conserving potential energy operator.

We therefore arrive at the general prescription for dealing with spin triplet states within the BCS-like approximation: Starting from the spin singlet formalism, replace Δ_{k} by a symmetric matrix $\hat{\Delta}_{\mathrm{k}}$ whose elements individually obey the matrix gap equation obtained from Eq. (5.50) by this replacement [with $\hat{E}_{\mathrm{k}} \equiv +(\epsilon_k^2 + \Delta_{\mathrm{k}}\cdot\Delta_{\mathrm{k}}^{\dagger})^{1/2}$], and replace F_k, $\Psi(\mathbf{n})$, etc. similarly by matrices as in Eq. (7.18). The free energy is then obtained by the prescription

$$\sum_{\mathrm{k}} f\{\Delta_{\mathrm{k}}\} \rightarrow \sum_{\mathrm{k}} \tfrac{1}{2}\,\mathrm{Tr}\hat{f}\{\hat{\Delta}_{\mathrm{k}}\}. \tag{7.24}$$

In particular the free energy in the Ginzburg–Landau region has the form

$$\begin{aligned}F\{\hat{\Psi}(\mathbf{n}),T\} &= \alpha[(T-T_c)/T_c]\int (d\Omega/4\pi)\tfrac{1}{2}\,\mathrm{Tr}\,|\hat{\Psi}(\mathbf{n})|^2\\ &\quad + \tfrac{1}{2}\beta\int (d\Omega/4\pi)\tfrac{1}{2}\,\mathrm{Tr}\,|\hat{\Psi}(\mathbf{n})|^4\end{aligned} \tag{7.25}$$

[where $|\hat{\Psi}(\mathbf{n})|^2 \equiv \hat{\Psi}(\mathbf{n})\hat{\Psi}^{\dagger}(\mathbf{n})$: in case of ambiguity $|\hat{\Psi}|^4$ is to be interpreted as $|\Psi|^2\cdot|\Psi|^2$, etc.] An ESP state is the special case where axes can be chosen so that $\hat{\Psi}(\mathbf{n})$ is diagonal for all $\mathbf{n}$. [In the future we shall refer to these axes as "the ESP axes" or "the proper axes": they are not necessarily unique (cf. below).]

We define a unitary triplet state quite generally as one

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[19] Note the convention for normalization of $\chi_{\downarrow\uparrow}$.

which satisfies the condition

$$\hat{\Delta}_{\mathrm{k}}\cdot\hat{\Delta}_{\mathrm{k}}^{\dagger} = c(\mathbf{n})\hat{1} \qquad (\text{or } \hat{\Psi}(\mathbf{n})\hat{\Psi}^{\dagger}(\mathbf{n}) = c'(\mathbf{n})\hat{1}), \tag{7.26}$$

where $\hat{1}$ is the unit matrix. For such a state the excitation energy matrix $\hat{E}_{\mathrm{k}}$ is a number, which means that although the "spin axes" of the condensate may vary over the Fermi surface, the excitation energy of a BP state for any particular $\mathbf{k}$ value is independent of spin. There is no special connection between the ESP property and the unitary one: states can be easily written down which have both properties, either one separately, or neither. In the unitary case we can define the function (a number, not a matrix)

$$|\Delta|(\mathbf{n}:T) \equiv +[\hat{\Delta}(\mathbf{n}:T)\hat{\Delta}^{\dagger}(\mathbf{n}:T)]^{1/2}, \tag{7.27}$$

which is then the energy gap for excitation of a BP state, independent of its spin. Fortunately it appears that a description of ³He-A and B can probably be given in terms of unitary states only, at least as regards the behavior to zeroth order in the external magnetic field.

At first sight the introduction of non-ESP states has merely complicated the notation without changing anything qualitatively. However, at least in the case of $l = 1$ pairing there is one interesting and important consequence. To see this, let us consider the problem of minimizing the free energy (7.25) in the GL region. If we write in analogy with Eqs. (5.65) and (5.73)

$$\begin{aligned}
&\hat{\Psi}(n) = \Psi\hat{f}(\mathbf{n}), \qquad \hat{f}(\mathbf{n}) = \sum_m \hat{a}_{lom} Y_{lom},\\
&\int (d\Omega/4\pi)\tfrac{1}{2}\,\mathrm{Tr}\,|\hat{f}(n)|^2 \equiv 1,\\
&\kappa \equiv \int (d\Omega/4\pi)\tfrac{1}{2}\,\mathrm{Tr}\,|\hat{f}(\mathbf{n})|^4
\end{aligned} \tag{7.28}$$

[where we note that the elements of the matrix $\hat{f}(\mathbf{n})$ must all be linear combinations of spherical harmonics of the same l, but not of course necessarily the *same* combination], then the problem reduces to finding the form of $\hat{f}$ which minimizes κ. Evidently, the optimum value of κ is 1, and this is attained if and only if $|\hat{f}(\mathbf{n})|^2 = \text{const.} = 1$. This condition can never be met by an ESP state for any odd l value. In fact, we easily verify that for $l = 1$ the minimum value of κ, is given by (in the ESP axes)

$$f(\mathbf{n}) = (3/2)^{1/2}\begin{pmatrix} n_x + in_y & 0 \\ 0 & n_y + in_z \end{pmatrix} \tag{7.29}$$

of any state of a similar form (including of course one in which the ↑↑ and ↓↓ elements are equal). The resulting value of κ is 6/5. On the other hand, if we allow also non-ESP states we have, for instance, the possibility

$$\hat{f}(\mathbf{n}) = \begin{pmatrix} -n_x + in_y & n_z \\ n_z & n_x + in_y \end{pmatrix} \tag{7.30}$$

which corresponds to $|(\hat{f}(\mathbf{n})|^2 = 1$ and hence $\kappa = 1$. This is the famous "Balian–Werthamer" (BW) or "isotropic" state: the function $|\Delta|(\mathbf{n}:T)$ is constant over the Fermi surface, just as in the $l = 0$ BCS state. Since the free energy in the GL region is proportional to κ^{-1} [cf. Eq. (5.77), which can be of course be taken straight over to the triplet case with the definition (7.28) of κ] it follows that the BW state is 20% lower in free energy (of condensation) than any ESP state, and is in fact the absolute free energy minimum for $l = 1$ pairing. It can be shown (Balian and Werthamer, 1963) that this last result holds at all temperatures below T_c, not just in the Ginzburg–Landau region: the degree of stability however decreases somewhat with temperature, falling to 12% at zero temperature.

For $l \geq 3$ it is not possible to get $\kappa = 1$ even with the help of non-ESP states. Nevertheless it seems very probable (cf. Barton and Moore, 1973) that the best state is still non-ESP: generally speaking, the more components of $\hat{f}$ one is allowed to use, the more uniform the function $|\hat{f}(\mathbf{n})|^2$ can be made over the Fermi surface. It is also interesting to observe that any pure-l ESP states apparently must have nodes of $|\Delta|(\mathbf{n})$ (or the eigenvalues of $\hat{\Delta}(\mathbf{n})$, in the non-unitary case) where it is very easy to construct non-ESP states, even for $l > 1$, for which $|\Delta(\mathbf{n})|$ is everywhere finite.

We will postpone consideration of the thermodynamic and correlation properties of general triplet states until we have introduced a concise and useful notation in the next subsection.

C. The vector notation

Although the matrix notation developed in the last section for general triplet states is quite concise, it is not as convenient as it might be, principally because the transformation properties of the matrices under rotation of the spin coordinates are somewhat complicated. A much more convenient notation, which replaces the matrix description by a vector one, was developed by Balian and Werthamer (1963) and is commonly used in recent papers.

Let $\hat{Q} \equiv Q_{\alpha\beta}$ be any of the symmetric matrices $\hat{F}_{\mathrm{k}}$, $\hat{F}(\mathbf{r})$, $\hat{\Psi}(\mathbf{n})$, $\hat{\Delta}(\mathbf{n})$, or $\hat{f}(\mathbf{n})$ introduced in the last subsection (but not $\hat{E}_{\mathrm{k}}$, which has somewhat different transformation properties). From the elements of $\hat{Q}$ we can form a complex vector $\mathbf{Q}$ by the prescription

$$\mathbf{Q} \equiv -\tfrac{1}{2}i\sum_{\alpha\beta}(\sigma_2\boldsymbol{\sigma})_{\alpha\beta}Q_{\alpha\beta}, \tag{7.31}$$

where the components σ_1, σ_2, σ_3 of the vector $\boldsymbol{\sigma}$ are Pauli matrices. Inversion of Eq. (7.31) gives after a little algebra

$$Q_{\alpha\beta} = i\sum_{i=1}^{3}(\sigma_i\sigma_2)_{\beta\alpha}Q_i \equiv i\sum_{i=1}^{3}(\sigma_i\sigma_2)_{\alpha\beta}Q_i, \tag{7.32}$$

or explicitly

$$\hat{Q} = \begin{pmatrix} -Q_x + iQ_y & Q_z \\ Q_z & Q_x + iQ_y \end{pmatrix}. \tag{7.33}$$

We notice that we have the relation

$$\tfrac{1}{2}\,\mathrm{Tr}\hat{Q}\hat{Q}^{\dagger} \equiv |\mathbf{Q}|^2. \tag{7.34}$$

[In the literature, various forms of the transformation (7.31) differing by overall factors and/or complex conjugations may be found. We shall see that such factors do not affect any of the physical results.]

In general we shall denote the transform of any matrix $\hat{Q}$ generated by Eq. (7.31) simply by the vector $\mathbf{Q}$—e.g., we write $\mathbf{F}(\mathbf{r})$, $\mathbf{\Delta}(\mathbf{n})$, etc. However, we shall reserve for the transform of the *normalized* quantity $\hat{\Psi}(\mathbf{n})$—that is, $\hat{f}(\mathbf{n})$ as defined in Eq. (7.28)—the notation $\mathbf{d}(\mathbf{n})$: then according to Eqs. (7.28) and (7.32)

$$\hat{\Psi}_{\alpha\beta}(\mathbf{n}) = \Psi i \sum_{i=1}^{3} (\sigma_i\sigma_2)_{\alpha\beta} d_i(\mathbf{n}) \tag{7.35}$$

and by definition we have

$$\int (d\Omega/4\pi) \mid \mathbf{d}(\mathbf{n}) \mid^2 \equiv 1. \tag{7.36}$$

This notation is not universal: one often uses $\mathbf{d}(\mathbf{n})$ for the transform of either $\Delta(\mathbf{n})$ or $\Psi(\mathbf{n})$. For unitary states (for given $\mathbf{n}$) all those quantities are strictly proportional, i.e., the vectors are parallel and differ only by an overall factor; for nonunitary states they may not be strictly parallel, but the error involved in taking them to be so is at most of the same order as that introduced by the mixing-in of $l \neq l_0$ spherical harmonics. We shall therefore always make this approximation in what follows.

What is the physical significance of the vector $\mathbf{d}(\mathbf{n})$? From Eq. (7.34) we have

$$\tfrac{1}{2} \operatorname{Tr} \mid \hat{\Psi}(\mathbf{n}) \mid^2 = \Psi^2 \mid \mathbf{d}(\mathbf{n}) \mid^2 \tag{7.37}$$

so that the magnitude of $\mathbf{d}(\mathbf{n})$ is evidently a measure of the total amplitude of condensation of the Cooper pairs at point $\mathbf{n}$ on the Fermi surface, irrespective of spin. Secondly, in the case of a *unitary* state we may easily verify that $\mathbf{d}(\mathbf{n})$ must be a real vector apart from an $\mathbf{n}$-dependent phase factor, so that we can associate with it a unique direction in spin space. To see the physical significance of this direction, we write out the spin wave function of the (triplet) Cooper pair explicitly in the form

$$\begin{aligned}\Psi(\sigma_1\sigma_2{:}\,\mathbf{n}) = {} & \Psi_{\uparrow\uparrow}(\mathbf{n}) \mid \uparrow\uparrow\rangle + \Psi_{\uparrow\downarrow}(\mathbf{n}) \mid \uparrow\downarrow + \downarrow\uparrow\rangle \\ & + \Psi_{\downarrow\downarrow}(\mathbf{n}) \mid \downarrow\downarrow\rangle\end{aligned} \tag{7.38}$$

and then verify explicitly that for real $\mathbf{d}(\mathbf{n})$ we have the *operator* relation

$$\mathbf{d}(\mathbf{n}) \cdot \hat{\mathbf{S}}\Psi(\sigma_1\sigma_2{:}\,\mathbf{n}) \equiv 0, \tag{7.39}$$

where $\mathbf{S} \equiv \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2$ is the total spin operator for the pair. This leads to the important conclusion that in a unitary state the pairs at any given point on the Fermi surface are condensed into a spin state which is an eigenstate of the spin projection along some axis with eigenvalue zero, and $\mathbf{d}(\mathbf{n})$ represents this axis. (For instance, in the case $\Psi_{\uparrow\uparrow} = -\Psi_{\downarrow\downarrow}$, $\Psi_{\uparrow\downarrow} = 0$, we have the pairs condensed into an eigenstate of S and S_x such that $S = 1$, $S_x = 0$.)

A number of other properties of the vector $\mathbf{d}(\mathbf{n})$ follow from the definition (7.35) [for further details see Mermin and Ambegaokar (1973)].

(1) For an ESP state (in the "proper" axes) the vector $\mathbf{d}(\mathbf{n})$ always lies in the xy plane for all $\mathbf{n}$. More generally, if for given $\mathbf{n}$ the "ESP axes" are chosen, then $\mathbf{d}(\mathbf{n})$ lies in the xy plane. For a nonunitary state this defines the ESP axes for the given $\mathbf{n}$ uniquely, but for a unitary state they are clearly defined only up to a rotation around the axis of $\mathbf{d}(\mathbf{n})$.

(2) In a nonunitary state the Cooper pairs at point $\mathbf{n}$ have a net average spin, in the sense that the expectation value of $\langle \mathbf{S} \rangle$ with respect to the spin function (7.38) is finite. In terms of $\mathbf{d}(\mathbf{n})$ it is given by

$$\langle \mathbf{S} \rangle = i\, \mathbf{d}(\mathbf{n}) \times \mathbf{d}^*(\mathbf{n}) \cdot \Psi^2. \tag{7.40}$$

However, it should be emphasized that this does *not* in itself imply that the total spin polarization associated with this point on the Fermi surface is finite: see subsection VII.D.

(3) While the second-order term in the GL free energy (7.25) is proportional to $\mid \mathbf{d}(\mathbf{n}) \mid^2$ (and hence simply gives Ψ^2 when integrated over the Fermi surface), the fourth-order term is given by

$$\tfrac{1}{2} \operatorname{Tr} \mid \hat{\Psi}(\mathbf{n}) \mid^4 = \{\mid \mathbf{d}(\mathbf{n}) \mid^4 - (\mathbf{d}(\mathbf{n}) \times \mathbf{d}^*(\mathbf{n}))^2\}\Psi^4 \tag{7.41}$$

(where the second term is positive, since the cross product is purely imaginary). Consequently, the quantity κ, which determines the overall magnitude of the fourth-order terms [Eq. (7.28)] is now written as

$$\kappa \equiv \int (d\Omega/4\pi) \{\mid \mathbf{d}(\mathbf{n}) \mid^4 - (\mathbf{d}(\mathbf{n}) \times \mathbf{d}^*(\mathbf{n}))^2\}. \tag{7.42}$$

Evidently, the second term is zero for a unitary state.

(4) To the order of approximation we are working to throughout, the components of $\mathbf{d}(\mathbf{n})$ must be composed of spherical harmonics corresponding to the same l value.

(5) The BW state has, according to Eq. (7.30), the simple description

$$\mathbf{d}(\mathbf{n}) = \mathbf{n} \qquad (\text{i.e., } d_i(\mathbf{n}) = n_i). \tag{7.43}$$

(6) The quasiparticle energy matrix $\hat{E}_{\mathrm{k}}$ can be written in terms of $\mathbf{\Delta}(\mathbf{n})$:

$$\hat{E}_{\mathrm{k}} = +\{\epsilon_k^2 + \mid \mathbf{\Delta}(\mathbf{n}) \mid^2 + i\boldsymbol{\sigma} \cdot (\mathbf{\Delta}(\mathbf{n}) \times \mathbf{\Delta}^*(\mathbf{n}))\}^{1/2} \tag{7.44}$$

showing that, as expected, in a nonunitary state the BP eigenstates correspond to quasiparticle spin along or against the direction of the "pair average spin" $\langle \mathbf{S} \rangle$.

(7) Finally, in the vector notation, the gap equation for a *unitary* state takes the simple form

$$\mathbf{\Delta}(\mathbf{n}) = \int (d\Omega'/4\pi) V(\mathbf{n}, \mathbf{n}') \boldsymbol{\Psi}(\mathbf{n}'), \tag{7.45}$$

$$\begin{aligned}\boldsymbol{\Psi}(\mathbf{n}) \equiv \Psi\, \mathbf{d}(\mathbf{n}) \equiv {} & \mathbf{\Delta}(\mathbf{n}) \cdot \tfrac{1}{2}(dn/d\epsilon) \\ & \times \int_{-\epsilon_c}^{\epsilon_c} d\epsilon_k (\tanh\tfrac{1}{2}\beta E_{\mathrm{k}}/2E_{\mathrm{k}}), \quad E_{\mathrm{k}} \equiv +[\epsilon_k^2 + \mid \mathbf{\Delta}(\mathbf{n}) \mid^2]^{1/2}\end{aligned} \tag{7.46}$$

[for a nonunitary state, some matrix notation is still strictly necessary in (7.46), but in almost all physically realistic cases the error introduced by using Eq. (7.46) instead of (7.44) is probably negligible].

As a kind of footnote to this section, it is worth remarking that the formalism developed in this and the last subsection can be extended to apply also to the case of mixed singlet-triplet pairing by introducing a *scalar* quantity $d_0(\mathbf{n})$ which describes the singlet part of the wave function and treating it on a footing similar to $\mathbf{d}(\mathbf{n})$. We refer to the original paper of Balian and Werthamer (1963) for the details. However, to the author's knowledge no-one has ever produced a case in which the absolute energy minimum as calculated from BCS-type theory corresponds to mixed singlet–triplet pairing (although there are plenty of free energy extrema which do); and while it is conceivable that in more general theories such a state might be the stable minimum, there seems at present no experimental evidence which requires this hypothesis.

D. Thermodynamic and correlation properties of triplet states

All those thermodynamic and correlation properties of superfluids which do not explicitly involve the spin coordinates, such as the specific heat, normal density, and relative orbital angular momentum, may be obtained for the general triplet case by a simple generalization of the corresponding expressions given in Sec. VI for the singlet case. All we need to do is to replace quantities such as $\Delta(\mathbf{n})$, $E_{\mathbf{k}}$, $\Psi(\mathbf{n})$, etc. by matrices as in Subsection VII.B and to make the substitution

$$\sum_{\mathbf{k}} \rightarrow \sum_{\mathbf{k}} \tfrac{1}{2}\,\mathrm{Tr}. \tag{7.47a}$$

If the expression involves only $E_{\mathbf{k}}$ (as in the case of specific heat or normal density) *and* the state is unitary we have the even simpler prescription: keep the same expression but with $E_{\mathbf{k}}$ now given by

$$E_{\mathbf{k}} = +[\epsilon_{\mathbf{k}}^2 + |\,\mathbf{\Delta}(n)\,|^2]^{1/2}. \tag{7.47b}$$

Notice in particular that for the BW state ($|\,\mathbf{\Delta}(n)\,|^2 = \Delta^2$) all such expressions reduce exactly to the corresponding results for a simple BCS superconductor. In particular the relative specific heat jump at T_c has the BCS value 1.42, and the normal density is isotropic and given by the original form of $Y(T)$, calculated by Yosida (1958) (with the appropriate Fermi-liquid correction).

The expression for the orbital angular momentum is now

$$\begin{aligned}\langle L_{\mathrm{rel}}\rangle &= -i\hbar n_c\cdot\tfrac{1}{2}\,\mathrm{Tr}\int (d\Omega/4\pi)\{\hat{f}^{\dagger}(\mathbf{n})[\mathbf{n}\times(\partial/\partial\mathbf{n})]\hat{f}(\mathbf{n}),\\ &\equiv -i\hbar n_c\sum_i\int (d\Omega/4\pi)\,d_i^*(\mathbf{n})[\mathbf{n}\times(\partial/\partial\mathbf{n})]\,d_i(\mathbf{n}),\\ n_c &\equiv \int d\mathbf{r}\,|\,\mathbf{F}(\mathbf{r})\,|^2,\end{aligned} \tag{7.48}$$

where the vector $\mathbf{F}(\mathbf{r})$ is defined from $\hat{F}(\mathbf{r})$ as above. We notice that the BW state, for example, has identically zero relative angular momentum since the angular momentum associated with the "up" pairs is exactly cancelled by the "down" pairs (and the $S_z = 0$ pairs have no angular momentum).

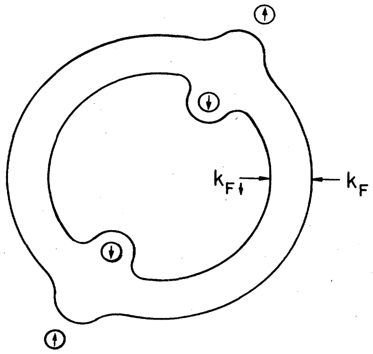

FIG. 7. Even differential polarization of the Fermi surface cannot inhibit parallel-spin pairing.

The spin-dependent properties of a general triplet superfluid are somewhat less trivial to calculate. Let us start with the spin susceptibility and neglect for the moment Fermi-liquid corrections. We will treat the Cooper pairs for the moment as having a definite spin configuration, i.e., treat $\mathbf{d}(\mathbf{n})$ as fixed independently of the weak external field (but see below). Now let us consider a given point $\mathbf{n}$ on the Fermi surface, and ask what is the *differential* susceptibility associated with this point, that is, the quantity $\chi(\mathbf{n})$ defined by (for the moment)

$$\begin{aligned}\mathbf{S}(\mathbf{n}) &= \chi(\mathbf{n})\mathbf{H}, \qquad \mathbf{S}(\mathbf{n}) \equiv \sum_{|k|}\hbar\boldsymbol{\sigma}_{\mathbf{k}}\\ &\equiv \tfrac{1}{2}\hbar(dn/d\epsilon)\int d\epsilon_k\boldsymbol{\sigma}_{\mathbf{k}},\end{aligned} \tag{7.49a}$$

where, as in Sec. II, $\boldsymbol{\sigma}_{\mathbf{k}}$ is the spin of the particles in the plane-wave state $\mathbf{k} \equiv \mathbf{p}/\hbar$ in units of $\hbar$. The total spin $\mathbf{S}$ is just the average of $\mathbf{S}(\mathbf{n})$ over the Fermi surface [a factor of 4π is incorporated in the definition (7.49a)] and so χ is related to $\chi(\mathbf{n})$ by

$$\chi = \int (d\Omega/4\pi)\chi(\mathbf{n}). \tag{7.49b}$$

To obtain $\chi(\mathbf{n})$ we argue as follows. Suppose that in the ESP axes for the point $\mathbf{n}$, the field $\mathbf{H}$ is along the z axis. Then we can apply the argument of Subsection VII.A: for even the *differential* polarization of the Fermi surface as in Fig. 7, provided that it is symmetric as in the figure, does not affect the possibility of ↑↑ or ↓↓ condensation (it merely shifts the "local" chemical potentials from the original value μ_0, so that ϵ_k must now be measured from $\mu_0 \pm \frac{1}{2}\gamma\hbar H$ for up and down spin particles, respectively, in all formulas). Consequently, the susceptibility with respect to the ESP z axis is equal to the normal value χ_n. On the other hand, it clearly cannot be equal to this if the field is along any arbitrary axis: for instance, if the state is unitary we can always choose the field along the direction of $\mathbf{d}(\mathbf{n})$, in which case (with respect to this axis) only $\Psi_{\uparrow\downarrow}(\mathbf{n})$ is nonzero. In this case we can apply exactly the same arguments as in the case of singlet pairing and obtain [cf. Eq. (6.13)]

$$\begin{aligned}\chi_{\perp}(\mathbf{n}) &= \tfrac{1}{4}\gamma^2\hbar^2(dn/d\epsilon)\int_0^{\infty} d\epsilon_k\tfrac{1}{2}\beta\,\mathrm{sech}^2\tfrac{1}{2}\beta E_k\\ &\equiv \tfrac{1}{4}\gamma^2\hbar^2(dn/d\epsilon)\,Y(\mathbf{n}\!:T) \equiv \chi_n Y(\mathbf{n}\!:T),\end{aligned} \tag{7.50}$$

where $Y(\mathbf{n}: T)$ is defined as in Eq. (6.14), with the replacement (7.47b).

The differential susceptibility must therefore be a *tensor* quantity which reduces to χ_n along an ESP z axis, i.e., for the direction(s) perpendicular to $\mathbf{d}(\mathbf{n})$, while for the direction along $\mathbf{d}(\mathbf{n})$ (in the unitary case) it reduces to $\chi_n Y(\mathbf{n}: T)$. Let us consider for a moment a unitary state and take $\mathbf{d}(\mathbf{n})$ arbitrarily along the x axis. Then for *any* choice of the y and z axes we must have

$$\chi_{xx}(\mathbf{n}) = \chi_n Y(\mathbf{n}: T), \qquad \chi_{yy}(\mathbf{n}) = \chi_{zz}(\mathbf{n}) = \chi_n. \quad (7.51)$$

Moreover, the fact that these equations must be true for any choice of the y and z axes implies that $\chi_{yz}(\mathbf{n}) \equiv \chi_{zy}(\mathbf{n}) \equiv 0$: and the argument leading to Eq. (7.50) plus the fact that $\chi(\mathbf{n})$ must be symmetric implies that all other off-diagonal components are also zero. A generalization of Eq. (7.51) to an arbitrary set of axes now leads to the result, valid for any unitary state,

$$\chi_{ij}(\mathbf{n}) = \chi_n \times \{\delta_{ij} - [1 - Y(\mathbf{n}: T)][d_i^*(\mathbf{n})\, d_j(\mathbf{n})/|\mathbf{d}(n)|^2]\} \quad (7.52)$$

(the complex conjugation is necessary because even for unitary states $\mathbf{d}$ may contain a complex overall phase factor). This formula has a simple interpretation: the second term essentially subtracts off the contribution of the Cooper pairs, which is proportional to $1 - Y(T)$, when the field is parallel to $\mathbf{d}$, i.e., the pairs are in an $S_z = 0$ state.

Integrating Eq. (7.52) over the Fermi surface, we obtain for the total susceptibility uncorrected for Fermi-liquid effects the result

$$\chi_{ij} = \chi_n\{\delta_{ij} - \int (d\Omega/4\pi)[1 - Y(\mathbf{n}: T)] \times \mathrm{Re}[d_i^*(\mathbf{n})\, d_j(\mathbf{n})/|\mathbf{d}(\mathbf{n})|^2]\} \equiv \chi_n \theta_{ij}(T), \quad (7.53)$$

where we have trivially taken the real part for reasons which will become clear. Equation (7.53) is a general expression for the susceptibility of a *unitary* state. It has not, as far as I know, been demonstrated for general nonunitary states [the proof given by Balian and Werthamer (1963) assumes the unitary condition]. Indeed, Takagi (1973) finds a much more complicated formula for the susceptibility of nonunitary states; however he shows that to lowest order in the departure from unitary (i.e., in $|\mathbf{d} \times \mathbf{d}^*|/|\mathbf{d}|^2$) Eq. (7.53) holds. Fortunately, the only nonunitary states we shall be interested in for present purposes are the ones occurring near the A transition in a magnetic field (see Sec. XIII) and the susceptibility of these is so close to the normal values that the difference can be neglected for most purposes (cf., however, Sec. IX.C below).

Fermi liquid corrections to formula (7.53) never involve the odd Z_l, since the polarization of the Fermi surface is always symmetric. If we neglect the even Z_l for $l \geq 2$, we find

$$\hat{\chi} = \chi_n(1 + \tfrac{1}{4}Z_0)[\hat{\theta}/(1 + \tfrac{1}{4}Z_0\hat{\theta})]. \quad (7.54)$$

This formula is exact in the limit $T \rightarrow T_c$, even if higher Z_l are nonzero. Notice that for an arbitrary state (unitary or not) we have

$$\mathrm{Tr}\hat{\theta} = 2 + Y(T) \geq 2. \quad (7.55)$$

In particular, the susceptibility of the BW state $[\mathbf{d}(\mathbf{n}) = \mathbf{n}]$ is isotropic and given in the approximation $Z_2 = 0$ by the formula

$$\chi_{\mathrm{BW}}/\chi_n = (1 + \tfrac{1}{4}Z_0)[\tfrac{2}{3} + \tfrac{1}{3}Y(T)]/\{1 + \tfrac{1}{4}Z_0[\tfrac{2}{3} + \tfrac{1}{3}Y(T)]\}. \quad (7.56)$$

This formula is plotted in Fig. 12 of Wheatley (1974) for $Z_0 \sim -3$. Actually, since the BW state has a variation of $\mathbf{d}(\mathbf{n})$ over the Fermi surface, the induced differential polarization $\mathbf{S}(\mathbf{n})$ is not uniform and hence Z_2 (but not higher Z_l's) does come into the result. At $T = 0$ the full corrected formula is (Czerwonko, 1967)

$$\frac{\chi_{\mathrm{BW}}}{\chi_n}(T = 0) = \tfrac{2}{3}(1 + \tfrac{1}{4}Z_0)/\{1 + \tfrac{1}{4}(\tfrac{2}{3}Z_0 + \tfrac{1}{15}Z_2)\}. \quad (7.57)$$

Evidently, Z_2 has to be very large to affect the result substantially.

According to Eq. (7.53), the susceptibility of a triplet state (even an ESP state) is in general *anisotropic*. This "susceptibility anisotropy" has been the subject of a certain amount of confusion (generated not least by the present author) in the last two years, so we shall try to discuss its meaning briefly. What we have calculated above is the susceptibility "at constant $\mathbf{d}(\mathbf{n})$," i.e., under the condition that the spin state of the Cooper pairs does not change in response to the weak external field. Now in general it *will* change, if the sample is allowed to come to complete equilibrium: for instance, if we take an ESP state and apply a field which is not perpendicular to the plane of $\mathbf{d}(\mathbf{n})$, then a bulk sample will simply rotate the spins of the Cooper pairs so that the new plane of $\mathbf{d}$ is perpendicular to the field (and the susceptibility, therefore, will be given by the normal value). More generally, a bulk sample will tend to adjust its Cooper pair spin state in any given external field so as to maximize the susceptibility. The experimentally measured susceptibility, therefore, will generally be the *largest* eigenvalue of χ_{ij} [note that in view of Eq. (7.55) this cannot be smaller than the expression (7.56)]. This is true even if the measurement is done by NMR techniques, provided only that the Hamiltonian is invariant against *total* rotation, that is, simultaneous rotation of spin and orbital coordinates [cf. Leggett (1973b)].

On the other hand, it would be incorrect to assume therefore that the anisotropy of the susceptibility has no physical meaning. There may well be other factors which help to determine the orientation of $\mathbf{d}(\mathbf{n})$, particularly in restricted geometries (see Sec. X) and if they are dominant the measured susceptibility will indeed by "at constant $\mathbf{d}(\mathbf{n})$": more generally, the anisotropy of the susceptibility will help determine the equilibrium state. Moreover, the finite-q susceptibility (cf. Sec. IX) will be truly anisotropic, in the sense that it will genuinely depend on the orientation of the measuring field relative to the "external" field, if any. In this respect the situation is very similar to that in an isotropic antiferromagnet: there, a static uniform external magnetic field will always tend to set the axis of the "staggered magnetization" $\mathbf{N}$ perpendicular to the field so as to maximize the susceptibility, and if we then apply a uniform small measuring field perpendicular to the external one, all that happens is that $\mathbf{N}$ rotates so as to be perpendicular to the sum of external and measuring field. Such an experiment therefore always measures the susceptibility perpendicular to the axis of $\mathbf{N}$. On the other hand, a finite-q probe would induce a different response for the measuring field parallel and perpendicular to the uniform one. The only difference in the case of the triplet superfluid is that "bulk" factors which in an antiferromagnet usually destroy the isotropy (crystalline anisotropy, etc.) are usually absent here.

One property of a triplet superfluid which has no analog in the singlet case is a quantity we might call the "spin superfluid density." To define this quantity, let us first consider an ESP state in the "proper" axes. Then, at least so long as there are no spin-nonconserving terms in the Hamiltonian, the "up" and "down" spin Cooper pairs may be regarded as independent systems for most purposes. In particular, we can imagine a situation in which they are flowing independently in opposite directions, so that (by symmetry) the total mass current is zero. In such a state the up and down order parameters have equal and opposite spatial variations of phase:

$$\begin{aligned}\Psi_{\uparrow\uparrow}(\mathbf{n}:\mathbf{R}) &= \Psi_{\uparrow\uparrow}(\mathbf{n}:0)\exp 2im\mathbf{v}_{\mathrm{sp}}\cdot\mathbf{R}/\hbar\\ \Psi_{\downarrow\downarrow}(\mathbf{n}:\mathbf{R}) &= \Psi_{\downarrow\downarrow}(\mathbf{n}:0)\exp -2im\mathbf{v}_{\mathrm{sp}}\cdot\mathbf{R}/\hbar.\end{aligned} \tag{7.58}$$

Here $\mathbf{R}$ is the center-of-mass coordinate of the pairs, and the variation is assumed to be slow on the scale of a correlation length. Comparing these formulae with Eq. (7.33), we see that the vector $\mathbf{d}(\mathbf{n})$ is rotating uniformly around the z axis as a function of $\mathbf{R}$, with the phase gradient related to $\mathbf{v}_{\mathrm{sp}}$ by

$$\mathbf{v}_{\mathrm{sp}} = -(\hbar/2m)\nabla\varphi, \tag{7.59}$$

which is reminiscent of the definition of the ordinary superfluid velocity for a simple superfluid. Clearly we can now consider more general states of an arbitrary (not necessarily ESP) superfluid, in which the system of $\mathbf{d}$ vectors as a function of position is rotating uniformly around some arbitrary axis with some phase gradient: and we can define a dyadic quantity $\Omega_{i\alpha}$ such that for small spatial increments $\delta\mathbf{R}$ we have (sums over repeated indices are implied)

$$\delta d_i = (2m/\hbar)\epsilon_{ijk}\, d_j\Omega_{k\alpha}\delta\mathbf{R}_\alpha. \tag{7.60}$$

Here Roman subscripts refer to spin space and Greek subscripts to coordinate space: we will use this convention hereafter whenever there is a danger of confusion. The dyadic $\Omega_{k\alpha}$ has the dimensions of a velocity, and in the special case considered above we evidently have $\Omega_{k\alpha} = \delta_{k,z}v_{\mathrm{sp},\alpha}$.

The spin current dyadic $\mathbf{J}^{\mathrm{sp}}$ is defined by the continuity equation

$$\partial S_i/\partial t + (\partial/\partial x_\alpha)J_{i\alpha}{}^{\mathrm{sp}} = 0. \tag{7.61}$$

Evidently, in the quasiequilibrium state of "spin counterflow" described by Eq. (7.60) the spin current should be proportional to the quantity $\mathbf{\Omega}$: we expect an equation of the form

$$J_{i\alpha}{}^{\mathrm{sp}} = (\hbar/2m)\rho_{ij,\alpha\beta}{}^{\mathrm{spin}}\Omega_{j\beta}, \tag{7.62}$$

where the factor of $\hbar/2m$ has been inserted for subsequent convenience, and where the "spin superfluid density" ρ^{spin} is a *fourth*-rank tensor.

To evaluate ρ^{spin} let us first neglect *all* Fermi-liquid effects, that is, consider a superfluid gas of particles of mass m (not m^*). Then let us start with an ESP state in the "proper" axes, and let $\Omega_{k\alpha}$ be given by $\delta_{k,z}v_{\mathrm{sp},\alpha}$ as above: that is, we suppose that the up and down spins are moving uniformly in counterflow. Now since the two sets of spins form two completely independent systems, we can associate with each its own superfluid velocity $\mathbf{v}_s$ and mass current $\mathbf{J}$: using the subscripts $\uparrow$, $\downarrow$ to denote the up and down spins, respectively, we obviously have

$$\begin{aligned}&\mathbf{v}_{s\uparrow} = -\mathbf{v}_{s\downarrow} = \mathbf{v}_{\mathrm{sp}}, \qquad \mathbf{J}_{z\uparrow}{}^{\mathrm{sp}} = -\mathbf{J}_{z\downarrow}{}^{\mathrm{sp}} = (\hbar/2m)\mathbf{J}_\uparrow,\\ &\mathbf{J}_x{}^{\mathrm{sp}} \equiv \mathbf{J}_y{}^{\mathrm{sp}} \equiv 0.\end{aligned} \tag{7.63}$$

Then comparing Eq. (7.62) with the definition of the ordinary superfluid density tensor when only superfluid is moving ($\mathbf{J} = \boldsymbol{\rho}_s\mathbf{v}_s$), we clearly have simply [cf. Eq. (6.22)]

$$\begin{aligned}&\rho_{zz:\alpha\beta}{}^{\mathrm{spin}} = \rho_{\alpha\beta}{}^{s} = \rho[1 - Y_{\alpha\beta}(T)],\\ &\rho_{xz}{}^{\mathrm{spin}} \equiv \rho_{yz}{}^{\mathrm{spin}} \equiv 0.\end{aligned} \tag{7.64}$$

Suppose now that our state is unitary and $\mathbf{d}$ lies along the y axis at the point in question. Then clearly the above argument goes through for any choice of the x and z axis, so that we must have for $i, j = x$ or z

$$\rho_{ij,\alpha\beta}{}^{\mathrm{spin}} = \delta_{ij}\rho_{\alpha\beta}{}^{s}, \qquad \rho_{yi,\alpha\beta}{}^{\mathrm{spin}} \equiv 0. \tag{7.65}$$

On the other hand, suppose that we now let $\Omega_{k\alpha}$ be given by $\delta_{k,y}v_{\mathrm{sp},\alpha}$. This means that the system of $\mathbf{d}$ vectors is rotating around the y axis: but this rotation has no physical significance (since all the $\mathbf{d}$'s are *along* the y axis) and hence cannot correspond to any physical currents. Consequently, $\rho_{iy,\alpha\beta}{}^{\mathrm{spin}} \equiv 0$, $i = x, y, z$. Putting these results together and writing them in a rotationally invariant form, we therefore find for any unitary ESP state of a superfluid *gas* the result

$$\begin{aligned}\rho_{ij,\alpha\beta}{}^{\mathrm{spin}} &= \rho(\delta_{ij} - d_i^*d_j/|\,\mathbf{d}\,|^2)[\delta_{\alpha\beta} - Y_{\alpha\beta}(T)]\\ &\equiv \rho\int(d\Omega/4\pi)(\delta_{ij} - d_i^*(\mathbf{n})\,d_j(\mathbf{n})/|\,\mathbf{d}(\mathbf{n})\,|^2)\\ &\quad\cdot 3n_\alpha n_\beta[1 - Y(\mathbf{n}:T)]\end{aligned} \tag{7.66}$$

where as above [see Eq. (7.50)]

$$Y(\mathbf{n}:T) \equiv \int_0^\infty d\epsilon_k \tfrac{1}{2}\beta\,\mathrm{sech}^2\tfrac{1}{2}\beta E_{\mathrm{k}}. \tag{7.67}$$

From the form of Eq. (7.66) it is intuitively plausible that it should apply more generally to all unitary states of a superfluid gas, whether or not they are ESP. This is in fact correct, although we shall not prove it here [a proof can be constructed by defining a "spin normal density" and calculating it in a way analogous to the calculations of ρ_n in Sec. VI, with separate consideration of each pair state $(\mathbf{k}, -\mathbf{k})$].

The question of Fermi-liquid corrections to Eq. (7.66) is considerably more delicate than in the case of χ or ρ_n, since the spin current, unlike the spin density or mass current, is not a conserved quantity. First, we have to remember that for a Fermi liquid, according to Eq. (2.15), the second part of Eq. (7.63) has to be multiplied by a factor of $(m/m^*)(1+\frac{1}{12}Z_1)$. Secondly, there is the by now standard molecular-field correction: this is actually slightly cumbrous and, since the existing evidence indicates that the factor $Z_1/12$ is rather small for ^{3}He, we shall put $Z_1 = 0$. In that case the only effect of Fermi-liquid corrections is to multiply Eq. (7.66) by (m/m^*). If we compare the resulting expression with (6.33) and (6.31), we see that *near* T_c (only) we have for an ESP state in the "proper" axis the relation

$$\rho_{zz,\alpha\beta}{}^{\mathrm{spin}} = \rho_{\alpha\beta}{}^{s} \qquad (T \to T_c) \tag{7.68}$$

(this result is actually not affected by the inclusion of a finite Z_1). However, as $T \to 0$ we get rather

$$\rho_{zz,\alpha\beta}{}^{\mathrm{spin}} = (m/m^*)\rho_{\alpha\beta}{}^{s}(\equiv m/m^*) \qquad (T \to 0). \tag{7.69}$$

Finally, we consider the problem of spin-dependent correlations in a triplet superfluid state. First, we see that "one-particle" averages of the form

$$\langle M^{(1)}\rangle = \sum_{i=1}^{N} f(\mathbf{r}_i, \mathbf{p}_i, \boldsymbol{\sigma}_i) \tag{7.70}$$

can be written for a uniform state of the system in the form

$$\langle M^{(1)}\rangle = \sum_{\mathbf{k}} \{ f_{\mathbf{k}}\langle n_{\mathbf{k}}\rangle + \mathbf{g}_{\mathbf{k}}\cdot\langle\boldsymbol{\sigma}_{\mathbf{k}}\rangle\} \tag{7.71}$$

and treated in analogy with the discussion of Sec. VI.D. Evidently, the expectation value (7.71) will be changed very little from its normal-state value unless $f_{\mathbf{k}}$ or $\mathbf{g}_{\mathbf{k}}$ is strongly varying near the Fermi surface. In particular, the total spin $\mathbf{S}$ of the system [and even the differential spin polarization $\mathbf{S}(\mathbf{n})$ defined in Eq. (7.49a)] will remain rigorously zero in the approximation of a constant density of states, *even if the superfluid state is nonunitary*. That is, the "spin of the Cooper pairs" $\mathbf{d}(\mathbf{n}) \times \mathbf{d}^*(\mathbf{n})$ bears *no* direct relation to the differential spin polarization $\mathbf{S}(\mathbf{n})$: the existence of a finite $\mathbf{d} \times \mathbf{d}^*$ indicates only that the up-spins (say) are more correlated than the down-spins, not that there are more of them.

As to the spin-dependent two-particle correlations, most operators of interest can be written in the general form

$$\begin{aligned} M^{(2)} &= \tfrac{1}{2}\sum_{ij:kl} g_{ij}(\mathbf{r}_k - \mathbf{r}_l)\sigma_i{}^{(k)}\sigma_j{}^{(l)}, \qquad g_{ij}(\mathbf{r}_k - \mathbf{r}_l) \\ &= g_{ji}(\mathbf{r}_k - \mathbf{r}_l) = g_{ij}(\mathbf{r}_l - \mathbf{r}_k) \end{aligned} \tag{7.72}$$

(where k, l number particles and i, j Cartesian components of spin in units of $\hbar$). We can argue as follows (see Appendix for a more general and rigorous derivation): Consider a system of two particles, described by the triplet wave function (7.21), and express the amplitudes $\chi_{\uparrow\uparrow}$, etc., in terms of a vector $\mathbf{d}'$ as in (7.31). Then we may easily verify, by direct calculation, the relation

$$\tfrac{1}{2}\langle\sigma_i{}^{(1)}\sigma_j{}^{(2)} + \sigma_j{}^{(2)}\sigma_i{}^{(1)}\rangle = \tfrac{1}{4}(\delta_{ij}\,|\,\mathbf{d}'\,|^2 - 2\,\mathrm{Re}d_i{}'^*d_j{}') \tag{7.73}$$

(where the factor of 1/4 comes from the fact that σ_k is the component of a spin vector in units of $\hbar$). Consequently, by analogy with the spin-independent result (6.46) we have

$$\begin{aligned} \langle M^{2)}\rangle = \tfrac{1}{4}\int d\mathbf{r} \sum_{ij} g_{ij}(\mathbf{r})\{\delta_{ij}\,|\,\mathbf{F}(\mathbf{r})\,|^2 \\ - 2\,\mathrm{Re}F_i{}^*(\mathbf{r})F^j(\mathbf{r})\}, \end{aligned} \tag{7.74}$$

where the notation $\mathbf{F}(\mathbf{r})$ is as explained in Subsection C above.

E. Summary

Since it appears very likely that a good description of ^{3}He–A and B can be given, for most purposes, in terms of unitary states alone, we assume in this summary that we are dealing with such states. (For the nonunitary states which occur near T_c in a magnetic field, see Sec. XIII).

In a unitary triplet state, a Cooper pair can form in any linear combination of the three spin states $|\uparrow\uparrow\rangle$, $|\uparrow\downarrow + \downarrow\uparrow\rangle$, $|\downarrow\downarrow\rangle$, as in Eq. (7.2), with amplitudes $\varphi_{\uparrow\uparrow}(\mathbf{r}_1 - \mathbf{r}_2)$, etc. The Fourier transforms of the amplitudes, $F_{\mathbf{k}\uparrow\uparrow}$ etc., can be regarded as the components of a symmetric matrix $\hat{F}_{\mathbf{k},\alpha\beta}$ [and can be expressed as the "anomalous averages" of annihilation operators $a_{\mathbf{k}\alpha}$ according to Eq. (7.22)]. This matrix, $\hat{F}_{\mathbf{k}}$, or equivalently the matrix $\hat{\Psi}(\mathbf{n}) \equiv \sum_{|\mathbf{k}|}\hat{F}_{\mathbf{k}}$, can be expressed in terms of an over-all amplitude Ψ and a normalized complex vector $\mathbf{d}(\mathbf{n})$ by the relation [Eqs. (7.35) and (7.36)]

$$\Psi_{\alpha\beta}(\mathbf{n}) = \Psi i \sum_{i=1}^{3} (\sigma_i\sigma_2)_{\alpha\beta} d_i(\mathbf{n}), \qquad \int \frac{d\Omega}{4\pi}\,|\,\mathbf{d}(\mathbf{n})\,|^2 = 1, \tag{7.75}$$

where $\mathbf{d}(\mathbf{n})$ satisfies the conditions $\mathbf{d}(\mathbf{n}) = -\mathbf{d}(-\mathbf{n})$ and (for a unitary state) $\mathbf{d}(\mathbf{n}) \times \mathbf{d}^*(\mathbf{n}) = 0$, (i.e., $\mathbf{d}$ is real apart from an over-all phase factor). The quantity $|\,\mathbf{d}(\mathbf{n})\,|^2$ then gives the relative number of Cooper pairs forming at point $\mathbf{n}$ on the Fermi surface, and the direction of $\mathbf{d}(\mathbf{n})$ gives the axis with respect to which they are in a state of spin projection zero. In a similar way the gap $\Delta_{\mathbf{k}} = \Delta(\mathbf{n})$ is a matrix and its elements $\Delta_{\mathbf{k},\alpha\beta}$ must individually obey the gap equation (5.52), i.e.,

$$\Delta_{\mathbf{k},\alpha\beta}(T) = -\sum_{\mathbf{k}'} V(\mathbf{k}, \mathbf{k}')\frac{\Delta_{\mathbf{k}',\alpha\beta}(T)}{2E_{\mathbf{k}'}(T)} \tanh \beta E_{\mathbf{k}'}(T)/2, \tag{7.76}$$

where the (scalar) excitation energy E_k is given by

$$E_k \equiv +(\epsilon_{\mathbf{k}}{}^2 + |\,\Delta(\mathbf{n})\,|^2)^{1/2}, \qquad |\,\Delta(\mathbf{n})\,|^2 \equiv \hat{\Delta}(\mathbf{n})\hat{\Delta}^\dagger(\mathbf{n}), \tag{7.77}$$

where the unitary condition guarantees that $|\Delta(\mathbf{n})|^2$ is a c number, not a matrix. In analogy to Eq. (7.75) we may write the matrix $\hat{\Delta}(\mathbf{n})$ in terms of $\mathbf{d}(\mathbf{n})$ and an "over-all" gap $\Delta(T)$

$$\Delta_{\alpha\beta}(\mathbf{n}) = \Delta i \sum_{i=1}^{3} (\sigma_i\sigma_2)_{\alpha\beta} d_i(\mathbf{n}) \tag{7.78}$$

so that

$$|\Delta(\mathbf{n})|^2 = \Delta^2 |\mathbf{d}(\mathbf{n})|^2. \tag{7.79}$$

[To be quite rigorous, the $\mathbf{d}(\mathbf{n})$ entering (7.78) has a slightly different $\mathbf{n}$-dependence from that entering (7.75), as in the case of $f(\mathbf{n})$ in the singlet case (cf. Sec. V.E). As usual we neglect the small corrections due to this.]

In the GL region the free energy can be written in terms of $\mathbf{d}(\mathbf{n})$ and either Ψ or Δ by combining Eqs. (7.25) and (7.41)

$$F\{\mathbf{d}(\mathbf{n}), \Psi, T\} = \alpha[(T - T_c)/T_c]\Psi^2 + \tfrac{1}{2}\beta\kappa\Psi^4, \tag{7.80}$$

where α and β have the same values as in the spin singlet case [see Eqs. (5.70–74)], and κ is given by Eq. (7.42) (with $\mathbf{d} \times \mathbf{d}^* \equiv 0$)

$$\kappa \equiv \int \frac{d\Omega}{4\pi} |\mathbf{d}(\mathbf{n})|^4. \quad \text{(unitary states)} \tag{7.81}$$

In terms of the over-all gap Δ Eq. (7.80) is rewritten explicitly [cf. (5.75) and (7.79)]

$$F\{\mathbf{d}(\mathbf{n}), \Delta, T\} = \tfrac{1}{2}(\mathbf{dn}/d\epsilon)\{-(1 - T/T_c)\Delta^2 + \tfrac{1}{2}\cdot\tfrac{7}{8}\zeta(3)(\pi k_B T_c)^{-2}\kappa\Delta^4\} \tag{7.82}$$

a form which is of course equally valid for singlet states, with κ then given by Eq. (5.76).

All properties of a triplet-paired superfluid which do not involve the spin (e.g., specific heat, normal density, pair relative angular momentum, etc.) are obtained trivially from the corresponding formulas for the singlet case (Sec. VI) by simply taking $|\Delta(\mathbf{n})|^2$ and $E_\mathbf{k}$ in these formulas to be given by Eq. (7.77) or (7.79). [Thus, they involve only the magnitude of $\mathbf{d}(\mathbf{n})$, not its direction]. The spin susceptibility and "spin superfluid density" do however depend on the direction of $\mathbf{d}(\mathbf{n})$ according to Eqs. (7.53), (7.54), and (7.66). In particular, in the ESP case [$\mathbf{d}(\mathbf{n})$ coplanar for all $\mathbf{n}$] the "experimental" susceptibility is unchanged from the normal state value.

Finally, in the $l = 1$ case the thermodynamically stable state [the BW state, Eq. (7.30)] is non-ESP, and there is reason to believe that this feature also applies for $l > 1$.

VIII. INTERLUDE: PRELIMINARY INDENTIFICATION OF ^{3}He-A AND B

Although not all of the results discussed in the previous sections had been written down explicitly before the new phases of liquid ^{3}He were discovered in 1972, they were all more or less implicit in the earlier work. This is therefore a good point at which to interrupt the theoretical development and ask: How far does the theory developed so far explain the experimental data available on ^{3}He? In particular, if ^{3}He–A and B are anisotropic superfluids of the general type discussed, how far can we identify them?

In this section we shall refer to the companion paper of Wheatley for the experimental data and concentrate on those quantities for which the theory has already been developed, in particular the phase diagram, specific heat, spin susceptibility, and normal density. Let us start with the spin susceptibility. The susceptibility of ^{3}He–A is very nearly equal to that of the normal liquid and is temperature-independent within the accuracy of the experiments: since both spin singlet pairing and non-ESP triplet pairing would result in decreased susceptibility [cf. Eqs. (6.13) and (7.53)], we must conclude that if ^{3}He–A is an anisotropic superfluid at all, it must be in an ESP triplet phase. As to the B phase, its susceptibility is decreased but apparently remains finite as the temperature tends to zero (in fact, it appears to be tending to a value about one-third of the normal state value). This rules out both spin singlet pairing [which would give $\chi = 0$ at $T = 0$, cf. Eq. (6.13)], and ESP triplet pairing (which would give $\chi = \chi_n$). So we can draw our first conclusion: Both the A and B phases correspond to triplet pairing, the A phase being ESP and the B phase non-ESP.

A second important conclusion follows from the fact that, apart possibly from the kink near the PCP (which remains a mystery, see Leggett, 1974b) the B–normal (B–N) phase coexistence curve appears to be a continuation of the A–normal (A–N) coexistence curve in the P–T plane. More generally, the thermodynamic and susceptibility measurements seem to imply that the A and B phases have the same transition temperature over a wide range of pressure (see Wheatley, Sec. IV). This indicates that the two condensed phases result from the same instability of the normal phase. Referring to the results of Sec. III, then, we see that they must correspond to pairing with the same (dominant) l value: this must of course be odd, since they are spin triplet phases.

Now we immediately encounter a substantial difficulty. Suppose for instance we assume that the l value is 1. Then, referring to the results of Sec. VII.B, we see that according to the theory developed there the non-ESP BW state should always be more stable (in zero magnetic field) than any state of the ESP type. How then can the A phase, which is indisputably of the ESP variety, be stable at all? At first sight one might try to resolve the difficulty by assuming a higher l value, say $l = 3$ or 5. However, explicit calculations seem to indicate (cf. Barton and Moore, 1973) that this does not improve the situation: the most stable states for any l value seem to be of the non-ESP type.

Putting this difficulty aside for the moment, let us examine how well the quantitative data fit the theory. First, the specific heat. As predicted, this has a substantial jump across both the A–N and B–N transitions, and then falls rapidly in both cases as the temperature is lowered. This is in qualitative agreement with the theory. However, the actual relative magnitude of the jump at T_c which according to Eq. (6.8) should be not more than 1.42, is actually between 1.6 and 1.9. Secondly, the spin susceptibility[20]: according

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[20] Note incidentally that the low temperature limit of the susceptibility, while in at least rough agreement with Eq. (7.56), is nowhere near the value ($\sim 0.1\chi_n$) which would be obtained by inserting the "paramagnon" value $Z_0/4 = -\bar{I} = -0.95$ (cf. Sec. II.D).

to Eq. (7.54) we find quite generally in the limit $T \to T_c$

$$\frac{\hat{\chi}'(T/T_c)}{\chi_n} = \frac{\hat{\theta}'(T/T_c)}{1+\frac{1}{4}Z_0}. \tag{8.1}$$

According to the considerations of Sec. VII.D, we will always measure the maximum eigenvalue of $\hat{\chi}$: hence, from Eq. (7.55), we find

$$\frac{\chi'(T/T_c)}{\chi_n} \leq \frac{\frac{1}{3}Y'(T/T_c)}{1+\frac{1}{4}Z_0} \approx \tfrac{2}{3}\kappa^{-1}(1+\tfrac{1}{4}Z_0)^{-1} \approx 2.5\kappa^{-1} \leq 2.5. \tag{8.2}$$

For the $l = 1$ BW state (and indeed any state in which the $S_z = 0$ component is populated equally to the other two) the inequality is an equality. The experimental value of 4.7 for χ'/χ_n is in serious violation of Eq. (8.2). Finally, the superfluid density: according to the theory this is in general a tensor quantity, but according to Eqs. (6.32) and (6.28) the slope of its *average* value $\bar{\rho}_s$ near T_c is given by

$$\frac{\bar{\rho}_s'(T/T_c)}{\rho} = (1+\tfrac{1}{3}F_1)^{-1}Y'(T/T_c) \approx 0.33\ \kappa^{-1} \leq 0.33. \tag{8.3}$$

The experimental value is more like 0.6. Note that even if we assume we measure the *maximum* eigenvalue (which is actually unlikely, see Sec. X) the estimate (8.3) is increased only by a factor of 6/5.

One prediction of the theory which appears in reasonable though not perfect agreement with experiment is that the pressure dependence of χ/χ_n and ρ_n/ρ *for given reduced temperature* $T/T_c(p)$ should be given entirely by the variation of the Landau parameters Z_0 and F_1, respectively. In particular, near T_c they should vary as $(1+\frac{1}{4}Z_0)^{-1}$ and $(1+\frac{1}{3}F_1)^{-1}$, respectively. If anything the two quantities seem to vary *less* than expected with pressure. Generally, we see that the absolute values of all three quantities $\Delta C/C_n$, $\Delta\chi$, and ρ_s are larger than predicted, very considerably so in the case of the last two. However, the *ratio* of the slope of χ to that of ρ_s near T_c is relatively well predicted by theory (see Wheatley, Sec. IV).

In the next section we shall discuss the beautiful idea of Anderson and Brinkman (1973) which goes, at least, a very long way towards resolving the difficulties encountered in this section.

IX. SPIN FLUCTUATIONS, FEEDBACK EFFECTS, AND "STRONG COUPLING"

The basic physical idea of the Anderson–Brinkman (AB) theory is that the formation of the superfluid state modifies the pairing interaction between quasiparticles, and that the precise nature of the modification depends on the particular kind of superfluid state formed. Crudely speaking, if for a given kind of state the relevant part of the pairing interaction is weakened (made less attractive) this will decrease its condensation energy below the BCS value, while if the pairing interaction is strengthened the condensation energy will exceed the BCS prediction. As a result, a state which is unstable in the BCS approximation may become the stable ground state (or free energy minimum) once such "feedback" effects are included.

The specific mechanism originally considered by Anderson and Brinkman (1973) and subsequently in more detail by them and others (see below) is the modification of the part of the effective pairing interaction associated with virtual spin polarization of the medium (or "exchange of spin fluctuations") as described in Sec. IV. However, while it seems probable that this is quantitatively the most important "feedback" effect, it is clear that in principle we cannot exclude modification of other parts of the interaction—e.g., that due to exchange of transverse current fluctuations, which will certainly also change in the superfluid state. Moreover, one can show that feedback effects of all types vanish in the "weak coupling limit" $k_BT_c/\epsilon_F \to 0$. If therefore we keep them, it is not obvious that we should not also keep other types of effect which would also vanish in this limit [and, of course, also effects which are proportional to (say) $(\ln\beta_c\epsilon_c)^{-1}$]. Such effects have become known generally as "strong coupling" effects. (The effects which are usually called "strong coupling" in the theory of superconductivity are one particular type, namely the variation of the pairing interaction with energy over a range $\sim k_BT_c$.) To a certain extent it is possible to take them into account quite generally (at least in the GL region) by a technique based on analysis of the possible forms of the free energy as a function of the order parameter. It turns out that these forms can actually be labelled by a small number of parameters, and that depending on the values of those parameters different types of state may be the stable free energy minimum. One can then try to determine the values of the parameters from some microscopic model, e.g., by assuming that the *only* important strong coupling effect is the feedback associated with spin fluctuation exchange. However, there are certain conclusions which can be drawn even without a detailed knowledge of their values.

In the first subsection we shall outline the physical basis of the theory along lines similar to those of the original AB paper. In the second, we explain the principle of the technique based on general forms for the free energy. In the third, we review the results of recent more sophisticated treatments of the spin fluctuation exchange process. Finally, in Subsection D we describe and discuss the states which are most widely believed to correspond to ³He-A and B.

A. Spin fluctuation feedback—the physical idea

As was pointed out in Sec. IV, in the normal phase of liquid ³He there is an effective "indirect" interaction between quasiparticles due to virtual spin polarization of the medium:

$$V_{\text{eff}}(\mathbf{r}'-\mathbf{r}, t'-t) = -\zeta_0^2\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'\cdot\tfrac{1}{2}[\chi(\mathbf{r}'-\mathbf{r}, t'-t) + \chi(\mathbf{r}-\mathbf{r}', t-t')]. \tag{9.1}$$

This interaction depends only on the relative spin orientation of the quasiparticles, not on the absolute orientation. Now, when we go over into the superfluid state the *static* susceptibility χ, as we have already seen, is modified and becomes in fact a *tensor* in spin space according to Eq. (7.53). Evidently, the dynamic susceptibility $\chi(\mathbf{r}'-\mathbf{r}, t'-t)$ is likewise modified. Following through an argument exactly similar to that of Sec. IV, we now obtain an effective interaction (where summation over repeated indices is implied)

$$V_{\text{eff}}(\mathbf{r}'-\mathbf{r},t'-t)=-\zeta_0{}^2\sigma_i\sigma_j{}'\tfrac{1}{2}[\chi_{ji}(\mathbf{r}'-\mathbf{r},t'-t)+\chi_{ij}(\mathbf{r}-\mathbf{r}',t-t')] \tag{9.2}$$

so that the interaction now depends on the absolute orientations. [There is nothing paradoxical in this, because the orientation of the Cooper pairs—or the vectors $\mathbf{d}(\mathbf{n})$—now define a special set of axes. We shall return briefly below to the question of why it is reasonable to use here the "susceptibility at constant $\mathbf{d}(\mathbf{n})$"—cf. Sec. VII.D.]

In order to see qualitatively the kind of effects which this anisotropy of the interaction is likely to have, we shall make a drastic approximation, which is similar though not quite identical to the original AB paper: We replace the *change* of the interaction (9.2), which is both nonlocal in time (retarded) and may depend on the *direction* of $\mathbf{r}-\mathbf{r}'$, by a simpler interaction of the form

$$\delta V_{\text{eff}}(\mathbf{r}'-\mathbf{r},t'-t)=-\zeta_0{}^2\sigma_i\sigma_j{}'(\gamma\hbar)^{-2}\delta\chi_{ij}g(|\mathbf{r}-\mathbf{r}'|), \tag{9.3}$$

where $\delta\chi_{ij}$ is the change of the static susceptibility tensor (we recall that the dynamic spin susceptibility $\chi(\mathbf{r}-\mathbf{r}',t-t')$ is defined with an extra factor of $(\gamma\hbar)^{-2}$ relative to the experimental static χ, see Sec. IV), and g is some function whose range and strength can be fitted so as to reproduce as closely as possible the true expression (9.2).

Now according to Eq. (7.4) the change in the expectation value of the potential energy of the Cooper pairs due to the change in the interaction is[21]

$$\Delta\langle V\rangle=-\tfrac{1}{8}(\gamma\hbar)^{-2}\zeta_0{}^2\delta\chi_{ij}[\int d\mathbf{r}g(\mathbf{r})\{\delta_{ij}|\mathbf{F}(\mathbf{r})|^2-2\,\mathrm{Re}F_i{}^*(\mathbf{r})F_j(\mathbf{r})]\}. \tag{9.4}$$

The integral can be converted [using the fact that the components of $\mathbf{d}(\mathbf{n})$ are spherical harmonics of one l value only, and the decomposition theorem for spherical harmonics] into the form

$$g_l\Psi^2\int(d\Omega/4\pi)\{\delta_{ij}|\mathbf{d}(\mathbf{n})|^2-2\,\mathrm{Re}\,d_i{}^*(\mathbf{n})\,d_j(\mathbf{n})\}, \tag{9.5}$$

where g_l is the appropriate spherical harmonic decomposition of $g(\mathbf{r})$ over the Fermi surface. As for the quantity $\delta\chi_{ij}$, it is given by Eq. (7.53) and (7.54). In particular, if we specialize to the region close to T_c and approximate the function $[1-Y(\mathbf{n}:T)]$ by its asymptotic value $\frac{7}{4}\zeta(3)|\mathbf{\Delta}(\mathbf{n})|^2/\pi^2k_B{}^2T_c{}^2$, we find

$$\delta\chi_{ij}=-\tfrac{1}{4}\gamma^2\hbar^2(dn/d\epsilon)(1+\tfrac{1}{4}Z_0)^{-2}\tfrac{7}{4}\zeta(3)(\pi k_BT_c)^{-2}\Delta^2\times\int(d\Omega/4\pi)\,\mathrm{Re}\,d_i{}^*(\mathbf{n})\,d_j(\mathbf{n}). \tag{9.6}$$

We combine Eqs. (9.5) and (9.6), express Ψ in terms of Δ by (5.68), and use the definition $\zeta_0\equiv(dn/d\epsilon)^{-1}Z_0$ (see Sec. IV). Then we finally get for $\Delta\langle V\rangle$ and hence for the change in the free energy ΔF the expression:

$$\Delta F=K(dn/d\epsilon)(k_BT_c)^{-2}\Delta^4\times[\langle\mathrm{Re}d_i{}^*d_j\rangle_{\text{av}}(\delta_{ij}-\langle 2\,\mathrm{Re}d_i{}^*d_j\rangle)] \tag{9.7}$$

where $\langle\ \rangle_{\text{av}}$ indicate averages over the Fermi surface, repeated indices are summed over, and the constant K is given by

$$K=(7/32\pi^2)\zeta(3)g_l[\tfrac{1}{4}Z_0/(1+\tfrac{1}{4}Z_0)]^2(\ln\zeta\beta_c\epsilon_c)^2. \tag{9.8}$$

The point, now, is that the factor in square brackets is *not* of the same form as the average (7.42) which determines the most stable state in BCS theory, and may favor a different type of state. The first term is always just 1, because of the normalization of $\mathbf{d}(\mathbf{n})$, and therefore always contributes a repulsive energy which for given Δ is independent of the configuration. The second term, however, favors states which are as anisotropic as possible in spin space. To see this, let us compare the "most isotropic" state for $l=1$ pairing, namely the BW state $\mathbf{d}(\mathbf{n})=\mathbf{n}$, with the highly anistropic ESP state (the so-called axial or ABM state, see below) given by (for example)

$$d_x(\mathbf{n})=(3/2)^{1/2}(n_y+in_z),\qquad d_y(\mathbf{n})\equiv d_z(\mathbf{n})\equiv 0. \tag{9.9}$$

We see that the term is the square brackets is $+\frac{1}{3}$ for the BW state but -1 for the state (9.9). Consequently, the spin fluctuation feedback effect, in this approximation, disfavors the BW state and favors the state (9.9), and if the constant K is large enough this effect may outweigh the effect of the BSC fourth-order terms [the integral (7.42)] which favor the BW state. That the (attractive) pairing energy due to exchange of spin fluctuations is reduced in the BW state is not surprising, since the susceptibility of this state and hence the "strength" of the spin fluctuations is reduced from the normal state values.

What is at first sight more difficult to understand intuitively is that the pairing energy is actually *increased* in the ABM state (9.9). The reason is as follows: in the normal state, there is a chance of finding a given pair of particles with antiparallel spins, and this will give a repulsive contribution to the energy due to exchange of spin fluctuations. Now in the ABM state all Cooper pairs are in a spin state with $S=1$ but $S_x=0$. In such a state the correlations of the y and z components of spin become more positive (i.e., the pairs form with parallel σ_y and σ_z) but the correlation of the x components is *negative*, i.e., the pairs have x component of spin antiparallel. The part of the potential energy arising from exchange of spin fluctuations then has a positive contribution from the y and z components but a negative contribution from the x components. The BCS approximation correctly estimates the attractive y and z contributions, but it overestimates the repulsive x contribution because the interaction is proportional to $\chi_{xx}\langle\sigma_x{}^{(1)}\sigma_x{}^{(2)}\rangle$, and χ_{xx} is *reduced* from its normal-state value [cf. Eqs. (9.6) and (9.9)].

The quantitative importance of this mechanism depends, of course, on the value of the constant K and hence of g_l. Since the ansatz (9.3) is anyway only a crude approximation introduced to illustrate the physical mechanism in-

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[21] The extra factor of $\frac{1}{2}$ arises because V_{eff} is a variational derivative. See subsection IX.C below.

volved, we will not discuss the value of g_l which might be thought plausible at this point. It is, however, worth mentioning that any reasonable estimate of g_l will result in the cancellation of the factor $[\ln(\zeta\beta_c\epsilon_c]^2$ and introduce at least one factor of (k_BT_C/ϵ_F), and that the factor in K involving Z_0 is large for ^{3}He ($\sim$10). This is important, because one might well ask why modification of the spin susceptibility should be so much more important than, say, that of the normal density? The latter will certainly affect that part of the "indirect" interaction between quasiparticles which is associated with exchange of transverse current fluctuations. (One may in fact show that it will tend to favor states which are as *orbitally* anisotropic as possible.) However, while other factors in the experession for the appropriate constant K' are much the same as in K, the factor involving Z_0 will be replaced by a similar one with $\frac{1}{4}Z_0$ replaced by $\frac{1}{3}F_1$. Since F_1 is large, this factor is then of order 1 rather than 10, and the effect associated with the normal density would therefore be expected to be only of order 10% of that due to the susceptibility modification.

B. The generalized free-energy approach

The approach based on generalized forms of the free energy was pioneered by Anderson and Brinkman (1973) and Mermin and Stare (1973). Subsequent work has been done by Brinkman and Anderson (1973), Mermin and Ambegaokar (1973b), Mermin (1973), Mermin and Stare (1974), and Barton and Moore (1974a,b); in this section we will discuss the principle of the method and some results, referring the reader to the above papers for details of the derivations [see also Ambegaokar (1974)].

Let us consider the GL region and imagine that we expand the free energy as a functional of the parameter $\mathbf{\Delta}(\mathbf{n})$ [or $\mathbf{d}(\mathbf{n})$] up to the fourth order in $\mathbf{d}$. As we saw in Sec. VII, in the BCS approximation the expansion has the simple form

$$\begin{aligned} F &= -\alpha(1-T/T_c)\Psi^2+\tfrac{1}{2}\kappa\beta\Psi^4 \\ &= \tfrac{1}{2}(dn/d\epsilon)\{-(1-T/T_c)\Delta^2+\tfrac{1}{2}\kappa\bar{\beta}\Delta^4\} \end{aligned} \tag{9.10}$$

$$\begin{aligned} \bar{\beta} &\equiv \tfrac{7}{8}\zeta(3)(\pi k_BT_c)^{-2}, \\ \kappa &\equiv \int (d\Omega/4\pi)\{|\mathbf{d}(\mathbf{n})|^4-[\mathbf{d}(\mathbf{n})\times\mathbf{d}^*(\mathbf{n})]^2\}. \end{aligned} \tag{9.10a}$$

More generally we can conceive that F can be a more complicated functional of $\mathbf{d}(\mathbf{n})$. However, the invariances of the Hamiltonian impose some strong constraints on the form of this general functional: F must be invariant against rotation in spin space and in ordinary (orbital) space separately and also of course gauge-invariant, i.e., invariant against multiplication of all $\mathbf{d}(\mathbf{n})$ by the same uniform phase factor. For an order parameter which is restricted to be composed of spherical harmonics of given l, there is only a finite number of invariant combinations I_i. We can therefore write the free energy in the form

$$F=\sum_i c_iI_i \tag{9.11}$$

and the coefficients c_i then form a parametrization of F. Depending on the values of the c_i, different states will be stable.

In what follows we will discuss the case of P-wave pairing, which has been studied in most detail among the spin triplet states. [For the case of spin singlet pairing the d-wave problem has been solved completely by Mermin (1974); however this unfortunately seems unlikely to be relevant to ^{3}He.] We take the quantities $\mathbf{d}(\mathbf{n})$, $\mathbf{\Delta}(\mathbf{n})$ and note that for $l=1$ pairing they can be written in the general form

$$d_i(\mathbf{n})=\sqrt{3}d_{\alpha i}n_\alpha, \qquad \Delta_i(\mathbf{n})=\sqrt{3}\Delta d_{\alpha i}n_\alpha. \tag{9.12}$$

There are nine complex quantities $d_{\alpha i}$ so that the order parameter has, in a sense, 18 real degrees of freedom. Nevertheless, the number of invariants which can occur in the free energy is limited. In fact, there is only one second-order invariant

$$I_0=d_{\alpha i}{}^*d_{\alpha i}\equiv 1 \tag{9.13}$$

(as usual, summation over repeated indices is implied). For the fourth-order invariants we have five possibilities (cf. Ambegaokar (1974)):

$$\begin{aligned} &I_1\equiv|\sum_{\alpha i}d_{\alpha i}{}^2|^2, \quad I_2\equiv d_{\beta i}{}^*d_{\beta j}{}^*d_{\alpha i}d_{\alpha j}, \quad I_3\equiv d_{\alpha i}{}^*d_{\beta i}{}^*d_{\alpha j}d_{\beta j} \\ &I_4\equiv(\sum_{\alpha i}|d_{\alpha i}|^2)^2(\equiv 1), \quad I_5\equiv d_{\alpha i}{}^*d_{\beta j}{}^*d_{\alpha j}d_{\beta i}. \end{aligned} \tag{9.14}$$

Using the fact that the average of $d_i{}^*(\mathbf{n})d_j(\mathbf{n})$ over the Fermi surface, which we write $\langle d_i{}^*d_j\rangle_{\text{av}}$ is just $d_{\alpha i}{}^*d_{\alpha j}$, we can write the invariants alternatively as

$$\begin{aligned} &I_1=|\langle\mathbf{d}^2\rangle_{\text{av}}|^2, \qquad I_2=\langle d_i{}^*d_j{}^*\rangle_{\text{av}}\cdot\langle d_id_j\rangle_{\text{av}}, \\ &I_3=\langle d_i{}^*d_j\rangle_{\text{av}}\cdot\langle d_i{}^*d_j\rangle_{\text{av}} \qquad I_4=(\langle|\mathbf{d}|^2\rangle_{\text{av}})^2\equiv 1, \\ &I_5=\langle d_i{}^*d_j\rangle_{\text{av}}\cdot\langle d_j{}^*d_i\rangle_{\text{av}}, \qquad I_0=\langle|\mathbf{d}|^2\rangle_{\text{av}}\equiv 1. \end{aligned} \tag{9.15}$$

The free energy is the written in the form

$$\begin{aligned} F &= -\alpha(1-T/T_c)\Psi^2+\tfrac{1}{2}(\sum_{i=1}^{5}a_iI_i)\beta\Psi^4 \\ &= \tfrac{1}{2}(dn/d\epsilon)\{-(1-T/T_c)\Delta^2+\tfrac{1}{2}(\sum_{i=1}^{5}a_iI_i)\bar{\beta}\Delta^4\} \end{aligned} \tag{9.16}$$

[for $\bar{\beta}$, see Eq. (9.10a)]. An alternative notation to be found at least equally frequently in the literature writes

$$d_\mu(\mathbf{n})=A_{\mu\nu}n_\nu, \tag{9.17}$$

regards the quantities $A_{\mu\nu}$ as forming a 3 × 3 matrix, and writes (apart from trivial variations for the quadratic term and equivalent forms for the traces)

$$\begin{aligned} F &= -\alpha(1-T/T_c)\,\mathrm{Tr}A^\dagger A+\beta_1|\mathrm{Tr}A\tilde{A}|^2+\beta_2(\mathrm{Tr}AA^\dagger)^2 \\ &\quad+\beta_3\,\mathrm{Tr}(A\tilde{A})(A\tilde{A})^*+\beta_4\,\mathrm{Tr}(AA^\dagger)^2 \\ &\quad+\beta_5\,\mathrm{Tr}AA^\dagger\cdot(AA^\dagger)^*, \end{aligned} \tag{9.18}$$

where $\tilde{A}$ and $A^\dagger$ are, respectively, the transpose and Hermitian conjugate of the matrix A. A comparison of Eqs. (9.15) and (9.17) and (9.18) gives the correspondences, apart from an overall factor

$$a_1=\beta_1,\ a_2=\beta_3,\ a_3=\beta_5,\ a_4=\beta_2,\ a_5=\beta_4. \tag{9.19}$$

We have chosen to define the parameters entering the free energy in terms of the expansion of the *normalized* order parameter $\mathbf{d}(\mathbf{n})$, or equivalently the normalized gap parameter: alternative conventions are of course equally possible and often used in the literature. The only effect is to multiply all the a_i's by a common overall factor relative to α. All considerations which follow in this section relate only to the *relative* magnitudes of the fourth-order coefficients a_i.

The BCS expression (9.10) for the free energy of a P-wave state may be easily verified to be a particular case of Eq. (9.16) corresponding to the values

$$a_2 = a_4 = a_5 = -a_3 = -2a_1 \equiv 2s (s \equiv 3/5). \qquad (9.20)$$

By comparison the naive spin fluctuation model result (9.7) gives a contribution δa_i to the a_i which we can express in terms of s and a single parameter δ' in the form [compare Eq. (9.7) with (9.15)]

$$\delta a_1 = \delta a_2 = 0, \quad \delta a_3 = -\delta a_4 = \delta a_5 \equiv -s\delta'. \qquad (9.21)$$

We already know that if δ' is sufficiently large (in fact, if $\delta' > \frac{1}{4}$) then the BW state is not the most stable. We are therefore led to ask how far one can analyze the stability of various P-wave states using the quite general form of free energy (9.16). Since we have essentially to find minima in an 18-dimensional space, the problem is not at all trivial.

At the time of writing, indeed, no complete analytic solution of the problem has appeared. We refer the reader to the papers by Mermin and Stare (1974) and Barton and Moore (1974a) for extended general discussions, and from now on shall restrict ourselves to the very much simpler problem (Mermin and Stare, 1973) in which the states of interest are restricted to obey the "unitary" condition $\mathbf{d}(\mathbf{n}) \times \mathbf{d}^*(\mathbf{n}) = 0$. It should be emphasized that there is no *a priori* reason to expect this condition to hold, but it appears probable (fortunately) that the states actually occurring in liquid ³He in zero magnetic field are in fact unitary.

For P states with the unitary restriction, then, the following results can be proved [for a proof see Mermin and Stare (1973) and Ambegaokar (1974)]: There exist exactly four classes of states which can be extrema of the free energy (9.16), states within a given class transforming into one another under rotation of the spin and/or orbital coordinates (which, of course, does not affect the values of the five invariants). We write down a typical representive of each class, arbitrarily normalizing $d_{\alpha i}$ for convenience as above so that $\sum_{\alpha i} | d_{\alpha i} |^2 = 1$.

(a) The "isotropic" or "Balian–Werthamer" (BW) state [Eq. (7.30) or (7.43)]

$$d_{\alpha i} = (1/\sqrt{3})\delta_{\alpha i}, \qquad \text{i.e.,} \quad \mathbf{d}(\mathbf{n}) = \mathbf{n}. \qquad (9.22)$$

(b) The "two-dimensional" (2D) or "planar" state

$$d_{\alpha i} = (1/\sqrt{2})\delta_{\alpha i}(1 - \delta_{iz}), \qquad \text{i.e.,} \quad d_x(\mathbf{n}) = (3/2)^{1/2} n_x,$$
$$d_y(\mathbf{n}) = (3/2)^{1/2} n_y, \qquad d_z(\mathbf{n}) = 0. \qquad (9.23)$$

TABLE I. Values of the five fourth-order invariants for the four unitary P-wave extrema of the free energy.

	BW	2D	ABM	Polar
I_1	1	1	0	1
I_2	$\frac{1}{3}$	$\frac{1}{2}$	0	1
I_3	$\frac{1}{3}$	$\frac{1}{2}$	1	1
I_4	1	1	1	1
I_5	$\frac{1}{3}$	$\frac{1}{2}$	1	1
F_{BCS}	$\frac{5}{3}s$	$2s$	$2s$	$3s$
F_{sf}	$\frac{1}{3}s\delta'$	0	$-s\delta'$	$-s\delta'$

(c) The "axial" or "Anderson–Brinkman–Morel" (ABM) state

$$d_{yx} = -id_{zx} = 1/\sqrt{2}, \qquad \text{all other components zero i.e.,}$$
$$d_x(\mathbf{n}) = (3/2)^{1/2}(n_y + in_z), \quad d_y(\mathbf{n}) = d_z(\mathbf{n}) = 0. \qquad (9.24)$$

(d) The "polar" or "one-dimensional" state

$$d_{zz} = 1, \qquad \text{all other components zero. i.e.,}$$
$$d_z(\mathbf{n}) = \sqrt{3} n_z, \quad d_x(\mathbf{n}) = d_y(\mathbf{n}) = 0. \qquad (9.25)$$

Substitution of the forms (9.22)–(9.25) into (9.14) enables us to construct a table (Table I) of values of the five fourth-order invariants for each of these states. (I_0 and I_4 are of course 1 by definition for each state.) We give also the fourth-order BCS free energy, F_{BCS}, from Eq. (9.20) and the spin fluctuation free energy F_{sf} as calculated from the naive model of the last subsection [see Eq. (9.21)].

We verify directly from Table I that the BW state is the most stable within the BCS approximation, but that when the "naive" spin-fluctuation energy F_{sf} is added the ABM state is more stable if $\delta' > \frac{1}{4}$. It can also be verified that, quite independently of any particular microscopic model (i.e., for quite arbitrary values of the parameters a_i), the "two-dimensional" state is *never* the stable free energy minimum. This is a somewhat surprising result, which could scarcely have been anticipated on any simple intuitive grounds. The polar state can in principle be the absolute minimum, but only if the non-BCS contributions to the free energy are at least comparable to the BCS ones. Of the unitary P states, therefore, the BW and ABM states seem the ones most likely to be realized in ³He and it is these which are commonly identified with ³He-B and ³He-A, respectively. (It should, however, be emphasized that up to this point we have presented no direct evidence either that these phases are P-wave phases, or that they are unitary.) In Sec. IX.D we discuss them in somewhat more detail.

Let us assume for the moment that the new phases of ³He are indeed P-wave phases (unitary or not). Then for any given pressure P and temperature T the constants $a_i(P, T)$ are fixed, and we can write the GL free energy in the form [cf. Eqs. (5.77) and (7.41)]

$$F(P, T) = -\alpha(1 - T/T_c)\Psi^2 + \tfrac{1}{2}\kappa\beta\Psi^4$$
$$= \tfrac{1}{2}(dn/d\epsilon)\{-(1 - T/T_c)\Delta^2 + \tfrac{1}{2}\kappa\bar{\beta}\Delta^4\}$$
$$[\bar{\beta} \equiv \tfrac{7}{8}\zeta(3)(\pi k_B T_c)^{-2}], \qquad (9.26)$$

where however κ is no longer given by (7.42) but by the more general expression

$$\kappa \equiv \sum_{i=1}^{5} a_i(P, T) I_i. \tag{9.27}$$

The I_i here are constant for any given state but different from one state to another (cf. Table I). The magnitude of the gap Δ is then given, for T near T_c, by the same expression as Eq. (5.76), namely,

$$\Delta(T) = 3.06\kappa^{-1/2}(k_B T_c)(1 - T/T_c)^{1/2} \tag{9.28}$$

and the specific heat jump is given again by

$$\Delta C_v / C_n = 1.42\kappa^{-1}. \tag{9.29}$$

There is now no reason why κ should not be less than one, and hence the specific heat jump can be *larger* than the BCS value. Since quantities like the susceptibility χ and the normal density ρ_n are determined near T_c for any given state by the value of Δ^2, they too depend only on κ; in fact they differ from their "weak-coupling" values by a factor $\kappa_{\mathrm{BCS}}/\kappa$.

Thus, the experimental quantities we have discussed so far give information only on the particular combinations of a_i's which appear in κ for the different states. However, we can in principle obtain information on different combinations—see especially Sec. XIII below.

C. Spin fluctuation feedback—more sophisticated treatment[22]

The treatment given in Subsection IX.A for the feedback effect shows up the qualitative nature of the effect, but it is based on the ansatz (9.3) which is not quantitatively valid. To obtain more reliable results one must work out more accurately the modification of the true (time-dependent) effective potential and its effects. Calculations which in effect do this have been carried out by Kuroda (1974a,b) and by Brinkman, Serene, and Anderson (1974) (referred to throughout this subsection as BSA); see also related work by Tewordt (1974a,b).[23] The physical basis of the two approaches appears to be essentially equivalent, and the crucial formula for the spin-fluctuation feedback modification to the free energy is common to both (apart, apparently, from an overall factor of 2) [BSA Eq. III.5 = Kuroda (1974b), Eq. (3.3)]; actual numerical results are slightly different but do not lead to qualitative discrepancies. Both calculations use the technique of analytic continuation in the complex plane; in the following, we shall use a rather more elementary approach (see Introduction).

The effective interaction $V_{ij}(\mathbf{r} - \mathbf{r}', t - t')$ is a *variational* derivative of the potential energy $\langle V \rangle$ with respect to variations in the spin density at points $\mathbf{r}$, t and $\mathbf{r}'$, t'. That is, a small *change* in the quantity $\langle \sigma_i(\mathbf{r}t)\sigma_j(\mathbf{r}'t') \rangle$ produces a change in $\langle V \rangle$ according to the formula

$$\delta\langle V \rangle = \iiint V_{ij}(\mathbf{r}' - \mathbf{r}, t' - t)\,\delta\langle \sigma_i(\mathbf{r}t)\sigma_j(\mathbf{r}'t') \rangle \, d\mathbf{r}\, d\mathbf{r}'\, dt'. \tag{9.30}$$

Let us Fourier transform both parts of this expression, using the variables $\mathbf{q}$ and ω in the standard way. Since V_{ij} is given by Eq. (9.2), and the Fourier transform $\chi_{ij}(\mathbf{q}\omega)$ satisfies the relation [a general one for response functions involving Hermitian operators, cf. Nozières (1964), p. 42] $\chi_{ji}(-\mathbf{q}, -\omega) = \chi_{ij}^*(\mathbf{q}\omega)$, the Fourier transform of V_{ij} can be written

$$V_{ij}(\mathbf{q}, \omega) = -\zeta_0^2 \,\mathrm{Re}\chi_{ij}(\mathbf{q}, \omega). \tag{9.31}$$

On the other hand, the Fourier transform of $\langle \sigma_i(\mathbf{r}t)\sigma_j(\mathbf{r}'t') \rangle$ can be related to the imaginary part of $\chi_{ij}(\mathbf{q}, \omega)$ by the fluctuation-dissipation theorem:

$$\langle \sigma_i \sigma_j \rangle(-\mathbf{q}, -\omega) = \mathrm{Im}\chi_{ji}(\mathbf{q}, \omega)[\exp(\beta\hbar\omega) - 1]^{-1}. \tag{9.32}$$

Consequently, using the symmetry property χ_{ij} quoted above and a little algebra we can rewrite (9.30) in the form

$$\delta\langle V \rangle = -\zeta_0^2 \sum_{\mathbf{q}} \int_0^\infty (d\omega/2\pi)\,\mathrm{Re}\chi_{ij}(\mathbf{q}\omega)\,\delta(\mathrm{Im}\chi_{ji}(\mathbf{q}\omega) \times \coth(\tfrac{1}{2}\beta\hbar\omega). \tag{9.33}$$

(Actually, it is probably necessary to replace $\mathrm{Im}\chi_{ij}$ by the unenhanced value $\mathrm{Im}\chi_{ji}^0$ so as to avoid double counting of the enhancement effect: cf. Introduction.)

Let us now imagine that we write

$$\chi_{ij}(\mathbf{q}\omega) = \delta_{ij}\chi_n(\mathbf{q}\omega) + \delta\chi_{ij}(\mathbf{q}\omega), \tag{9.34}$$

where $\chi_n(\mathbf{q}\omega)$ is the (isotropic) normal-state dynamic susceptibility, and consider the change in $\langle V \rangle$ as a functional of $\delta\chi_{ij}(\mathbf{q}\omega)$. The first-order term in $\delta\chi$ contains simply $\mathrm{Re}\chi_n(\mathbf{q}\omega)\,\mathrm{Im}\delta\chi_{ii}(\mathbf{q}\omega)$; this is precisely the term we would have got by allowing $\langle \sigma_i(\mathbf{r}t)\sigma_j(\mathbf{r}'t') \rangle$ to change but keeping $V(\mathbf{r}' - \mathbf{r}, t' - t)$ at its normal state value, that is, by making the ordinary BCS approximation. This term is already handled in the weak-coupling free energy. The second-order term in $\delta\chi$ is proportional [because of the differential form of Eq. (9.30)] not to $\mathrm{Re}\delta\chi\,\mathrm{Im}\delta\chi$ but to $\int \mathrm{Re}\delta\chi\, d\,(\mathrm{Im}\delta\chi)$: since the real and imaginary parts vary in the same way (e.g., with Δ), this is equal to $\frac{1}{2}\,\mathrm{Re}\delta\chi\,\mathrm{Im}\delta\chi$. Consequently, the spin fluctuation feedback correction to $\langle V \rangle$ and hence to the free energy F is

$$\Delta F_{\mathrm{sf}} = -\tfrac{1}{2}\zeta_0^2 \sum_{\mathbf{q}} \int_0^\infty (d\omega/2\pi)\,\mathrm{Re}\delta\chi_{ij}(\mathbf{q}\omega)\,\mathrm{Im}\delta\chi_{ji}(\mathbf{q}\omega) \times \coth(\tfrac{1}{2}\beta\hbar\omega). \tag{9.35}$$

Now if we neglect all molecular-field parameters except $Z_0/4$ (the quantity called $-\bar{I}$ by BSA and Kuroda), we can express the true susceptibility tensor $\chi_{ij}(\mathbf{q}\omega)$ in terms of the "free-superfluid-gas" susceptibility tensor $\chi_{ij}^0(\mathbf{q}\omega)$ by an argument exactly similar to that leading to Eq. (2.38).

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[22] See Introduction.

[23] The important paper of Tewordt *et al.* (1974) arrived, in final version, too late for discussion here.

In fact, if we define a tensor $\hat{g} \equiv g_{ij}(\mathbf{q}\omega)$ by

$$\chi_{ij}{}^0(\mathbf{q}, \omega) \equiv \tfrac{1}{4}(dn/d\epsilon) g_{ij}(\mathbf{q}\omega) \qquad (9.36)$$

then we have [cf. Eq. (2.38)]

$$\hat{\chi} = \tfrac{1}{4}(dn/d\epsilon)\{\hat{g}(q\omega)/[1 + \tfrac{1}{4}Z_0\hat{g}(q\omega)]\}. \qquad (9.37)$$

Now it is found by a detailed examination of the form of $\chi_{ij}{}^0(q\omega)$ that most of the contribution to the integral (9.35) comes from regions where it is only slightly changed from the normal state value, i.e., $\delta\chi_{ij}(q\omega) \ll \chi(q\omega)$. Moreover, in almost all of these regions we have $\omega \leq T_c$, $\Delta/v_F \ll q \ll k_F$, and hence $\omega \ll v_F q$. In that case we can replace the normal-state value of g, $g_N(\mathbf{q}\omega)$, by its static value $g_N(q) \equiv g_N(\mathbf{q}, 0)$. With these approximations we get

$$\Delta F_{\text{sf}} = -\tfrac{1}{2}\zeta_0{}^2 \int [d\mathbf{q}/(2\pi)^3] \int_0^\infty (d\omega/2\pi) \times \frac{\mathrm{Re}\delta\chi_{ij}{}^0(\mathbf{q}\omega)\ \mathrm{Im}\delta\chi_{ji}{}^0(\mathbf{q}\omega)}{[1 + \frac{1}{4}Z_0 g_N(q)]^4} \coth(\tfrac{1}{2}\beta\hbar\omega). \qquad (9.38)$$

This is almost but not quite equivalent to BSA's Eq. (III.5) or Kuroda's (3.3); the difference is that they have the expression $[1 + \frac{1}{4}Z_0 g_N(q)]$ in the denominator only squared, not to the fourth power. BSA also apparently have an overall factor of 2 [cf. their Eq. (IV.5)]. Replacement of $\mathrm{Im}\chi$ by $\mathrm{Im}\chi^0$ in Eq. (9.33) (cf. above) would remove the major discrepancy. In any case, as we shall see, it affects only the overall magnitude of the corrections, not their form.

The problem therefore reduces to the calculation of the quantity $\delta\chi_{ij}{}^0(\mathbf{q}\omega)$ (or its continuation in the complex plane) as a function of the $d_{\alpha i}$ (or more generally as a function of $\mathbf{d}(\mathbf{n})$: note that so far we have nowhere assumed P-wave pairing). The calculation, which is analogous to the calculation of the frequency- and wave-vector-dependent conductivity in the theory of superconductivity, is straightforeward but tedious; it is carried out, for unitary states only, by both Kuroda and BSA for T near T_c and also by BSA for $T = 0$. We will just quote the principal features of the results for the region near T_c: Over most of the region which contributes appreciably to the integral (9.38), $\delta\chi_{ij}{}^0(\mathbf{q}, \omega)$ can be written up to terms of order Δ^2 in the form (for P-wave pairing)

$$\delta\chi_{ij}{}^0(\mathbf{q}, \omega) = \tfrac{1}{4}(dn/d\epsilon)(\Delta^2/\hbar v_F q \cdot k_B T) f_{ij}(\hbar\omega/k_B T, \hat{\mathbf{q}}, \{d_{\alpha i}\}), \qquad (9.39)$$

where $\hat{\mathbf{q}}$ is a unit vector along $\mathbf{q}$, and f_{ij} is a dimensionless function which falls off fast for $\hbar\omega \ll k_B T$. The integration over the magnitude of q in Eq. (9.38) can then be performed; if we define the dimensionless factor:

$$C \equiv k_F{}^{-1} \int \{dq/[1 + \tfrac{1}{4}Z_0 g_N(q)]^n\}, \qquad (9.40)$$

where n is 2 for the BSA–Kuroda calculation and 4 for the calculation leading to Eq. (9.38), then C evidently depends on the specific assumptions we make about the way in which $g_N(q)$ falls off at high q; this cannot be obtained from Landau theory (cf. Sec. II) and is the source of uncertainty in the actual magnitude of ΔF_{sf}, but clearly whatever (reasonable) assumptions we make C will increase strongly as the system tends towards the ferromagnetic instability ($Z_0 \to -4$). The integral over ω will, on dimensional grounds, contribute a factor proportional to T. Finally, using the fact that $k_F/v_F{}^2$ is of order $\hbar^2\epsilon_F{}^{-1}(dn/d\epsilon)$, we obtain a result of the general form

$$\Delta F_{\text{sf}} = \text{const.}\ (dn/d\epsilon)(k_B T \cdot \epsilon_F)^{-1}\Delta^4\Phi\{d_{\alpha i}\} \qquad (9.41)$$

where the value of the constant depends on detailed assumptions about $g_N(q)$ and on the details of the ω integration, etc., and where Φ is a dimensionless function of the $d_{\alpha i}$'s of order unity. (BSA incorporate a factor of Δ in the definition of the $d_{\alpha i}$.) The important features of this result are: (1) If compared with the BCS fourth-order terms, Eq. (5.75), the spin fluctuation terms contain a factor $k_B T_c/\epsilon_F$, that is, they represent a genuine "strong-coupling" effect which would vanish in the limit $T_c/\epsilon_F \to 0$. (2) However, the small factor $k_B T_c/\epsilon_F$ can be compensated if C is sufficiently large, i.e., in the limit of large enhancement of the susceptibility. It is this feature (the large enhancement) which distinguishes the particular feedback effect associated with exchange of spin fluctuations from other feedback effects, e.g., that associated with transverse current fluctuations. (3) The coefficient of Δ^4 in Eq. (9.41) is proportional to T^{-1}. On the other hand, the coefficient of the BCS fourth-order terms, Eq. (5.75), actually goes as T^{-2} (although we have replaced it by $T_c{}^{-2}$). Hence the importance of the spin fluctuation terms relative to the BCS ones decreases as T falls below T_c.

The quantity Φ in Eq. (9.41) can be written, of course, in the form corresponding to (9.16), i.e.,

$$\Delta F_{\text{sf}} = \tfrac{1}{2}(dn/d\epsilon)\tfrac{1}{2}(\sum_{i=1}^{5} a_i I_i)\bar{\beta}\Delta^4 \qquad [\bar{\beta} \equiv \tfrac{7}{8}\zeta(3)(\pi k_B T_c)^{-2}]. \qquad (9.42)$$

While the absolute values of the a_i's are subject to some uncertainty, the calculation should give their *relative* values fairly accurately, at least for the unitary states which have been explicitly considered. BSA quote the results (where $a_i \equiv a_{i(\text{sf})}$ and $a_1 < 0$)

$$a_1 : a_2 : a_3 : a_4 : a_5 = -1 : -0.5 : -7.0 : +2.0 : -5.5. \qquad (9.43)$$

Then according to Table I the spin fluctuation corrections to the BW, 2D, ABM, and polar states are, respectively, proportional to $-\frac{10}{3}$, $-\frac{11}{2}$, $-\frac{21}{2}$, -12. Kuroda, in (1974a) finds for these contributions values in the ratio 0, -0.3, -1.05, -1.2 [but in (1974b) apparently finds a negative spin fluctuation contribution to both ABM and BW states]. It is interesting that all calculations appear to give the same ordering of the spin fluctuation contribution to the free energy of the four states, and that (contrary to the result of Sec. IX.A) the free energy of the BW state is now *reduced* by spin fluctuation effects (or at least not increased). Apart from this feature the ratios of the a_i's are not greatly different from those obtained in Sec. IX.A. As to the absolute magnitude of the spin fluctuation effects relative to the BCS terms, BSA find the result[24]

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[24] BSA call the quantity defined by Eq. (9.44) δ. Note that it is *minus* the δ defined by Anderson and Brinkman (1973). Cf. Sec. XIII.

$$\delta'' \equiv \frac{\Delta F_{\text{sf}}(\text{BW})}{F_{\text{BCS}}(\text{BW}) - F_{\text{BCS}}(\text{ABM})} = 140\left(\frac{k_B T_c}{\epsilon_F}\right)\alpha^{-1/2}\left(\frac{Z_0/4}{1+Z_0/4}\right)^{3/2}, \tag{9.44}$$

where α is a parameter used to characterize $g_N(q)$, which would be $\frac{1}{3}$ in the free quasiparticle gas approximation. Note that apart from the factor involving Z_0, which was expected to be large in the nearly ferromagnetic limit, the numerical factor multiplying $k_B T_c/\epsilon_F$ is unexpectedly large.

A word of caution: both the calculation outlined here and the graph-summation procedure of BSA and Kuroda take into account only some of the most obvious effects connected with the modification of the spin-fluctuation spectrum. Even if modification of other responses is neglected, it is not at all obvious that the assumption implicit in the above calculation—namely that the effective potential can be treated as a classical quantity, neglecting the quantum correlations between V_{ij} and the operator $\sigma_i(\mathbf{r}t)\sigma_j(\mathbf{r}'t')$ in (9.30)—is a valid one. Equally, there may be other kinds of corrections, e.g., to the quasiparticle effective mass, which are possibly of the same order of magnitude (cf. Kuroda, 1974b). Moreover the calculations consider only unitary states, and it is not entirely clear what error, if any, this involves in the values of the a_i's for general states. This is of some importance since we wish subsequently (in Sec. XIII) to use the general strong-coupling theory to discuss the (nonunitary) A_1 and A_2 states.

Note: some further types of correction to the BCS model from spin fluctuations have now been considered by Tewordt *et al.* (1974). Their result for the ratios of the a_i's is in general different from Eq. (9.43); see their Eq. (41).

Note added in proof: The conjecture made in the text that $\text{Im}\chi_{ji}$ in Eq. (9.33) should be replaced by $\text{Im}\chi_{ji}^{(0)}$ to avoid overcounting is almost certainly correct. The factors of 2 by which the various calculations disagree however remain baffling.

D. Properties of the BW and ABM states

As we saw in Subsection B, provided the non-BCS contributions to the free energy are not too large compared with the BCS ones, the stable state (in zero field) of an anisotropic superfluid with P-wave pairing will be either the BW state or the ABM state, and it is these states which are commonly identified with ^{3}He-B and ^{3}He-A, respectively. We shall discuss later how far the evidence supports this assignment; in this subsection we shall review the theoretical predictions, for these states, as to those thermodynamic and other quantities which have been discussed so far (NMR properties are discussed in Sec. XI). In each case the overall magnitude of the gap, $\Delta(T)$, will be taken as a phenomenological parameter: near T_c, as we have seen, it will be proportional to $\kappa^{-1/2}(1-T/T_c)^{1/2}$, where κ is defined by (9.27) (and is possibly pressure-dependent). In each case we must remember that there is actually a whole class of states which transform into one another under rotation and/or inversion of the spin and orbital coordinates, and which are degenerate in the approximation, always made so far, of neglecting weak "orienting" effects (for these, see next section).

The BW state is given by Eq. (7.43),

$$\mathbf{d}(\mathbf{n}) = \mathbf{n} \quad [\text{i.e., } d_x(\mathbf{n}) = n_x, \text{ etc.}] \tag{9.45}$$

or by any state related to this by a *relative* rotation of the spin and orbital coordinate axes, i.e.,

$$\mathbf{d}'(\mathbf{n}) = \mathbf{d}(\hat{R}\mathbf{n}) \equiv \hat{R}\mathbf{d}(\mathbf{n}) \equiv \hat{R}\mathbf{n}, \tag{9.46}$$

where $\hat{R}$ is a rotation matrix.[25] [Note that overall rotation of the spin and orbital axes together does not change the form of $\mathbf{d}(\mathbf{n})$.] It has a strictly isotropic energy gap:

$$|\Delta(\mathbf{n})|^2 = \Delta^2. \tag{9.47}$$

Its spin-independent thermodynamic properties (expressed in terms of Δ) are therefore exactly the same as those of a BCS superconductor: in particular the specific heat at low temperatures is exponential and the normal density is isotropic:

$$\rho_{n,\alpha\beta} = \delta_{\alpha\beta}\rho_n,$$
$$\rho_n/\rho \equiv Y(\Delta^2, T)(1+\tfrac{1}{3}F_1)/[1+\tfrac{1}{3}F_1 Y(\Delta^2, T)], \tag{9.48}$$

where $Y(\Delta^2, T)$ is the Yosida function defined by (6.14). (We write it explicitly as a function of Δ^2 to emphasize that while its functional form in terms of Δ^2 and T is given by (6.14), the resulting temperature dependence may be different from that given by BCS theory.)

The spin susceptibility of the BW state is isotropic and given if we neglect Z_2 by Eq. (7.56):

$$\chi_{\text{BW}}/\chi_n = \frac{(1+\frac{1}{4}Z_0)[\frac{2}{3}+\frac{1}{3}Y(\Delta^2, T)]}{1+\frac{1}{4}Z_0[\frac{2}{3}+\frac{1}{3}Y(\Delta^2, T)]}. \tag{9.49}$$

Near T_c we can use the fact that $1 - Y(T) \approx \frac{7}{4}\zeta(3)\pi^{-2}\Delta^2/k_B^2 T_c^2$, together with formula (6.5) for the specific heat jump in terms of Δ^2, to predict relations between the specific heat jump, spin susceptibility change $\Delta\chi/\chi_n \equiv 1 - \chi(T)/\chi_n$, and superfluid density $\rho_s \equiv \rho - \rho_n$

$$(1+\tfrac{1}{4}Z_0)(\Delta\chi/\chi_n) = (1+\tfrac{1}{3}F_1)(\rho_s/\rho)$$
$$= 2(1 - T/T_c)(\Delta C/1.42 C_n) \qquad (T \to T_c). \tag{9.50}$$

We refer to Wheatley (Sec. III and V) for a comparison with the experimental data on ^{3}He-B.

The above properties are independent of the choice of the rotation $\hat{R}$. The "spin superfluid density" (Sec. VII.D) however does depend on $\hat{R}$. The simplest way of expressing the result is to assume that we simply choose a new set of orbital axes so that (9.45) is preserved: these are rotated relative to the old ones (which coincided with spin coordinate axes) by $\hat{R}^{-1}$. Then, from Eqs. (9.45) and (7.67) we find for all T (under the assumption that $Z_1 = 0$ but

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[25] The notation is somewhat condensed here: $\hat{R}$ is allowed to act in either spin or orbital space as appropriate. "Rotation" here and subsequently includes "improper" rotations, i.e., operations involving inversion of the coordinates.

including the effective-mass correction, cf. Sec. VII.D)

$$\frac{\rho_{ij,\alpha'\beta'}{}^{\mathrm{spin}}}{(m/m^*)\rho} = 3[1 - Y(T)\langle(\delta_{ij} - n_i n_j) n_{\alpha'} n_{\beta'}\rangle_{\mathrm{av}}$$
$$= [1 - Y(T)](\delta_{ij}\delta_{\alpha'\beta'} - \tfrac{1}{5}[\delta_{ij}\delta_{\alpha'\beta'} + \delta_{i\alpha'}\delta_{j\beta'} + \delta_{i\beta'}\delta_{i\alpha'})] \tag{9.51}$$

(the primes indicate that the orbital coordinate system has been rotated). For any given direction in orbital space $\hat{\rho}^{\mathrm{spin}}$ therefore has two eigenvalues of $\frac{4}{5}[1 - Y(T)](m/m^*)$ and one of $\frac{2}{5}[1 - Y(T)](m/m^*)$.

Finally we notice that the relative orbital angular momentum of the BW state is zero, according to Eq. (7.48). This is a result of cancellation of the angular momenta due to the up and down spin pairs. Clearly, however, the *magnitude* of relative angular momentum is not zero; in fact it is $\hbar$. The magnitude of total spin is also $\hbar$. (The pairs form with $L = S = 1$.) For the original state (9.45), L and S are coupled to give[26] $J = 0$, but for more general states of the form (9.46) J does not always have a unique value.

Turning now to the ABM state, we recall that a typical state is of the form

$$d_x(\mathbf{n}) = (3/2)^{1/2}(n_y + in_z), \qquad d_y(\mathbf{n}) = d_z(\mathbf{n}) = 0 \tag{9.52}$$

or [cf. Eq. (7.33)] $\Delta_{\uparrow\uparrow}(\mathbf{n}) = -\Delta_{\downarrow\downarrow}(\mathbf{n}) = -(3/2)^{1/2}\Delta(n_y + in_z)$. In this state *all* the pairs, wherever they may be on the Fermi surface, are in the same spin state, namely that corresponding to $S = 1$, $S_x = 0$. The orbital wave function, on the other hand, corresponds to angular momentum $\hbar$ around the x axis. Obviously, a general ABM state can have $\mathbf{d}$ along any axis and the orbital angular momentum along any axis: if we represent the latter by a unit vector $\mathbf{l}$, therefore, we can formally write the general ABM state in the form

$$\mathbf{d}(\mathbf{n}) = (3/2)^{1/2}\mathbf{d}(\boldsymbol{\alpha}_1\cdot\mathbf{n} + i\boldsymbol{\alpha}_2\cdot\mathbf{n}), \tag{9.53}$$

where $\mathbf{d}$ is now a real unit vector and $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ are two arbitrary mutually orthogonal unit vectors. We then have $\mathbf{l} = \boldsymbol{\alpha}_1 \times \boldsymbol{\alpha}_2$. It is convenient to express the properties of the ABM state in terms of the unit vectors $\mathbf{d}$ and $\mathbf{l}$.

Since (to the extent that we neglect mixing-in of $l \neq 1$ spherical harmonics) the gap has two nodes, the specific heat at low temperature is not exponential but proportional to T^3: in fact, it is given by Eq. (6.4) with the factor $\sum_i a_i^{-1}b_i^{-1}$ equal to $\frac{4}{3}$. However, as pointed out in Sec. VI, this result should probably not be taken too seriously, as in reality mixing-in of $l \neq 1$ harmonics may well wash out the nodes.

The superfluid (or normal) density is anisotropic, the principal axes of the tensor $\hat{\rho}_s$ being determined by the vector $\mathbf{l}$ and any pair of axes perpendicular to $\mathbf{l}$. In particular near T_c we have [cf. Eq. (9.50)]

$$\rho_{\alpha\beta}{}^{s}/\rho = \tfrac{6}{5}(1 + \tfrac{1}{3}F_1)^{-1}(\Delta C/1.42C_n)$$
$$\times (2\delta_{\alpha\beta} - l_\alpha l_\beta)(1 - T/T_c) \tag{9.54}$$

Thus, near T_c, the eigenvalues are in the ratio 2:2:1, the small eigenvalue being associated with motion along the direction of $\mathbf{l}$. At lower temperatures the superfluid density may be a good deal more strongly anisotropic (cf. the discussion in Sec. VI.C and Fig. 6, which is in fact drawn for the ABM state).

From Eqs. (7.53) and (7.54) we find for the spin susceptibility

$$\chi_{ij}/\chi_n = (\delta_{ij} - d_i d_j)$$
$$+ d_i d_j\{(1 + \tfrac{1}{4}Z_0)Y(T)/[1 + \tfrac{1}{4}Z_0 Y(T)]\}. \tag{9.55}$$

As remarked in Sec. VII.D, the experimentally measured susceptibility usually corresponds to the maximum eigenvalue of χ, that is, χ_n (cf. however Sec. XIII). The spin superfluid density may be conveniently written in the form

$$\frac{\rho_{ij,\alpha\beta}{}^{\mathrm{sp}}}{\rho} = (\delta_{ij} - d_i d_j)\left(\frac{m}{m^*}\right)\frac{\rho_{\alpha\beta}{}^{s(0)}}{\rho} \tag{9.56}$$

(where the zero indicates that ρ_s/ρ is not corrected for Fermi-liquid effects, that is, is calculated from Eq. (6.26). Near T_c the right-hand side is just $(\delta_{ij} - d_i d_j)$ times the *true* superfluid density.

Finally we note again that the ABM state possesses a finite relative orbital angular momentum, of order $(N\hbar/2)(\Delta/\epsilon_F)$. This angular momentum is evidently directed along $\mathbf{l}$.

For completeness we should also mention briefly the third unitary state which can be stable, the so-called "polar" state:

$$d_x(\mathbf{n}) = \sqrt{3}n_x, \qquad d_y = d_z = 0 \tag{9.57}$$

or any state obtained from this by rotation of the spin and/or coordinate axes. This is very similar to the ABM state except that the pairs do not have a finite relative orbital angular momentum. The eigenvalues of the superfluid density tensor near T_c are in the ratio 3:1:1.

X. ORIENTATION AND "TEXTURES"

Up to now we have been neglecting small terms in the Hamiltonian such as the dipole energy, and also have implicitly considered a uniform system, so that any effect of the boundaries is neglected. The Hamiltonian is therefore invariant with respect to rotation of the spin and orbital coordinates separately as well as with respect to gauge transformation, and as a result, the considerations about the free energy developed in Secs. V and IX do not define a unique thermodynamic state but rather a class of states which transform into one another under these transformations. On the other hand, we have already seen that some physical properties of the system, e.g., the superfluid density tensor, depend on which particular state we select. The question therefore arises, which one of the (so far) equivalent states in a given class is in fact the true stable thermo-

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[26] This may be seen by direct inspection of Eq. (7.30).

dynamic state? Or equivalently, how does the system "orient" the wave function of its Cooper pairs? To study this question we need to consider the small terms, so far neglected, which break the invariance under spin and spatial rotation (gauge invariance remains unbroken so long as particle number is strictly conserved).

The symmetry-breaking (orienting) effects usually considered are those associated with: (1) magnetic fields, (2) the dipole forces, (3) walls, (4) currents, (5) electric fields. (There may well be other more subtle effects which have so far escaped detection or theoretical discussion.) Crudely speaking the first orients the spin of the Cooper pairs, the last three their relative orbital angular momentum, and the dipole force orients the spin *relative* to the orbital angular momentum. In Subsection A we shall discuss (1) and (2), in Subsection B, (4) and (5) and in Subsection C, (3); for the sake of concreteness we usually consider explicitly the ABM and BW states. In the last subsection we shall discuss how the system behaves when the various effects are in competition.

A. Magnetic fields and dipole forces

The effect of the external magnetic field is rather simple: it tends to orient the spin quantization axes $\mathbf{d}(\mathbf{n})$ in such a way that the eigenvalue of the susceptibility tensor corresponding to the direction along the field $\mathbf{H}$ is a maximum. We have in fact quite generally for the external field contribution to the free energy, F_{magn}

$$F_{\text{magn}} = -\tfrac{1}{2}\chi_{ij}H_iH_j \equiv -\tfrac{1}{2}\chi_n H^2 + \Delta F_{\text{magn}}, \tag{10.1}$$

where χ_{ij} is given in terms of the vectors $\mathbf{d}(\mathbf{n})$ by Eq. (7.53) and χ_n is the normal state susceptibility. For the BW state χ is isotropic whatever the orientation, so an external magnetic field by itself has no orienting effect. For the ABM state, on the other hand, insertion of Eq. (7.53) into (10.1) gives [cf. Eq. (9.55)]

$$\Delta F_{\text{magn}}^{(\text{ABM})} = \tfrac{1}{2}\chi_n\{[1 - Y(T)]/[1 + \tfrac{1}{4}Z_0 Y(T)]\}(\mathbf{d}\cdot\mathbf{H})^2. \tag{10.2}$$

At low temperatures this energy is of order $\tfrac{1}{2}\chi_n H^2 \sim 5 \times 10^{-8}$ ergs/cm^3 gauss^2, while for T near T_c we have approximately

$$\begin{aligned}\Delta F_{\text{magn}}^{(\text{ABM})} &= [\kappa^{-1}(1 - T/T_c)/(1 + \tfrac{1}{4}Z_0)]\chi_n H^2 \\ &\approx 5(1 - T/T_c)\chi_n H^2 \\ &\approx 5 \times 10^{-7}(1 - T/T_c) \text{ ergs/cm}^3 \text{ gauss}^2,\end{aligned} \tag{10.3}$$

where we used Eq. (9.29) to estimate κ for the A phase (if it is assumed to be ABM). Evidently, the effect of this energy is to force $\mathbf{d}$ into the plane perpendicular to $\mathbf{H}$.

The dipole energy of the system has the standard form

$$H_D = \tfrac{1}{2}\gamma^2\hbar^2 \sum_{kl} \left\{ \frac{\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_l}{|\mathbf{r}_k - \mathbf{r}_l|^3} - \frac{3\boldsymbol{\sigma}_k\cdot(\mathbf{r}_k - \mathbf{r}_l)\boldsymbol{\sigma}_l\cdot(\mathbf{r}_k - \mathbf{r}_l)}{|\mathbf{r}_k - \mathbf{r}_l|^5} \right\}. \tag{10.4}$$

According to Eq. (7.74), the contribution of the Cooper pairs to this expression is

$$H_D = -\tfrac{1}{2}\gamma^2\hbar^2 \int d\mathbf{r} \left\{ \frac{|\mathbf{F}(\mathbf{r})|^2 - 3|\hat{\mathbf{r}}\cdot\mathbf{F}(\mathbf{r})|^2}{r^3} \right\}. \tag{10.5}$$

Since (all components of) the pair wave function $\mathbf{F}(\mathbf{r})$ fall off as $r \to 0$ or $r \to \infty$, there are no divergences in this expression. After a little algebra we find quite generally (see Appendix)

$$\begin{aligned}H_D = {} & (2\pi\gamma^2\hbar^2/3)\Psi^2 \int (d\Omega/4\pi) \int (d\Omega'/4\pi)\{\mathbf{d}^*(\mathbf{n})\cdot\mathbf{d}(\mathbf{n}') \\ & - 3\hat{\mathbf{q}}\cdot\mathbf{d}^*(\mathbf{n})\hat{\mathbf{q}}\cdot\mathbf{d}(\mathbf{n}')\}\end{aligned} \tag{10.6}$$

$$\hat{\mathbf{q}} \equiv (\mathbf{n} - \mathbf{n}')/|\mathbf{n} - \mathbf{n}'|$$

and for the case of a pure P-wave state this reduces further (see Leggett, 1974a, Appendix) to

$$H_D = \tfrac{1}{2}\pi\gamma^2\hbar^2\Psi^2 \int (d\Omega/4\pi)\{3|\mathbf{n}\cdot\mathbf{d}(\mathbf{n})|^2 - |\mathbf{d}(\mathbf{n})|^2\}. \tag{10.7}$$

For $l > 1$ states we get the same expression but with the constant in front replaced by $\pi\gamma^2\hbar^2/l(l+1)$ (see Takagi, 1974a).

Actually, the above derivation assumes that the dipole energy of two *quasiparticles* is of the form (10.4), which is not entirely obvious. It can be shown however that any effects connected with the renormalization of the dipole energy by the "dressing" process which turns a real particle into a quasiparticle merely have the effect of multiplying Eq. (10.7) by some factor $\langle R^2\rangle_{\text{av}}$ (see Leggett, 1947a). We shall see in the next section that the experimental NMR result suggest that $\langle R^2\rangle_{\text{av}}$ is close to 1; it may in fact be identically 1, although this has not been proved to my knowledge.

Let us define a quantity $g_D(T)$ by

$$g_D(T) \equiv \tfrac{1}{2}(\pi\gamma^2\hbar^2)\Psi^2(T)\langle R^2\rangle_{\text{av}}. \tag{10.8}$$

This quantity is a measure of the possible contribution of the Cooper pairs to the dipole energy. At low temperatures, it is of order $\gamma^2\hbar^2(k_BT_c)(dn/d\epsilon)^2(\ln\beta_c\epsilon_c)^2$, and near T_c we have explicitly

$$\begin{aligned}g_D(T) &= \tfrac{1}{8}[9.3\pi\gamma^2\hbar^2(k_BT_c)^2]\langle R^2\rangle_{\text{av}}(dn/d\epsilon)^2\kappa^{-1} \\ &\quad\times (\ln 1.14\beta_c\epsilon_c)^2(1 - T/T_c) \\ &\approx 10^{-3}(1 - T/T_c) \text{ ergs/cm}^3,\end{aligned} \tag{10.9}$$

where we took for definiteness $\epsilon_0 \sim 0.7$°K. (For a possible justification of this estimate, see Leggett, 1972.) It is interesting that, at least in the simple model we are using throughout, quantities like $g_D(T)$ which are essentially expectation values over the Cooper pair wave function *do* depend on the cutoff energy ϵ_c, though only logarithmically. It is not entirely clear whether a proper calculation of the normalization constant $\bar{R}^2$ would cancel this dependence or not. Notice that $g_D(T)$ is roughly equivalent to the energy due to susceptibility anisotropy, Eq. (10.3) in a field of the order of 50 G; near T_c this "characteristic field" is approximately temperature-independent.

Substituting Eq. (10.9) in (10.7), we find for the dipolar free energy

$$\Delta F_{\text{dip}} = \text{const.} + g_D(T) \int (d\Omega/4\pi) 3 \,|\,\mathbf{n}\cdot\mathbf{d}(\mathbf{n})\,|^2, \qquad (10.10)$$

where the constant is independent of orientation. Evidently the effect of ΔF_{dip} will be to push $\mathbf{d}(\mathbf{n})$ as nearly perpendicular to $\mathbf{n}$ as possible. For the ABM state we easily find apart from an orientation-independent constant,

$$\Delta F_{\text{dip}}{}^{(\text{ABM})} = -\tfrac{3}{5} g_D(T) (\mathbf{d}\cdot\mathbf{l})^2 \qquad (10.11)$$

so that the dipolar energy tends to force $\mathbf{l}$ to lie along $\mathbf{d}$, and hence to lie in the plane perpendicular to the external magnetic field.

The effect of the dipole forces on the BW state is somewhat more subtle. Before considering it explicitly, let us briefly note that for the "2D" state, Eq. (9.23), the dipole energy is clearly minimized by a rotation of (9.23) through 90° around the z axis; this leaves $\mathbf{d}(\mathbf{n})$ exactly perpendicular to $\mathbf{n}$ for all points on the Fermi surface and hence must minimize the positive-definite second term in (10.10). Clearly this result can be generalized: if within a given plane we start with $\mathbf{d}(\mathbf{n}) \sim \mathbf{n}$, then the dipole energy is minimized by a rotation of 90° around an axis perpendicular to this plane.

Returning now to the BW state, we first notice that inversion of the coordinate system will not change the dipole energy; hence in writing down the general BW state (9.46) we can confine ourselves to the case of a pure rotation. If we represent $\hat{R}$ as an (orthogonal) rotation matrix in the usual way, then we find from (10.10) (apart from the constant) with summation over repeated indices as usual implied

$$\begin{aligned}\Delta F_{\text{dip}}{}^{(\text{BW})} &= \tfrac{1}{5} g_D(T) \{R_{\alpha\alpha}R_{\beta\beta} + R_{\alpha\beta}R_{\alpha\beta} + R_{\alpha\beta}R_{\beta\alpha}\} \\ &= \text{const.} + \tfrac{1}{5} g_D(T) \{(\text{Tr}R)^2 + \text{Tr}(R^2)\}, \qquad (10.12)\end{aligned}$$

where we took into account that $R_{\alpha\beta}R_{\alpha\beta} \equiv 1$. Now for a rotation around an axis $\hat{\boldsymbol{\omega}}$ through an angle θ we have, independently of $\hat{\boldsymbol{\omega}}$,

$$\text{Tr}R = (1 + 2\cos\theta), \qquad \text{Tr}(R^2) = (1 + 2\cos 2\theta) \qquad (10.13)$$

and hence, again omitting an orientation-independent constant

$$\Delta F_{\text{dip}}{}^{(\text{BW})} = \tfrac{4}{5} g_D(T) \{\cos\theta + 2\cos^2\theta\}. \qquad (10.14)$$

We therefore reach the result that the dipole energy of the BW state is minimized by a rotation, around *any* axis $\hat{\boldsymbol{\omega}}$, by the "magic angle"

$$\theta_0 \equiv \cos^{-1}(-\tfrac{1}{4}) \approx 104°. \qquad (10.15)$$

Clearly the arbitrariness of the rotation axis $\hat{\boldsymbol{\omega}}$ leaves a large degeneracy in the problem.[27] Is there any way to break this degeneracy, i.e., to determine which direction $\hat{\boldsymbol{\omega}}$ will actually take? In a magnetic field there is: one can argue that the field will tend to depress the number of $S_z = 0$ pairs relative to the $S_z = \pm 1$ pairs, i.e., to reduce d_z relative to d_x and d_y. In that case it is more important to get the orientation of $\mathbf{d}(\mathbf{n})$ relative to $\mathbf{n}$ right for the xy components of $\mathbf{d}$ than for the z component, which is slightly smaller; hence, we would expect $\hat{\boldsymbol{\omega}}$ to lie along the z axis (cf. the comments above on the 2D state: the xy components of the BW state are exactly the same as those of the 2D state, apart from normalization). Since the relative depression of the z component is expected (by analogy with known results on depairing in superconductors) to be of order $[\mu H/\Delta(T)]^2$, we expect an extra energy tending to orient $\hat{\boldsymbol{\omega}}$ of the form

$$\Delta F'_{\text{magn}}{}^{(\text{BW})} \sim -g_D(T)[\mu/\Delta(T)]^2(\hat{\boldsymbol{\omega}}\cdot\mathbf{H})^2. \qquad (10.16)$$

Since the temperature dependence of $g_D(T)$ is approximately the same at that of $\Delta^2(T)$ we find the remarkable result that $\Delta F'$ is temperature-*independent* to a first approximation.

A quantitative calculation of this effect has been carried out by Engelsberg, Brinkman, and Anderson (1974), by adding to the general free energy expression (9.16) terms explictly representing the magnetic field and dipole energies, and finding the values of the $d_{\alpha i}$ which minimize the total expression. They find using weak coupling values of a_i's that the $\hat{\boldsymbol{\omega}}$-orienting energy is of order 10^{-18} °K/atom G^2, i.e., $\sim 4 \times 10^{-12}$ ergs/cm^3 G^2. This is several orders of magnitude smaller than most of the other orientational energies we shall meet in this section; however, it is generally believed that it is the principal factor determining $\hat{\boldsymbol{\omega}}$ for the BW phase in bulk.

B. Currents and electric fields

It was long ago recognized (Glassgold and Sessler, 1961) that the fact that the superfluid density of an anisotropic superfluid is itself in general anisotropic implies that flow of the superfluid component alone, or a relative counterflow of the superfluid and normal components, would tend to orient the Cooper-pair wave function. The simplest way of producing a counterflow is to establish a thermal gradient across the sample, so that two-fluid convection takes place just as in superfluid ^{4}He (see, e.g., Khalatnikov, 1965). Very probably, such a situation has been produced involuntarily in some experiments with ^{3}He. A flow of the superfluid component alone can occur in a narrow channel ("superleak") in which the normal component is clamped by viscous forces.

In liquid ^{4}He the kinetic energy of counterflow is given by

$$E_{\text{flow}} = \tfrac{1}{2}\rho_s \mathbf{v}^2{}_s + \tfrac{1}{2}\rho_n \mathbf{v}_n{}^2. \qquad (10.17)$$

In ^{3}He we may easily derive the appropriate generalization of this (cf. Subsection D below):

$$E_{\text{flow}} = \tfrac{1}{2}\rho_{\alpha\beta}{}^{(s)} v_{s\alpha} v_{s\beta} + \tfrac{1}{2}\rho_{\alpha\beta}{}^{(n)} v_{n\alpha} v_{n\beta}. \qquad (10.18)$$

Assuming that there is no mass flow (which is true in typical experimental counterflow situations) we can eliminate $\mathbf{v}_n$ by putting $\mathbf{P} = 0$ in (the generalization of) Eq.

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[27] In the literature the vector $\hat{\boldsymbol{\omega}}$ is often denoted $\hat{\mathbf{n}}$. Since we have used $\mathbf{n}$ above with a different sense, we avoid this notation.

(6.23), thereby obtaining

$$\Delta F_{\text{flow}} = \tfrac{1}{2}\rho(\hat{\rho}_s/\hat{\rho}_n)_{\alpha\beta}v_{s\alpha}v_{s\beta} \approx \tfrac{1}{2}\rho_{\alpha\beta}{}^{(s)}v_{s\alpha}v_{s\beta}(T \to T_c), \tag{10.19}$$

where the last expression applies only near T_c. In most experimental situations it is $\mathbf{v}_s$ which is fixed (e.g., by the condition that it is equal to the critical velocity): therefore the flow energy is minimized (at any temperature) by minimizing the component of ρ_s along the direction of flow. In a "superleak" the situation is identical except that in Eq. (10.19) the coefficient is $\rho_{\alpha\beta}{}^{(s)}$ at *all* temperatures.

For the ABM state using the fact that the eigenvalue of ρ_s corresponding to the direction of $\hat{\mathbf{l}}$ is always the least of the three, we evidently have apart from an orientation-independent constant (where $\hat{\mathbf{v}}_s$ is a unit vector along $\mathbf{v}_s$)

$$\Delta F_{\text{flow}}{}^{\text{ABM}} = -\text{const.}(\mathbf{l}\cdot\hat{\mathbf{v}}_s)^2, \tag{10.20}$$

where the constant is of order $\tfrac{1}{2}\rho v_s^2$. More specifically, near T_c we find from Eq. (9.54)

$$\begin{aligned}\Delta F_{\text{flow}}{}^{\text{ABM}} &= -\tfrac{3}{10}\{[1 - Y(T)]/(1 + \tfrac{1}{3}F_1)\}v_s^2(\mathbf{l}\cdot\hat{\mathbf{v}}_s)\\ &= -A(\mathbf{v}_s)(1 - T/T_c)(\mathbf{l}\cdot\hat{\mathbf{v}}_s)^2\\ A &\approx 10^{-2}\mathbf{v}_s^2 \text{ erg sec}^2/\text{cm}^5.\end{aligned} \tag{10.21}$$

If we assume that critical velocities in ³He-A are of order of magnitude 0.05 cm/sec (cf. Wheatley, Sec. VII), then the maximum orientation energy due to flow near T_c is comparable to that due to a field of the order of a few gauss, and is about two orders of magnitude smaller than the dipole energy. At lower temperatures the flow orientation effect becomes somewhat more important relative to field effects (by a factor of order 20) because the factor $(1 + \tfrac{1}{4}Z_0) \times (1 + \tfrac{1}{3}F_1)^{-1}$ which occurs in their ratio near T_c [compare Eqs. (10.3) and (10.21)] no longer occurs at lower temperatures.

For the BW state the superfluid density is isotropic, and consequently, a heat current has no direct effect on the orientation. However, just as an external magnetic field tends to depopulate ("depair") the $S_z = 0$ component of the wave function and therefore makes it advantageous for the rotation vector $\hat{\omega}$ to lie along the field, so a heat current will tend to depopulate the component $L_z = 0$ along the current direction [to an extent of order $(p_F v_s/\Delta)^2$] and therefore to orient $\hat{\omega}$ along the current direction. By analogy with Eq. (10.16) we expect the orientation energy to be of the form

$$\Delta F'_{\text{flow}}{}^{(\text{BW})} \sim -g_D(T)[p_F v_s/\Delta(T)]^2(\hat{\omega}\cdot\hat{\mathbf{v}}_s)^2 \tag{10.22}$$

which may be comparable to the magnetic orientation energy in weak fields [note that the critical velocity for ³He-B seems to be about 0.5 cm/sec, that is, about ten times larger than in ³He-A (see Wheatley)].

Qualitatively similar remarks apply to orientation by spin convection currents (spin counterflow, cf. Sec. VII.D); we shall not work out the results in detail as it may not be easy to produce well-defined macroscopic spin currents in practice.

Apart from passing heat currents through the liquid, it should be possible to orient bulk superfluid ³He by applying an electric field (Delrieu, 1974). The basis of the mechanism envisaged is that the electric field induces on each He atom a small but finite electric dipole moment, which because of the very short characteristic relaxation time associated with the electronic processes responsible may be assumed to be always in the direction of the electric field. These electric dipoles then experience a dipole interaction given by a formula identical to (10.4) except that $\mu\boldsymbol{\sigma}$ is replaced by the induced electric dipole moment $\mathbf{p} = \alpha\mathbf{E}$ (α = polarizability of ³He atom). Thus the spins of the atoms are not involved. Then using Eq. (6.43) and following through a derivation analogous to the one above for the dipole energy, we find apart from an orientation-independent constant

$$\begin{aligned}\Delta F_{\text{el}} &= -(4\alpha^2/\mu^2)E^2 g_D(T)\int(d\Omega/4\pi)3(\mathbf{n}\cdot\hat{\mathbf{E}})^2\,|\,\mathbf{d}(\mathbf{n})\,|^2\\ &\quad(\hat{\mathbf{E}} = \text{unit vector along } \mathbf{E})\end{aligned} \tag{10.23}$$

(where we neglect the fact that R^2 may not be the same for electric as for magnetic moments).

The factor $4\alpha^2E^2/\mu^2$ is approximately equal to 1 when $E \sim 40\,000$ V/cm, so that for fields of this order the electric dipole energy is comparable to the magnetic dipole contribution.

For the ABM state we find from Eq. (10.23)

$$\Delta F_{\text{el}}{}^{(\text{ABM})} = +(4\alpha^2E^2/\mu^2)g_D(T)\tfrac{1}{5}(\mathbf{l}\cdot\hat{\mathbf{E}})^2 \tag{10.24}$$

indicating that **l** tends to lie perpendicular to **E**. Since we already know that the combined effects of susceptibility anisotropy and the magnetic dipole energy force **l** to lie perpendicular to **H**, we see that in the presence of crossed electric and magnetic fields **l** and **d** will be forced to lie in the unique direction (apart from a sign) perpendicular to both. This seems to be the most promising way of producing a well defined single "domain" in bulk superfluid ³He-A (Delrieu, 1974).

For the BW state the effect of an electric field should be very small indeed: because of the form of ΔF_{el} [Eq. (10.23)] this energy plays no role at all in determining the axis or angle of rotation $\hat{R}$ discussed in Subsection A above, and while there is in principle a depairing effect associated with the electric field which will tend to orient $\hat{\omega}$, an electric field of the order of 40 000 V/cm is equivalent to a magnetic field of only ~50 G.

We would like to emphasize that all the bulk orientation energies we have considered in this section and the last one usually are extremely small compared to the "gross" condensation energy differences between the A and B phases; the latter are typically of the order of a few ergs/cm³ (cf. Wheatley Sec. IV), while the orientation energies we have been considering are rarely greater than one-thousandth of this at fields of less than 1 kG. (The wall energies we shall be considering in the next section are by contrast quite comparable to condensation energies but act only over very small regions.) However, it should perhaps be pointed out that extremely near the critical temperature the orientation energies, which generally speaking are proportional to $(1 - T/T_c)$, will outweigh the A–B condensation energy difference, which varies as $(1 - T/T_c)^2$. The most obvious example of the effect of this is of course the "profound effect" of a magnetic field on the phase diagram near the PCP (Wheatley, Secs. III and IV). This results, *inter alia*, in the interposition in a magnetic field of a thin slice of A phase between the N and B phases below the pressure of PCP (see Wheatley, Fig. 1). In this connection it is amusing to note that for very low external magnetic fields (much less than 30 G) the *dipole* energy (10.10) will be more important than the external field energy (10.1) and will actually favor neither the BW nor the ABM phases, but the normally unstable "2D" phase (9.23).[28] Consequently, if the temperature is lowered in zero (or very small) external magnetic field at pressures below the PCP, we expect not the series of transitions $\mathrm{N} \to \mathrm{A} \to \mathrm{B}$ but rather $\mathrm{N} \to 2\mathrm{D} \to \mathrm{B}$ (the second transition probably occurring continuously). Although it is unlikely that this behavior will be directly visible, it might have some interesting implications for some of the "memory" effects which can be observed in superheating and supercooling (Osheroff, 1973; Wheatley, ch. XI). At higher pressures we expect $\mathrm{N} \to 2\mathrm{D} \to \mathrm{A} \to \mathrm{B}$.

C. Walls

So far we have neglected the boundaries of the liquid ³He, whether they be solid walls, free surfaces, or, possibly, an interface with a ⁴He-rich solution. If we consider for definiteness a solid plane boundary, it is intuitively plausible that the Cooper pairs near the wall should tend to orbit in a plane parallel to the wall rather than one perpendicular to it, since in the latter case they will bump against it. Consequently, one might naively expect that in (say) the ABM state the $\mathbf{l}$ vector would orient itself *perpendicular* to a wall (and more generally, perpendicular to a boundary of any sort).

This conclusion has been confirmed in an elegant calculation by Ambegaokar, de Gennes and Rainer (1974, hereafter Ambegaokar *et al.*, 1974). Their approach, which relates the equation obeyed by the order parameter when it is varying in space to the current–current response function in the normal state, requires more advanced formal techniques than are assumed in this review, so we shall not try to reproduce it here. One can get some idea of the physics behind the argument from the following considerations (which however are a considerable oversimplification): Consider the ABM phase, and imagine a Cooper pair approaching a perfect plane boundary which will reflect the atoms specularly (Fig. 8). In the bulk liquid the phase of the pair wave function is fixed as a function of the relative position vector $\mathbf{r}_1 - \mathbf{r}_2$ of the pair (and is constant as a function of the center-of-mass position, since there is assumed to be no net current). For S-wave pairing this phase is independent of the direction of $\mathbf{r}_1 - \mathbf{r}_2$, but for P-wave pairing it is a function of direction: in fact since the angular dependence of the wave function is of the form $\sin\theta \exp(i\phi)$, where θ, φ are polar angles measured in a system with $\mathbf{l}$ along the z axis, it follows that the phase is unchanged if the component of $\mathbf{r}_1 - \mathbf{r}_2$ along $\mathbf{l}$ is inverted but in general changes if either of the components perpendicular to $\mathbf{l}$ is inverted. Suppose now that our particles are on paths which, if the wall were not there, would take them to P_1 and P_2. If they are in the correct "bulk" wave function, they should arrive there with the correct phase of their relative wave function. On the other hand, since the wall reflects them specularly, they will in fact arrive instead respectively at P_1' and P_2', the "reflected" positions. This means that the component of $\mathbf{r}_1 - \mathbf{r}_2$ perpendicular to the wall is inverted, while those parallel to the wall are unchanged. If $\mathbf{l}$ is perpendicular to the wall this does not matter, since then only the component parallel to $\mathbf{l}$ is changed and, as we have seen, this does not affect the phase of the relative wave functions; the unreflected and reflected waves can therefore continue to interfere constructively and no harm is done. On the other hand, if $\mathbf{l}$ is parallel to the wall a component of $\mathbf{r}_1 - \mathbf{r}_2$ perpendicular to $\mathbf{l}$ is changed and the particles therefore arrive with an incorrect phase of their relative wave function; that is, the unreflected and reflected waves interface destructively and the total amplitude is reduced. In fact Ambegaokar *et al.* find that for $\mathbf{l}$ parallel to the wall the amplitude of the pair wave function tends to zero at the wall, resulting in a loss of condensation energy, whereas for $\mathbf{l}$ perpendicular it remains the same as in bulk. For diffuse reflection the wave function is reduced for both perpendicular and parallel $\mathbf{l}$, but more so in the parallel case. The general conclusion, therefore, is that $\mathbf{l}$ will tend to be anchored *perpendicular* to the wall. One finds that the distance over which the pair amplitude is depressed near the wall in "unfavorable" cases is of the order of the temperature-dependent coherence length(s) $\xi(T)$ (see next subsection). The same general conclusions are expected to apply for the behavior near a free surface (Privorotskii, 1974).

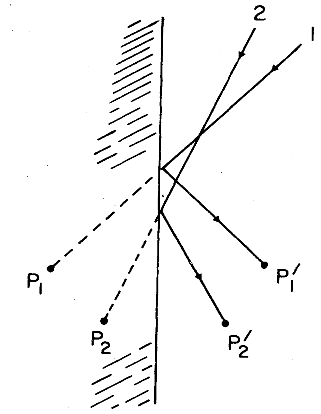

FIG. 8. Effect of specular reflection on the phase coherence of a Cooper pair (for explanation, see text).

As in most other orientation problems, the question of the favored orientation for the BW state near a wall is a good deal more complex than the corresponding problem for the ABM state. It is, indeed, fairly obvious that the character of this state must change somewhat as we approach the wall, since some of the angular momentum components will be suppressed more strongly than others. This problem has been discussed by Brinkman, Smith, Osheroff and Blount (1974; hereafter Brinkman *et al.*, 1974) on the basis of Ambegaokar *et al.*'s results for the ABM phase. They introduce a trial wave function of the form

$$d_{\alpha i}(\mathbf{r}) = (1/\sqrt{3})[\delta_{\alpha\beta} - f(r_\perp)\hat{s}_\alpha\hat{s}_\beta]R_{\beta i}(\mathbf{r}), \tag{10.25}$$

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[28] This was pointed out to me by K. Maki; quantitative calculations confirming this conclusion have been done by S. Takagi (unpublished).

where $\hat{\mathbf{s}}$ is a unit vector normal to the wall, R is an arbitrary rotation matrix specified by an angle θ and an axis $\hat{\boldsymbol{\omega}}$ as in Subsection B, and $f(r_\perp)$ is a function of the distance $r_\perp$ from the wall which tends to 1 as $r_\perp \to 0$ (i.e., as we approach the wall) and to zero as $r_\perp \to \infty$ (far into the bulk liquid). The form (10.25) automatically guarantees that the component of the BW wave function corresponding to angular momentum parallel to the wall is suppressed as we approach the wall; in fact in the limit as we approach the wall the wave function essentially becomes of the "2D" type. Brinkman *et al.* (1974) find that the rotation angle θ is essentially independent of $\mathbf{r}$, and in the absence of an external magnetic fields the axis $\hat{\boldsymbol{\omega}}$ lies perpendicular to the wall. In the presence of a magnetic field $\mathbf{H}$ there is a "field-dependent surface energy," since owing to the effect of suppression of one angular momentum component the susceptibility becomes anisotropic. As a result, for $\mathbf{H}$ in the plane of the wall and large the axis $\hat{\boldsymbol{\omega}}$ no longer lies perpendicular to the wall: in fact it makes an angle of $\cos^{-1}(1/5^{1/2})$ with both the normal to the wall and the field.

In connection with calculations of wall energies one point should be emphasized: although the wall may in general strongly distort the bulk Cooper pair wave function [cf. Eq. (10.25)] involving an energy comparable to the condensation energy, this distortion takes place over a distance of the order of the temperature-dependent coherence length(s) $\xi(T)$ to be introduced in the next section. Generally speaking (except very close to T_c) these lengths are of the order of the pair radius $\xi_0 \sim \hbar v_F/\pi k T_c$, that is, a few hundred Å [cf. Eq. (10.29) below]. Beyond this distance the wall has no longer any distorting effect on the pair wave function, but it *does* have an orienting effect: that is, beyond a distance $\sim\xi_0$ from the wall the pair wave function is as in an infinite bulk medium, but its orientation is (at first) that which is most favorable from the point of view of the wall energy: e.g., for an ABM state $\mathbf{l}$ points perpendicular to the wall. As we go further out into the bulk liquid, the orientation will in general gradually change in accordance with bulk orientation effects, but as we shall see in the next section the scale of the variation is generally very much larger than ξ_0, in fact sometimes of the order of mm. Therefore, generally speaking, to take the effect of the wall into account it is adequate simply to impose the boundary condition that the orientation at the wall is the "most favorable" one.

As a footnote to this section, it should be pointed out that although wall energies, like bulk orientation energies, are usually a small perturbation on the "gross" free energy difference between the ABM and BW phases, they may have a large effect when one is very near the A–B transition. In particular, in a thin slab geometry, the effect of the walls on the phase diagram is expected to be qualitatively like that of a magnetic field (Privorotskii, 1974; cf. Wheatley, Fig. 1).

D. Gradient energies: "Textures"

We have now found rather a large number of different effects which will tend to orient the superfluid phases. Evidently, there will be circumstances in which they are in competition. For instance, if we imagine a situation in which there exists in the ABM phase a heat current and also a magnetic field parallel to it (say, both along the z axis), then the magnetic field will want $\mathbf{d}$ to lie in the xy plane, the heat current will want $\mathbf{l}$ to lie along the z axis, but the dipole forces will try to orient $\mathbf{d}$ parallel to $\mathbf{l}$. This particular example has been studied in detail by Takagi (1973), who shows that a number of different equilibrium configurations are possible depending on the relative magnitudes of the energies involved. Similar considerations should apply to any case in which *bulk* energies are in competition.

A rather more difficult problem arises when the wall orientation effect is in competition with bulk effects. Such a situation would arise, for example, near a wall in a magnetic field perpendicular to the wall. Clearly in general in such situations the equilibrium orientation will be a function of position; near the wall $\mathbf{l}$ will be perpendicular to the wall (hence parallel to the magnetic field), but as we go into the bulk liquid it will gradually rotate so as to be perpendicular to the field. To study these effects in detail one must investigate gradient energies, that is, the energies which arise when the order parameter is varying in space (either in magnitude or orientation). In what follows, we shall always assume that the variation is over a scale long compared to the quantity $\xi_0 \equiv \hbar v_F/\pi k_B T_c$ which is effectively the "size of a Cooper pair" (cf. Sec. VI.D).

In a simple BCS superconductor the gradient energy has an extremely simple form provided the variation of the order parameter in space is sufficiently slow (see below). In fact, if we write them in terms of a spatially varying gap $\Delta(\mathbf{r})$ (we could of course equally well use the order parameter $\Psi(\mathbf{r})$, but this is the conventional choice) we have (see, e.g., de Gennes 1966, Sec. 6.2)

$$\Delta F_{\text{grad}} = \int d\mathbf{r}\gamma \mid \nabla\Delta(r) \mid^2. \tag{10.26}$$

By using the fact that superfluid flow with velocity $\mathbf{v}_s$ corresponds to the variation $\Delta(\mathbf{r}) \sim \Delta \exp(2im\mathbf{v}_s\cdot\mathbf{r}/\hbar)$, and that such a flow introduces an extra energy $\frac{1}{2}\rho_s v_s^2$, we can find the coefficient γ:

$$\begin{aligned}\gamma &= (\hbar^2/8m^2)(\rho_s/\Delta^2)\\ &\approx (N\hbar^2/8m)(m/m^*)[7\zeta(3)/4\pi^2](k_BT_c)^{-2} \equiv \gamma_0,\end{aligned} \tag{10.27}$$

where the second expression is valid in the limit $T \to T_c$, and we used Eqs. (6.32), (6.16), and (2.42) (which of course assumes translational invariance). In a situation where Δ is varying in space, the rest of the GL energy is obtained by treating $F\{\Delta(\mathbf{r})\}$ obtained from (the S-wave version of) Eq. (5.75) as a "local" free energy and integrating it over space. Thus the total GL free energy is

$$\begin{aligned}F_{\text{GL}} = \int d\mathbf{r}[\tfrac{1}{2}(dn/d\epsilon)\{-(1 - T/T_c) \mid \Delta(\mathbf{r}) \mid^2\\ + \tfrac{1}{2}\bar{\beta} \mid \Delta(\mathbf{r}) \mid^4\} + \gamma \mid \nabla\Delta(\mathbf{r}) \mid^2]\\ [\bar{\beta} \equiv \tfrac{7}{8}\zeta(3)(\pi k_BT_c)^{-2}].\end{aligned} \tag{10.28}$$

Clearly we can define a characteristic length $\xi \equiv [2\gamma/(dn/d\epsilon)(1 - T/T_c)]^{1/2}$ from the coefficients of the first and third terms; this has the significance of the length which the order parameter needs to recover its bulk value after being depressed, e.g., by the presence of a wall (cf. De Gennes, 1966, Chap. 6). It should be carefully distinguished from the quantity $\xi_0 \equiv \hbar v_F/\pi k_B T_c$ which has the significance of the radius of a Cooper pair, which is *not* significantly temperature-dependent. Using expression (10.27) for γ, we have

$$\begin{aligned}\xi(T) &\equiv \left[\frac{2\gamma}{(dn/d\epsilon)(1-T/T_c)}\right]^{1/2}\\ &= \left[\frac{N\hbar^2}{4m(dn/d\epsilon)}\left(\frac{m}{m^*}\right)\frac{7\zeta(3)}{4\pi^2(k_BT_c)^2}\right]^{1/2}(1-T/T_c)^{-1/2}\\ &= [7\zeta(3)/48]^{1/2}(\hbar v_F/\pi k_BT_c)(1-T/T_c)^{-1/2}\\ &\equiv \text{const. }\xi_0(1-T/T_c)^{-1/2} \qquad (10.29)\end{aligned}$$

[where we also used Eq. (2.5)]. For a BCS superconductor without Fermi-liquid interactions the expression (10.26) for the gradient terms is actually valid (for sufficiently small gradients, that is $\xi_0\,|\,\nabla\Delta\,| \ll \Delta$) at all temperatures and not just in the GL region; however, γ is then a function of T and is not equal to γ_0. If Fermi-liquid interactions are introduced it seems probable that one needs different coefficients, for T far from T_c, for amplitude and phase variations of the gap.

In the case of an anisotropic superfluid things are somewhat more complicated, since the order parameter is a more complicated object and, correspondingly, the superfluid density is a tensor. We will treat the case of P-wave pairing, which is the only one which has been discussed extensively in the literature (de Gennes, 1973a,b; Wölfle 1974a; Ambegaokar, de Gennes and Rainer, 1974; Brinkman, Smith, Osheroff and Blount, 1974; Brinkman and Smith, 1974). Let us introduce the quantity

$$\bar{d}_{\alpha i} \equiv \Delta d_{\alpha i} \qquad (10.30)$$

so that $\bar{d}_{\alpha i}$ is in fact the expansion of $\boldsymbol{\Delta}(\mathbf{n})$. (In the literature this quantity is often called $d_{\alpha i}$ or A_{pi}; in the latter notation one should note that p labels the *spin* indices and i the *orbital* ones, in contrast to our notation in which α labels orbital indices and i spin ones). If one wants to construct an expression which is of second order in the spatial gradients of the $\bar{d}_{\alpha i}$ and has the correct invariance properties under gauge transformations and spin and spatial rotations, there are only three possibilities ($\partial_\alpha \equiv \partial/\partial x_\alpha$, etc.):

$$\text{(a) } \partial_\alpha\bar{d}_{\alpha i}\partial_\beta d_{\beta i}^*,\quad \text{(b) }\partial_\alpha\bar{d}_{\beta i}\partial_\alpha\bar{d}_{\beta i}^*,\quad \text{(c) } \partial_\alpha d_{\beta i}\partial_\beta\bar{d}_{\alpha i}^*. \qquad (10.31)$$

These three are actually not quite independent, in the sense that (c) can be written as (a) plus a term of the form div **S**. This divergence is often neglected in the literature, presumably on the grounds that when integrated it will give a surface term which vanishes if the boundary conditions derived in the last section are applied (this follows from an examination of the concrete form of **S**). However, it is not completely clear that this neglect may not lead to subtle errors in some circumstances. In particular, it may change the expression for the local mass current (cf. Wölfle, 1974a). If we keep this term, then we can evidently write the gradient energies in the form

$$\begin{aligned}\Delta F_{\text{grad}} &= \gamma_1(\partial_\alpha\bar{d}_{\alpha i}\partial_\beta\bar{d}_{\beta i}^*) + \gamma_2(\partial_\alpha\bar{d}_{\beta i}\partial_\alpha\bar{d}_{\beta i}^*)\\ &\quad + \gamma_3(\partial_\alpha\bar{d}_{\beta i}\partial_\beta\bar{d}_{\alpha i}^*) \equiv \gamma_1 a + \gamma_2 b + \gamma_3 c. \qquad (10.32)\end{aligned}$$

One often also meets the alternative expression, which is equivalent to within the divergence mentioned above:

$$\Delta F_{\text{grad}}' = \sum_p \{\tfrac{1}{2}K_L\,|\,\text{div}\,\mathbf{A}_p\,|^2 + \tfrac{1}{2}K_T\,|\,\text{curl}\,\mathbf{A}_p\,|^2\}, \qquad (10.33)$$

where $\mathbf{A}_p$ is the vector with components A_{pi} ($\equiv \bar{d}_{\alpha i}$, cf. above). A comparison of Eqs. (10.32) and (10.33) under the assumption $a = c$ leads to the relations

$$K_T = 2\gamma_2, \qquad K_L = 2(\gamma_1+\gamma_2+\gamma_3). \qquad (10.34)$$

The coefficients γ_i can be related to the superfluid density $\rho_{\alpha\beta}{}^s$. To do this, we notice that in the special case in which we assume a *uniform* variation of the phase of all the components of $\bar{\mathbf{d}}_{\alpha i}$, that is, put

$$\bar{d}_{\alpha i}(\mathbf{R}) = \Delta d_{\alpha i}\exp i(2m\mathbf{v}_s\cdot\mathbf{R}/\hbar), \qquad (10.35)$$

the resulting state corresponds to a uniform superfluid flow with velocity $\mathbf{v}_s$, and consequently from the definition of the superfluid density tensor, [cf. Eq. (10.18)] the gradient energy must reduce to the form

$$\Delta F_{\text{grad}} = \tfrac{1}{2}\rho_{\alpha\beta}{}^s v_{s\alpha}v_{s\beta}. \qquad (10.36)$$

A comparison of this expression with the one obtained by substituting Eq. (10.35) in (10.32) leads to the relation

$$\rho_{\alpha\beta}{}^s = (8m^2/\hbar^2)(\gamma_1 d_{\alpha i}d_{\beta i}^* + \gamma_2\delta_{\alpha\beta} + \gamma_3 d_{\beta i}d_{\alpha i}^*)\Delta^2 \qquad (10.37)$$

[where we took into account the normalization of the $d_{\alpha i}$, Eq. (9.12)]. On the other hand, an explicit evaluation of Eq. (6.32) for $\rho_{\alpha\beta}{}^s$ in terms of $d_{\alpha i}$ leads, using (6.27), (6.15) and the prescription $|f(\mathbf{n})|^2 \to |\,\mathbf{d}(\mathbf{n})\,|^2$ for triplet states [cf. Subsection VII.E] to

$$\rho_{\alpha\beta}{}^s = \tfrac{3}{5}[1-Y(T)](\delta_{\alpha\beta} + d_{\alpha i}d_{\beta i}^* + d_{\beta i}d_{\alpha i}^*)(1+\tfrac{1}{3}F_1)^{-1}\rho. \qquad (10.38)$$

Comparing (10.37) and (10.38), and using formula (6.16) for $Y(T)$, we conclude (but see note at end)

$$\begin{aligned}\gamma_1 = \gamma_2 = \gamma_3 &= \tfrac{3}{5}(\hbar^2/8m^*)[N/(k_BT_c)^2][7\zeta(3)/4\pi^2]\\ &\equiv \tfrac{3}{5}\gamma_0, \qquad (10.39)\end{aligned}$$

where γ_0 is the coefficient which would arise for an S-wave (BCS) superfluid. (Cf. Eq. (10.27) above.] Equivalently we have from Eq. (10.34)

$$K_T = \tfrac{1}{3}K_L = \tfrac{6}{5}\gamma_0. \qquad (10.40)$$

Since K_T and K_L are not equal we have a different "bending energy" corresponding to bending of the vector components $\mathbf{A}_p$ of the order parameter perpendicular and parallel to their directions (or, equivalently, the energy associated with a given gradient of $d_{\alpha i}$ *along* the direction α is not equal to the energy associated with the same gradient in a transverse direction). So we can define *two* characteristic lengths[29]

$$\xi_T \equiv \left(\frac{K_T}{(dn/d\epsilon)(1-T/T_c)}\right)^{1/2},$$
$$\xi_L \equiv \left(\frac{K_L}{(dn/d\epsilon)(1-T/T_c)}\right)^{1/2}. \tag{10.41}$$

From Eqs. (10.41) and (10.40) and (10.27) we have

$$\xi_T = \xi_L/\sqrt{3} = (3/5)^{1/2}[7\zeta(3)/48]^{1/2}(\hbar v_F/\pi k_B T_c) \times (1-T/T_c)^{-1/2} \equiv \text{const. } \xi_0(1-T/T_c)^{-1/2}, \tag{10.42}$$

where the constant is of order one. Patton (1974) estimates the coefficient of $(1-T/T_c)^{-1/2}$ to be $(\frac{3}{5})^{1/2} \times 124$ Å.

The gradient terms in the free energy in the GL region can be written, then, in the simple form

$$\Delta F_{\text{grad}} = \tfrac{3}{5}\gamma_0(\partial_\alpha \bar{d}_{\alpha i}\partial_\beta \bar{d}_{\beta i}^* + \partial_\alpha \bar{d}_{\beta i}\partial_\alpha \bar{d}_{\beta i}^* + \partial_\alpha \bar{d}_{\beta i}\partial_\beta \bar{d}_{\alpha i}^*)$$
$$\equiv \tfrac{3}{5}\gamma_0 \sum_p \{3\,|\operatorname{div}\mathbf{A}_p|^2 + |\operatorname{curl}\mathbf{A}_p|^2\}. \tag{10.43}$$

It is important to realize that although we derived the coefficients in (10.43) by considering simple superfluid flow, which corresponds to a simple variation in space of all $d_{\alpha i}$ by the same phase factor, the expression (10.43) is applicable to much more general situations. Some examples of other types of variation we may want to consider are: (1) A variation of the overall *magnitude* of $\bar{d}$ (i.e., Δ) in space. This corresponds to a spatial variation of the actual number of condensed pairs: one might for example have to consider such a situation in the presence of very strong thermal inhomogeneity. (2) A rotation of $\mathbf{d}$ in spin space only, without change of magnitude. This corresponds to "spin counterflow" or more generally superfluid spin currents [cf. Sec. VII.D and Vuorio (1974)]. (3) A rotation of $d_{\alpha i}$ in orbital space only. In the ABM phase this would correspond to a change (rotation) of the vector $\mathbf{l}$ in space. This situation is in some ways the most interesting of all, and is generally believed to give rise to the so-called "orbital supercurrents" (Anderson and Morel, 1961; de Gennes, 1973a; Wölfle, 1974a; Ambegaokar *et al.*, 1974). However, there are a number of questions concerning this type of situation which seem to me still unclear, so I shall not discuss it in detail here (cf. Introduction).

Before we apply Eq. (10.43) it should be pointed out that it has been derived above only for the GL region [and, strictly speaking, only then under the assumption that the allowed invariants are restricted to those in Eq. (10.31)]. I do not believe it is more generally valid, even if γ_0 is allowed to depend on temperature. To see that this is so, we can substitute in it a form of variation corresponding to "spin superflow" rather than ordinary superflow: we then reach the prediction that there is a unique and temperature-independent relation between $\rho_{\alpha\beta}{}^s$ and the "spin superfluid density" $\rho_{ij,\alpha\beta}{}^{\text{spin}}$ [Eq. (7.62)]. This is true if there are no Fermi-liquid effects, but as pointed out in Sec. VII.D, in the presence of such effects it is true *only* in the GL region. Therefore, if we apply Eq. (10.43) at arbitrary temperature we reach a contradiction. Nevertheless, it is probably valid for purposes of order-of-magnitude estimation.

Using the expression (10.43) for the gradient free energy, we can in principle solve the problem of the behavior of the space-dependent order parameter $d_{\alpha i}(\mathbf{R})$ [or $\mathbf{d}(\mathbf{n}:\mathbf{R})$] when there are competing orientation effects. What we have to do is to write down the sum of the bulk orientation energies and the gradient energy, impose the correct boundary conditions (e.g., for the ABM state, $\mathbf{l}$ near a wall must be perpendicular to the wall; if the liquid is in a wide tube and known to be flowing, far from the walls, with velocity $\mathbf{v}_s$, then far from the walls $\mathbf{l}$ must be parallel to $\mathbf{v}_s$; etc.), and minimize the whole expression with respect to $d_{\alpha i}(\mathbf{R})$. Since all terms are quadratic in the $d_{\alpha i}$, this gives a set of simultaneous linear second-order differential equations which can in principle be solved to give the complete behavior of $d_{\alpha i}(\mathbf{R})$, and hence the complete position-dependent orientation of the anisotropic superfluid. Such a program has been carried out, for example, by de Gennes and Rainer (1974) for the A phase flowing in a narrow tube, and by Brinkman *et al.* (1974) for the somewhat more complicated problem of the B phase oriented by walls and by a magnetic field. We will not try to reproduce the details of those calculations, but want to comment on one important general point which emerges: whether we are talking about the A or the B phase, very close to a wall it is always the wall which determines the orientation, while if we go out sufficiently far into the bulk liquid obviously the system will have "forgotten" the wall and taken up the orientation determined by bulk effects. The question then arises, what is the order of magnitude of the "healing length" R_c such that for distances from the wall larger than R_c we have essentially bulk behavior? Evidently, this is determined by the competition of the bulk orientation effects, whatever they are, and the gradient energies. In fact, we can get the order of magnitude R_c from the following argument: Suppose that the healing length is of order R. Then the spatial derivatives of the $\bar{d}_{\alpha i}$ are of order Δ/R, and accordingly the gradient free energy per unit area is of order $\gamma_0 R(\Delta/R)^2 \sim \gamma_0\Delta^2/R$. On the other hand, if the bulk orientation energy per unit volume is ΔF_b, by allowing the orientation to differ from the value preferred in bulk over a distance of order R we have increased the free energy per unit area by an amount of order $R\Delta F_b$. The total added free energy is therefore

$$\Delta F' \sim \gamma_0\Delta^2/R + R\Delta F_b \tag{10.44}$$

and the value of R_c, which is found by minimizing this expression, is given by

$$R_c \sim (\gamma_0\Delta^2/\Delta F_b)^{1/2}. \tag{10.45}$$

Since, from Eq. (10.27), $\gamma_0\Delta^2$ is of order $(N\hbar^2/m)(\rho_s/\rho)$, we have generally

$$R_c \sim \left(\frac{N\hbar^2}{m}\right)^{1/2}\left(\frac{\rho_s/\rho}{\Delta F_b}\right)^{1/2} \sim 10^{-4}\left(\frac{\rho_s/\rho}{\Delta F_b}\right)^{1/2} \text{erg}^{1/2}\,\text{cm}^{-1/2}. \tag{10.46}$$

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[29] This definition is the same as that of Ambegaokar *et al.*, 1974 [their a is in fact just $(dn/d\epsilon)(1-T/T_c)$].

For the ABM phase the bulk energies ΔF_b are usually of the order 10^{-3} ergs/cm^3 and hence the healing length is of order 10^{-3}–10^{-2} cm (or more).[30] [Note that if, for instance, it is possible to choose a configuration which satisfies the wall boundary condition and minimize either the external field energy ΔF_{magn} or the dipole energy ΔF_{dip} but not both, then the system will already orient itself near the wall so as to minimize the larger of ΔF_{magn} and ΔF_{dip}, so that it is the *smaller* of the bulk orientation energies which enters the estimate (10.46). Thus the above estimate is valid even in very high magnetic fields.] We notice that for the ABM phase the healing length for given field, etc. does not depend substantially on temperature near T_c, since both ρ_s/ρ and ΔF_b are proportional to $(1 - T/T_c)$.

For the BW phase we may consider specifically competition between the wall orientation of the rotation axis $\hat{\boldsymbol{\omega}}$ and its orientation in bulk by a magnetic field $\mathbf{H}$. Using the estimate of the bulk orientation energy made in subsection A, i.e., about $4 \times 10^{-12} H^2$ ergs/cm^3 G^2 (with no temperature dependence near T_c), we find

$$R_c \sim 100(1 - T/T_c)^{1/2} H^{-1} \text{ cm G}. \qquad (10.47)$$

A more quantitative analysis by Brinkman *et al.* (1974) reduces this estimate (in effect) by a factor of order four or five. The result is still a *macroscopic* length if H is not too large: hence, one reaches the somewhat surprising conclusion that for liquid ^{3}He-B (if it is indeed a BW phase) contained in a typical experimental sample chamber (dimensions $\sim$1 cm) in fields of the order of a few G and not too near T_c, the orientation should be determined principally by the walls! While the quantitative details of the above argument clearly cannot be trusted too far, the conclusion that the characteristic healing length is proportional to $H^{-1}(1 - T/T_c)^{1/2}$ should be independent of them. One can then see that at high fields the fraction of the liquid which has an orientation differing substantially from the bulk value is proportional to H^{-1}.

When the orientation of the Cooper pairs is changing over lengths very long compared to the pair radius, in the way described above, we say the liquid forms "textures" (de Gennes, 1973a; the name comes from liquid crystal theory). Depending on the geometry of the sample chamber, the field, etc., different types of texture may be stable (cf. Brinkman *et al.*, 1974); in some there may occur singularities. For instance, if we consider the ABM phase in an infinite cylinder (with field, etc. equal to zero) then it is clearly impossible on topological grounds for the vector $\mathbf{l}$ to be everywhere perpendicular to the walls and at the same time continuous everywhere in the bulk liquid. If, however, one is allowed to introduce lines parallel to the cylinder axis on which $\mathbf{l}$ is not defined, there is no problem.

Such a line is known as a "disgyration" and is somewhat similar to the vortex cores well known in the theory of type-II superconductors or liquid ^{4}He. The study of this and other types of singularity in the textures of an anisotropic superfluid is still in its infancy (cf. de Gennes, 1973a; Fujita and Tsuneto, 1974).

Note: the argument used in this subsection to obtain the γ_i's [Eq. (10.39)] cannot exclude an extra "antisymmetric" term $\delta\gamma_i$ such that $\delta\gamma_2 = 0$, $\delta\gamma_1 = -\delta\gamma_3$. [Such a term would give zero contribution to (10.32) when the spatial variations is of the form (10.35)]. This problem has recently been studied in detail by M. C. Cross (unpublished) who concludes that the extra term vanishes under the normal assumption of "particle-hole symmetry" (i.e., if we neglect the variation of the density of states near the Fermi surface) and more generally is at most of order $(T_c/\epsilon_F)^2$. Cf., P. W. Anderson and W. F. Brinkman, Lecture Notes of the 1974 Scottish Universities Summer School (to be published).

XI. NUCLEAR MAGNETIC RESONANCE

Considering the substantial amount of theoretical work which had been done on anisotropic superfluids before the experimental discovery of ^{3}He-A and B in 1972 (cf. Chaps. V–VII) it is somewhat ironical that it should be an area which had attracted virtually no theoretical attention at all, the nuclear magnetic resonance properties, which has turned out to provide some of the most fruitful opportunities for the comparison of theory and experiment. There are a number of reasons for this. On the experimental side, NMR experiments are possible in a wide variety of experimental geometries, do not put any constraints on the cooling technique, etc., and moreover, can be carried out with extremely high precision. On the theoretical side, it turns out that while experiments on the static susceptibility are sensitive to the behavior of the Cooper pair spins, and experiments on the superfluid density to their orbital behavior, NMR experiments are sensitive to the correlations between spin and orbital motion, which involve as it were many more degrees of freedom and therefore show up as an increased richness and complexity of the NMR behavior.

In the theoretical literature two main approaches have been taken to the study of NMR phenomena. One is based on the microscopic equations of motion or Green's functions (Maki and Ebisawa, 1973, 1974a,b,c; Takagi, 1974a; Combescot and Ebisawa, 1974); generally speaking the equations of motion must be solved by some approximation, e.g., the generalization of the random-phase approximation to the superfluid case (or by an equivalent assumption about the regularity, etc., of quantities occurring in the Green's function). Such methods have the advantage that they automatically permit the study of the *linear* behavior of the system [that is, essentially the behavior which shows up in the unsaturated continuous-wave (c.w.) resonance] over a wide range of variation of parameters such as external field and collision time; in particular, they enable an account to be given of the damping of the c.w. resonance. However, these methods are formally very complex, and because of this (and perhaps also for deeper reasons) it has not so far proved possible to generalize them to deal with nonlinear phenomena, which from an experimental point of view are at least as interesting as the c.w. behavior. The second method was a semiphenomenological approach based on an adiabatic (Born–Oppenheimer) type of approximation (Leggett, 1973a, 1974a; Engelsberg, Brinkman, and Anderson, 1974; Maki and Tsuneto, 1974a; cf. Anderson, 1973). While this method is certainly not valid for arbitrary values of parameters such as collision time, it seems fairly well

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[30] De Gennes and Rainer (1974) assume that $\mathbf{v}_s$ can be as large as 1 cm/sec in small capillaries and thereby find a value of R_c of $\sim 2 \times 10^{-4}$ cm.

established that it is valid at least as a first approximation for the values which actually occur in liquid ^{3}He-A and B; moreover, it can handle the nonlinear phenomena just as easily as linear ones, and is formally a great deal simpler than the microscopic approach. The main drawback of this approach is that it has not so far proved possible to incorporate in it a consistent account of relaxation effects (see however Subsection F below).

In this section I shall use the second, semiphenomenological approach, as being more consistent with the spirit of this review. I shall concentrate on the derivation and discussion of the basic equations rather than on their application to specific experimental situations, since the latter has been extensively discussed in the companion paper by Wheatley (Sec. VIII). Moreover, since I have presented many of the ideas to be discussed here rather more fully in two other papers (Leggett, 1973b, 1974a), I shall often refer to these papers for details of the argument.

A. Dipole forces and broken spin-orbit symmetry

Let us start by considering the implications of the earliest discovered NMR anomaly in the new phases of ^{3}He—the observation that the ordinary c.w. resonance frequency in ^{3}He-A shifts away from the Larmor value $\omega_L \equiv \gamma H_0$, by an amount which (as we now know) may be of the order of at least several tens of kHz in low external fields (see Wheatley, Sec. VIII). It may easily be seen that no such shift is possible if we take into account only forces which conserve the total spin S and its components. For in that case energy eigenstates of the system in an external field H_0 may be labelled by total spin S, projection of spin S_z (along the field axis), and a whole set of other quantum numbers which we collectively label n; and since only the external field (Zeeman) energy depends on S_z, we can write

$$E(n, S, S_z) = E_0(n, S) - \gamma H_0 S_z \tag{11.1}$$

(where γ is the gyromagnetic ratio). But a weak rf field perpendicular to the z axis has the effect of changing S_z by $\pm\hbar$ (according to the usual selection rules) and so we see from Eq. (11.1) that it must gain or lose an energy equal to $\gamma\hbar H_0$, i.e., the resonance frequency is exactly equal to $\gamma H_0 \equiv \omega_L$. (For a more formal version of this argument, see Leggett, 1973b.)

Consequently, it is essential to invoke some force which does *not* conserve total spin. The most obvious candidate is the interaction between the nuclear magnetic moments, which was already mentioned in Sec. X in connection with orientation. [It is also conceivable that an induced interaction between nuclear spins due to polarization of the electronic shells might play a role (Anderson and Varma, 1973), though so far there is no direct evidence for this. If such an interaction does exist, it should presumably have the same symmetry as the dipole forces and the remarks below would apply equally to it.] The dipole force is actually extremely small, in fact, the order of magnitude of the interaction between two neighboring ^{3}He nuclear magnetic moments is $\sim 10^{-7}$°K in temperature units, and the field induced by one nuclear dipole at the site of its neighbor is of order 1 G, corresponding to a precession frequency of about 3 kHz. It is therefore at first sight somewhat mysterious that it should have such large effects, especially at temperatures as high as 10^{-3}°K.

The reason for this is essentially as follows (for a very much more extensive discussion, see Leggett, 1973b): If we have a pair of particles, say a diatomic molecule, subject to spin-conserving forces, then the total spin $\mathbf{S}$ and relative orbital angular momentum $\mathbf{L}$ are separately conserved and moreover, classically speaking, may point in any direction relative to one another without affecting the energy of the system: the analogous quantum-mechanical statement is that the energy is independent of total angular momentum quantum number J. If now we switch on the weak nuclear dipole forces, then provided that there is no accidental degeneracy with respect to L in the original system, we may to a good approximation assume that both $|L|$ and $|S|$ are still conserved; however, they now precess around one another as in the well-known vector model of atomic physics, and the total energy now depends on the direction of $\mathbf{L}$ relative to $\mathbf{S}$, i.e., on the total angular momentum J. In fact, classically speaking, the nuclear dipoles prefer to lie end-to-end rather than side-to-side, which means that $\mathbf{L}$ tends to lie perpendicular to $\mathbf{S}$: correspondingly, in quantum mechanics (for $S = 1$) the state with $J = L$ lies lower than either of the states $J = L \pm 1$. However, the splitting of the levels is clearly of the order of the nuclear dipole energy, that is, of order 10^{-7}°K. Now, if we take a statistical ensemble of such pairs of particles at a temperature $T \gg 10^{-7}$°K, evidently all the states $J = L - 1, L, L + 1$ will be almost equally populated; that is, any correlating effect of the dipole force will be completely disrupted by thermal fluctuations.

The reason why this argument does not work for an anisotropic superfluid is that because of the "Bose-condensed" nature of the system wave function (5.2), all pairs of particles are forced, at least at $T = 0$, to have *the same* relative wave function $\varphi(\mathbf{r}_1 - \mathbf{r}_2, \sigma_1\sigma_2)$. (At $T \neq 0$ those particles which form Cooper pairs must satisfy this condition.) This means that *all* Cooper pairs must have the same value[31] of J. Once this condition is imposed, the thermal fluctuation energy is no longer competing with an energy of order 10^{-7}°K but rather with one of order 10^{-7}°K $\times N_c$, where N_c is the total number of condensed particles in the system. It is therefore unable to prevent the dipole forces from selecting that orientation of the pair wave function (within a given class, e.g., the class of ABM states or of BW states) which minimizes the dipole energy. This situation (which was already implicitly assumed in the discussion of Sec. X.A) is a particular case of a more general property which I have elsewhere called "spontaneously broken spin–orbit symmetry" (Leggett, 1973b).

The upshot of this argument is that the anomalous NMR behavior must be closely associated with the nuclear dipole orientation energy (10.10). We will use this expression in the following: as will be seen, the orientation-independent constant does not affect the results.

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[31] Or more strictly the same linear combination of eigenfunctions of $\hat{J}$, since because of the nonlinear nature of the BCS equation the Cooper pair wave function need not correspond to a definite J-value.

B. The adiabatic approximation[32]

It is by now well established experimentally that a "longitudinal" ("parallel") vibration, that is, a vibration of the component of magnetization *parallel* to the external field, can occur in both ^{3}He-A and B, and that at least for small vibration amplitudes (as in c.w. resonance) its frequency is essentially independent of the (constant) external field (Wheatley, Sec. VIII; cf. Osheroff, 1974). This may reasonably be interpreted as meaning that in such a case the only function of the magnetic field is to establish an orientation for the *equilibrium* state, and that it has no subsequent effect on the motion (since it cannot exert a torque on the component of magnetization parallel to itself). This then is a particularly simple situation, and for concreteness we shall refer explicitly to such a case in the ensuing argument, although all considerations are in fact quite generally valid.

The observation which is the key to the construction of a sensible semiphenomenological approach to the longitudinal resonance (and more generally to the NMR properties) is that the resonance frequency, though unexpectedly large, is still small compared to almost any other characteristic frequency in ^{3}He. For instance, the normal state relaxation frequency τ^{-1} may be estimated to be of order 100 MHz (cf. Wheatley, Appendix) and the "gap frequency" Δ/h is also of order 100 MHz except close to T_c; whereas the maximum value of the (A-phase) longitudinal resonance frequency is of order 100 kHz. This means that it is reasonable to assume that any variable which is not conserved (or very nearly conserved) will come to equilibrium in a time short compared to a period of the vibration. In that case we may hope to describe the system entirely in terms of the "quasi-conserved" variables. For present purposes, there are two sets of quasi-conserved variables of interest: the components of the total spin $\mathbf{S}$, which would be rigorously conserved were it not for the dipole forces, and which therefore change over a characteristic time of the order of the period of the vibration, and the magnitude and orientation of the set of vectors $\mathbf{d}(\mathbf{n})$ which describe the Cooper pair wave function. The question of the conservation of $\mathbf{d}(\mathbf{n})$ is somewhat delicate: suppose for instance that we start with a particular member of the class of states which happens to be thermodynamically stable for the values of pressure and temperature in question—e.g., we start with the ABM state with a given orientation of $\mathbf{l}$ and $\mathbf{d}$. Then any attempt to change the basic nature of the state—e.g., to perturb it so that it becomes more like the 2D state—will clearly produce very large restoring forces, so that this possibility can be neglected for present purposes. On the other hand, it can be *rotated*—or more generally reoriented—at a very much lower cost in energy, in fact at the cost only of the dipole energy. One can say, therefore, that the basic nature of the state is rigorously conserved to all intents and purposes, while its orientation is quasi-conserved, that is, it changes only over a time of the order of the longitudinal vibration period. As we shall see below, this second conclusion is not quite correct under all circumstances; nevertheless, this does not invalidate the method to be used, as we shall see.

The conclusion of these considerations, then, is that we can construct a theory of the longitudinal vibration in terms of the variables $\mathbf{S}$ and $\mathbf{d}(\mathbf{n})$, or more accurately the *orientation* of $\mathbf{d}(\mathbf{n})$. What we must do is to express the energy in terms of those variables alone, find the kinematic relations between them, and then apply the standard techniques of quantum mechanics to work out the equations of motion. The underlying philosophy is very similar to that used in the Born–Oppenheimer theory of molecular vibrations: there, one actually has a very complicated problem involving all the electronic coordinates as well as the nuclear ones. However, the period of the molecular vibrations is known to be very long compared to characteristic electronic relaxation times, and one therefore assumes that the electrons adjust adiabatically to the motion of the nuclei. Then the electronic degrees of freedom can be eliminated from the problem, and we can describe the vibrational motion purely in terms of the nuclear coordinates and momenta. However, the nuclei now move in an effective potential which is provided, *inter alia*, by the electrons. In just the same way, in the NMR problem, we assume that all degrees of freedom other than $\mathbf{S}$ and the orientation of $\mathbf{d}(\mathbf{n})$—e.g., the normal quasiparticle distribution function—can adjust essentially instantaneously to the motion of the quasi-conserved variables, $\mathbf{S}$ and $\mathbf{d}$, and that their whole effect is to provide an effective potential for this motion.

It should be emphasized that this adiabatic assumption is only a first approximation and cannot be expected to hold to arbitrary accuracy. In particular, while it should be adequate to describe the gross features of the ^{3}He NMR behavior, by its very construction there is no room for irreversible relaxation processes, and it cannot therefore say anything about the damping of the resonances (cf., however, Subsection F below). In fact, if τ is a characteristic relaxation time (say, the relaxation time of normal quasiparticles), and ω_D is the frequency of the longitudinal resonance, the adiabatic approximation is equivalent to the assumption $\omega_D\tau \to 0$, and we would expect the importance of corrections to it to be measured by the magnitude of $\omega_D\tau$ (cf. Subsection F below). If we estimate τ from the normal-state relaxation time, then we always have $\omega_D\tau \leq 10^{-3}$, but as we go down into the superfluid phases we expect that the normal-quasiparticle relaxation time increases quite rapidly, and therefore the corrections become relatively more important. It is therefore not altogether surprising that the B-phase longitudinal resonance appears to be quite highly damped. However, we should not expect this damping to increase indefinitely as we lower the temperature, since presumably it is associated with the normal component; at zero temperature there *is* no normal component and we expect no damping. Consequently, if we lower the temperature sufficiently the adiabatic approximation should improve again.

Although the above considerations were developed with specific reference to the longitudinal vibration, clearly they apply more generally. True, in the case of more general (e.g., transverse) motion in an external field H_0, the characteristic period of the motion is $\omega_L \equiv \gamma H_0$ rather than ω_D and one might therefore at first sight suspect that the corrections to the adiabatic approximation would be of order $\omega_L\tau$ (which is not always small) rather than $\omega_D\tau$. However, all the effects we are going to consider are associated with the dipole forces, and as we shall see below

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[32] A more extensive discussion is given in Leggett, 1974a, Sec. 4.

(Subsection E) the importance of the dipole forces relative to the external field is of order $(\omega_D/\omega_L)^2$. Consequently we guess that at high ω_L the correction is not of order $\omega_L\tau$ but rather at most of order $(\omega_D/\omega_L)^2\omega_L\tau$, which is always small. This guess is strongly confirmed, at least in the case of c.w. phenomena, by the results to be reported in Subsection F).

We must now write down the Hamiltonian as a function of $\mathbf{S}$ and $\mathbf{d}(\mathbf{n})$ in the adiabatic approximation. Since it is only the orientation of $\mathbf{d}$ which is varying, we need only the terms in the energy which depend on the orientation, namely the dipole energy. This has already been written down for a P-wave state [Eq. (10.10)]:

$$\hat{H}_D\{\mathbf{d}(\mathbf{n})\} = g_D(T) \int 3\,|\mathbf{n}\cdot\mathbf{d}(\mathbf{n})|^2\,(d\Omega/4\pi),$$
$$g_D \equiv \tfrac{1}{2}\pi\gamma^2\hbar^2\Psi^2(T)\,\langle R^2\rangle_{\mathrm{av}}. \qquad (11.2)$$

As for the S-dependent energy, this can be obtained directly from thermodynamic considerations:

$$E(\mathbf{S}) = \tfrac{1}{2}\gamma^2\chi^{-1}\mathbf{S}^2 - \gamma\mathbf{S}\cdot\mathbf{H}(t), \qquad \mathbf{H}(t) \equiv \mathbf{H}_0 + \mathbf{H}_{\mathrm{rf}}(t), \qquad (11.3)$$

where $\mathbf{H}(t)$ is the total external magnetic field, which we now allow to be a function of time since we may want to include the rf field, H_{rf}. The susceptibility χ here is the "*experimental*" susceptibility, that is, the quantity which would normally be measured in a static magnetization measurement (cf. Sec. VII.D) (see below). The total adiabatic Hamiltonian (for a P-wave state) is then

$$\hat{H}\{\mathbf{S}, \mathbf{d}(\mathbf{n})\} = \tfrac{1}{2}\gamma^2\chi^{-1}\mathbf{S}^2 - \gamma\mathbf{S}\cdot\mathbf{H}(t) + g_D \int 3\,|\mathbf{n}\cdot\mathbf{d}(\mathbf{n})|^2\,(d\Omega/4\pi) \qquad (11.4)$$

and we claim that this Hamiltonian will describe the NMR behavior to a first approximation.

In connection with expression (11.4) at least two questions may be raised. The first concerns the role of the orbital coordinates. Evidently, the dipole energy (11.2) depends on both spin and orbital orientation, e.g., for the ABM phase it is proportional to $-(\mathbf{d}\cdot\mathbf{l})^2$ [Eq. (10.11)]. What should we assume about $\mathbf{l}$, or more generally about the orientation of the *orbital* wave function? At the beginning of the NMR experiment the orbital wave function will presumably have its equilibrium orientation, which is determined by various (possibly competing) effects, including the dipole energy, as described in Sec. X. Now one may argue intuitively that because of the large moment of inertia ($\sim m\xi_0^2$) associated with the orbital motion of a Cooper pair, this quantity will be difficult to change and may be assumed to be held constant during a typical NMR experiment. Another way of putting this is to say that since the typical fluctuation frequency of $\mathbf{l}$ [the frequency of an "orbital wave" (cf. Anderson, 1973; Wölfle 1974a)] is on any account very small compared to a typical longitudinal resonance frequency, the orbital angular momentum has no time to follow the spin vector $\mathbf{d}$ during this period of vibration and is effectively fixed in space.

The second question—which has caused a certain amount of confusion—is connected with the anisotropy of the susceptibility. As we saw, the susceptibility in the superfluid state is actually a tensor χ_{ij} with axes depending on the orientation of the $\mathbf{d}(\mathbf{n})$ [Eq. (7.52)]. Should one not, therefore, replace the $\chi^{-1}\mathbf{S}^2$ in Eq. (11.2) by $(\chi)_{ij}{}^{-1}S_iS_j$ and express the matrix χ^{-1} in terms of $\mathbf{d}(\mathbf{n})$? If so, this would give terms in the Hamiltonian which couple $\mathbf{S}$ and $\mathbf{d}(\mathbf{n})$ directly. This procedure is quite possible and straightforward (Engelsberg, Brinkman, and Anderson, 1974; Leggett, 1974a) and in the most general case gives results which are not identical to those obtained from Eq. (11.3) as it stands. However, it can be shown (Leggett, 1974a) that the corrections vanish in all the following cases: (a) any state in the limit $T \to T_c$, (b) longitudinal resonance in any state, (c) high-field effects in any state, (d) the BW state and, more generally, any state with isotropic susceptibility, (e) the ABM state and, more generally, any state with $\mathbf{d}(\mathbf{n}) = \mathbf{d}f(\mathbf{n})$. This covers the overwhelming majority of cases of practical interest; in particular, the conventional identification of the A phase as ABM and the B phase as BW on the basis of their NMR behavior can be carried out entirely on the basis of results obtained close to T_c, so that there is no question of corrections from susceptibility anistropy invalidating these identifications. Moreover, if the identifications are correct and ³He-A and B are indeed, respectively, ABM and BW phases, no corrections are needed under any circumstances.[33] I shall therefore assume that Eq. (11.3) does not need to be corrected for susceptibility anisotropy for any case of practical interest.

C. A-phase longitudinal resonance

To obtain equations of motion from the Hamiltonian (11.3), we need to supplement it with the appropriate kinematic relations between $\mathbf{S}$ and $\mathbf{d}(\mathbf{n})$—that is, in quantum mechanics, by the commutation relations, or equivalently, in classical mechanics, by the Poisson brackets. In the present subsection we shall consider for simplicity the A-phase longitudinal resonance, obtain the kinematic relations for this case in a rather intuitive way, and briefly discuss the equations of motion which result; in the next subsection we rederive the kinematic relations in a rather more rigorous manner. As far as possible in this subsection we follow the notation of Wheatley.

If we assume that the A phase is of the ABM type [or more generally that it is described by $\mathbf{d}(\mathbf{n}) = \mathbf{d}f(\mathbf{n})$], then in the ESP axes we have according to Eqs. (7.33) and (7.35)

$$\Psi_{\uparrow\uparrow}(\mathbf{n}) = f(\mathbf{n})\{-d_x + id_y\}, \quad \Psi_{\downarrow\downarrow}(\mathbf{n}) = f(\mathbf{n})\{d_x + id_y\}, \qquad (11.5)$$

where $\mathbf{d}$ is a real unit vector. Let us introduce the angle θ used by Wheatley:

$$d_x \equiv \cos\theta, \qquad d_y \equiv \sin\theta. \qquad (11.6)$$

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[33] I find the statement in the abstract of Engelsberg, Brinkman, and Anderson (1974a) that Eq. (11.3) should be corrected for the ABM and BW states by replacing χ by the "correct temperature-dependent susceptibilities," somewhat confusing. Provided that χ in (11.3) is correctly interpreted as the "experimental" susceptibility (which is temperature dependent in the BW state but not in the ABM state), the results obtained from Eq. (11.3) agree with those of the above authors.

Then the Cooper pair wave function can be written, apart from an overall phase factor, in the general form

$$\Psi(\mathbf{n}:\sigma_1\sigma_2) = f(\mathbf{n})\{\exp(-2i\theta)\,|\uparrow\uparrow\rangle - |\downarrow\downarrow\rangle\} \qquad (11.7)$$

so that we see explicitly that the spin part of the wave function is a linear superposition of $S_z = +1$ and $S_z = -1$ components, with a definite phase relation $\Delta\varphi = \pi - 2\theta$ between them (cf. the end of Sec. VII.A). If, with Wheatley, we choose the orbital angular momentum vector **l** to lie along the x axis at the beginning of the experiment, then provided the dipole energy is the dominant factor in choosing the orientation **d** will also lie along the x axis in equilibrium; that is, the equilibrium value of θ is zero. However, we are also interested in situations in which it deviates from this value.

Evidently, if the pairs are in a Cooper pair wave function of the form (11.7) the total z component of spin S_z is not a good quantum number (cf. the end of Sec. VII.A). We should in fact expect that S_z and θ are in some sense conjugate variables. To obtain the exact kinematic relation between them we may argue in the following somewhat nonrigorous way: Imagine for the moment that S_z represented an *orbital* angular momentum, and that correspondingly **d** was a vector in ordinary coordinate space. We know that in classical mechanics the Poisson bracket of two dynamical quantities A and B is given in terms of the change of A under an infinitesimal transformation generated by B [see, for instance, Kibble (1966), Chap. 13.6]. But the z component of angular momentum generates an infinitesimal rotation around the z axis (Kibble, *loc. cit.*) i.e., θ changes to $\theta + \delta\theta$. Hence the classical Poisson bracket of S_z and θ would be simply 1. Making the transition to quantum mechanics in the standard way, we find for the commutator:

$$[S_z, \theta] = -i\hbar. \qquad (11.8)$$

This commutation relation, together with the adiabatic Hamiltonian (11.4), gives a complete description of the problem of longitudinal resonance in the ABM phase. [If the derivation of (11.8) seems inadequate, see the next subsection.]

To derive the equations of motion we first rewrite the dipole energy explicitly in terms of θ, using Eqs. (10.11) and (11.6). We have

$$H_D = -\tfrac{3}{5}g_D(T)(\mathbf{d}\cdot\mathbf{l})^2 \equiv -\tfrac{1}{2}\lambda\cos^2\theta, \qquad \lambda \equiv \tfrac{6}{5}g_D(T). \qquad (11.9)$$

Consequently, the complete adiabatic Hamiltonian (11.4) becomes

$$H = H(S_z, \theta) = \tfrac{1}{2}\gamma^2\chi^{-1}S_z^2 - \gamma S_z H_z(t) - \tfrac{1}{2}\lambda\cos^2\theta, \qquad (11.10)$$

where we have assumed that the field $\mathbf{H}(t)$ is always along the z axis and that consequently the x and y components of **S** remain zero. From Eqs. (11.10) and (11.8) we easily obtain

$$dS_z/dt = -\partial H/\partial\theta = -\lambda\sin\theta\cos\theta \qquad (11.11)$$

$$d\theta/dt = \partial H/\partial S_z = -\gamma[H_z(t) - \gamma S_z/\chi]. \qquad (11.12)$$

The first of these equations corresponds to Eqs. (8.2) and (8.13) of Wheatley's paper (bearing in mind that for the case considered $\mathbf{S}\times\mathbf{H} \equiv 0$) and the second is equivalent to his (8.12) under the conditions envisaged there. We refer to Wheatley for a discussion of the consequences of Eqs. (11.11)–(11.12) and a comparison with experiment.

It is instructive to rewrite the equations of motion in terms of the phase difference $\Delta\varphi = \pi - 2\theta$ of the "up" and "down" spin components of the Cooper pair wave function [cf. Eq. (11.7)]. We have

$$dS_z/dt = -\tfrac{1}{2}\lambda\sin\Delta\varphi \qquad (11.13)$$

$$d(\Delta\varphi)/dt = 2\gamma[H_z(t) - \gamma S_z/\chi]. \qquad (11.14)$$

One can interpret these quations as describing a sort of Josephson effect between the particles in the up and down spin bands. (See Leggett, 1974a; Maki and Tsuneto, 1974a.) Just as in the Josephson case we have two bulk systems (the bulk superconductors) which to a first approximation are noninteracting, so here we have two systems (the up- and down-spin bands) which are completely noninteracting to the extent that spin is conserved. The weak dipole interaction however drives Cooper pairs between the up- and down-spin bands, just as the weak tunneling processes drive them, in the Josephson case, across the junction between the bulk superconductors. Equation (11.13) corresponds to the Josephson equation for the current across the barrier (i.e., the rate of change of the *difference* in the number of particles on the two sides). As to Eq. (11.14) let us suppose that we start with $S_z = H_z = 0$ and suddenly apply a field H_0. Then the rate of change of $\Delta\varphi$ immediately subsequently is given by

$$d(\Delta\varphi)/dt = 2\gamma H_0 \qquad (11.15)$$

which is precisely the analog of the Josephson phase equation $d(\Delta\varphi)/dt = 2eV/\hbar$, since just as $2eV$ is twice the difference of chemical potential across the Josephson junction so $2\gamma\hbar H_0$ is twice the difference $\Delta\mu$ of chemical potential between the up- and down-spin bands. Indeed, more generally this difference is $\gamma\hbar(H - \gamma S_z/\chi)$, so that quite generally Eq. (11.14) can be written

$$d(\Delta\varphi)/dt = -2(\Delta\mu/\hbar). \qquad (11.16)$$

It is interesting to notice, however, that the second term on the right-hand side of (11.14) has no direct analog in the Josephson situation. It corresponds to the fact that when pairs flow from the down band into the up band, they push up the chemical potential in the latter. The corresponding term in the Josephson situation is inversely proportional to the relative capacitance of the two bulk superconductors and hence is usually negligibly small; however, in this case there are of course capacitance and screening effects associated with the junction itself, which give rise among other things to the "Josephson plasma resonance" (see, e.g., Josephson, 1969). In this respect and others, an important difference between the Josephson effect and the longitudinal resonance in ³He is that in the former case we have a real "geographical" transport of particles across a physical barrier, and the parameters entering the equations depend, *inter alia*, on the detailed properties of the junction itself (size, geometry, etc.). On the other hand, in ^{3}He, the "transport" is only in spin space (between two "interpenetrating" superfluids) and all parameters entering (λ, χ, etc.) depend only on the thermodynamic state of the bulk system.

It is worth remarking, incidentally, that experimental observation of the prediction that if a field ΔH much larger than $(\lambda/\chi)^{1/2}$ is suddenly turned off, then the resultant ringing of the magnetization is at a frequency $2\gamma\Delta H$ [Maki and Tsuneto 1974a; see Wheatley's discussion of his Eq. (8.32)] would constitute a rather satisfying direct test of the assumption that superfluidity in ^{3}He is a consequence of the formation of Cooper *pairs* (and not, say, quadruples, which would give a ringing frequency $4\gamma\Delta H$).

D. General equations of spin dynamics

Having explored the physical nature of the anomalous magnetic resonance behavior in ^{3}He in one particularly simple case, let us now go on to consider the general case. The ensuing discussion follows fairly closely that given in Leggett (1974a)[34] and it may therefore be helpful to comment on the relation between the notations used here and in that paper. In both cases we denote by $\mathbf{S}$ the actual total spin angular moment of the system, i.e., it has the dimension of $\hbar$. In the (1974a) paper, a vector operator $\mathbf{T}(\mathbf{n})$ is defined by Eqs. (2.2) and (2.18) of that paper, i.e.,

$$T_i(\mathbf{n}) \equiv (2\pi\hbar)^{-2} \int d^3\mathbf{k}'\delta(\mathbf{n}-\mathbf{n}')T_i(\mathbf{k}') \equiv \tfrac{1}{2}\hbar \sum_{|k|} T_i(\mathbf{k}), \tag{11.17}$$

$$T_i(\mathbf{k}) \equiv \sum_{\alpha\beta} a_{-\mathbf{k}\alpha}(\sigma_2\sigma_i)_{\alpha\beta}a_{\mathbf{k}\beta}, \tag{11.18}$$

where (as in Eq. (7.31)) the σ_i are Pauli matrices. On the other hand, in the present paper we have defined, in Eqs. (7.22) and (7.18),

$$F_{\mathbf{k}\alpha\beta} \equiv \langle a_{-\mathbf{k}\alpha}a_{\mathbf{k}\beta}\rangle, \tag{11.19}$$

$$\Psi_{\alpha\beta}(\mathbf{n}) \equiv \sum_{|k|} F_{\mathbf{k}\alpha\beta}, \tag{11.20}$$

and by inverting (7.35) [cf. Eqs. (7.31)–(7.32)] we have

$$d_i(\mathbf{n}) = -\tfrac{1}{2}i\Psi^{-1}\sum_{\alpha\beta}(\sigma_2\sigma_i)_{\alpha\beta}\Psi_{\alpha\beta}(\mathbf{n}). \tag{11.21}$$

Consequently, $\mathbf{d}(\mathbf{n})$ as so far defined is given in terms of $\mathbf{T}(\mathbf{n})$ by the relation

$$\mathbf{d}(\mathbf{n}) = -(i/\hbar)\Psi^{-1}\langle\mathbf{T}(\mathbf{n})\rangle. \tag{11.22}$$

The correct expression for the matrix $\Psi_{\alpha\beta}(\mathbf{n})$ is[35]

$$\Psi_{\alpha\beta}(\mathbf{n}) = \Psi\begin{pmatrix} -d_x+id_y & d_z \\ d_z & d_x+id_y \end{pmatrix} = \hbar^{-1}\begin{pmatrix} T_y+iT_x & -iT_z \\ -iT_z & T_y-iT_x \end{pmatrix}. \tag{11.23}$$

The dipole energy is given in the (1974a) paper in the form [Eqs. (4.4)–(4.5)]

$$H_D = \tfrac{2}{3}\pi\gamma^2 \int (d\Omega/4\pi) \int (d\Omega'/4\pi)\{\mathbf{T}^\dagger(\mathbf{n})\cdot\mathbf{T}(\mathbf{n}) - 3\hat{\mathbf{q}}\cdot\mathbf{T}^\dagger(\mathbf{n})\hat{\mathbf{q}}\cdot\mathbf{T}(\mathbf{n}')\} \qquad \hat{\mathbf{q}} \equiv (\mathbf{n}-\mathbf{n}')/|\mathbf{n}-\mathbf{n}'|. \tag{11.24}$$

Since the $a_{k\alpha}$'s in Eqs. (11.18) and (11.19) are actually quasiparticle operators, it is necessary to multiply the integrand of (11.24) by some renormalization factor $R^2(|\mathbf{n}-\mathbf{n}'|)$ [see Sec. VII of (1974a)]. If we assume a pure P-wave state, then it is shown in the Appendix of (1974a) that Eq. (11.24) including the renormalization factor can be rewritten in the simpler form (where we trivially complex conjugate a real quantity)

$$H_D = \tfrac{1}{2}\pi\gamma^2\langle R^2\rangle_{\text{av}} \int (d\Omega/4\pi)\{3|\mathbf{n}\cdot\mathbf{T}(\mathbf{n})|^2 - |\mathbf{T}(\mathbf{n})|^2\}. \tag{11.25}$$

Using Eq. (11.22) we see that this agrees with the expression (10.7) of the present paper. However, since it is precisely the NMR behavior which has provided the most solid evidence in favor of the P-wave hypothesis, we shall use for the present the more general expression (11.24) in terms of $\mathbf{d}(\mathbf{n})$:

$$H_D = \tfrac{2}{3}\pi\gamma^2\hbar^2\Psi^2(T) \int (d\Omega/4\pi) \int (d\Omega'/4\pi)\{\mathbf{d}^*(\mathbf{n})\cdot\mathbf{d}(\mathbf{n}') - 3\hat{\mathbf{q}}\cdot\mathbf{d}^*(\mathbf{n})\hat{\mathbf{q}}\cdot\mathbf{d}(\mathbf{n}')\}. \tag{11.26}$$

The total spin operator $\mathbf{S}$ has the explicit representation

$$S_i = \tfrac{1}{2}\hbar \sum_{\mathbf{k}\alpha\beta} a_{\mathbf{k}\alpha}{}^+\sigma_{\alpha\beta}{}^{(i)}a_{\mathbf{k}\beta} \tag{11.27}$$

(where $\sigma_{\alpha\beta}{}^{(i)}$ are matrix elements of the Pauli matrices σ). As in (1974a), we find the commutation relations of $\mathbf{S}$ and $\mathbf{T}(\mathbf{n})$ directly from the definitions (11.17)–(11.18) and the standard Fermi commutation relations obeyed by the $a_{\mathbf{k}\alpha}$'s:

$$[S_i, T_j(\mathbf{n})] = i\hbar\epsilon_{ijk}T_k(\mathbf{n}) \tag{11.28a}$$

$$[S_i, T_j{}^\dagger(\mathbf{n})] = i\hbar\epsilon_{ijk}T_k{}^\dagger(\mathbf{n}), \tag{11.28b}$$

where ϵ_{ijk} is the standard Levi-Civita symbol. Also we have as usual

$$[S_i, S_j] = i\hbar\epsilon_{ijk}S_k. \tag{11.29}$$

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[34] Throughout this subsection and the next, this paper is generally referred to simply as "1974a."

[35] Equation (2.13) of the (1974a) paper contains some sign errors. This does not affect any of the subsequent results. Note also that in (1974a), the notation $\mathbf{d}(\mathbf{n})$ is used for the value of $\langle\mathbf{T}(\mathbf{n})\rangle$ inequilibrium.

It is also possible to calculate the commutation relations of the T_i among themselves [see Eq. (2.26) of (1974a)]. However these are not actually needed for purposes of NMR calculations [cf. the discussion on p. 27 of (1974a)].

Now we have so far been regarding $\mathbf{d}(\mathbf{n})$ as an expectation value. However, to obtain its equations of motion we must treat it as a quantum-mechanical operator, i.e., take away the brackets in Eq. (11.22). Since $\mathbf{d}(\mathbf{n})$ is a *macroscopic* quantity, once we have got the equations of motion we can treat it as a classical variable. (We could, of course, equally well proceed by calculating the classical Poisson bracket of $\mathbf{S}$ and $\mathbf{d}$—see last subsection.) If then we treat $\mathbf{d}(\mathbf{n})$ as an operator, we obtain from Eq. (11.28)

$$[S_i, d_j(\mathbf{n})] = i\hbar\epsilon_{ijk}d_k(\mathbf{n}) \qquad (11.30\text{a})$$

$$[S_i, d_j^\dagger(\mathbf{n})] = i\hbar\epsilon_{ijk}d_k^\dagger(\mathbf{n}). \qquad (11.30\text{b})$$

From (11.30) it follows immediately that $\mathbf{S}$ commutes with $\mathbf{d}^\dagger(\mathbf{n})\cdot\mathbf{d}(\mathbf{n}')$ and therefore any term proportional to this will have no effect on the motion (in the present approximation, in which we neglect the terms arising from the communication of the d_i's among themselves—cf. above). We can also verify that for the case $\mathbf{d}(\mathbf{n}) = \mathbf{d}f(\mathbf{n})$ Eqs. (11.30) with (11.6) give back (11.8).

In view of the importance of Eqs. (11.30), it is worth pointing out that they can be obtained quite directly, without explicit use of the second-quantized expression for $\mathbf{T}(\mathbf{n})$, simply from the fact that $\mathbf{d}(\mathbf{n})$ is a vector in spin space and must therefore transform as such under rotation of spin coordinates. Since it is precisely the total spin operator $\mathbf{S}$ which generates such rotations ($\psi \to \psi' = \exp(i\delta\boldsymbol{\omega}\cdot\mathbf{S}/\hbar)\psi$), when the rotation is described by a small vector $\delta\boldsymbol{\omega}$ we must have the relation between the new value $\mathbf{d}'(\mathbf{n})$ and the old one $\mathbf{d}(\mathbf{n})$

$$\mathbf{d}'(\mathbf{n}) = \exp(i\delta\boldsymbol{\omega}\cdot\mathbf{S}/\hbar)\mathbf{d}(\mathbf{n})\exp(-i\delta\boldsymbol{\omega}\cdot\mathbf{S}/\hbar) \qquad (11.31)$$

so that the change $\delta\mathbf{d}(\mathbf{n}) \equiv \mathbf{d}'(\mathbf{n}) - \mathbf{d}(\mathbf{n})$ is given (for $\delta\boldsymbol{\omega} \to 0$) by expanding this expression in $\delta\boldsymbol{\omega}$:

$$\delta\mathbf{d}(\mathbf{n}) = i\hbar^{-1}[\delta\boldsymbol{\omega}\cdot\mathbf{S}\mathbf{d}(\mathbf{n}) - \mathbf{d}(\mathbf{n})\delta\boldsymbol{\omega}\cdot\mathbf{S}] \qquad (11.32)$$

On the other hand, we also obviously have by the usual formulas for transformation of vectors under rotation

$$\delta\mathbf{d}(\mathbf{n}) = \delta\boldsymbol{\omega} \times \mathbf{d}(\mathbf{n}). \qquad (11.33)$$

The expressions (11.32) and (11.33) are compatible only if (11.30a) holds; (11.30b) then follows simply by Hermitian conjugation.

The adiabatic Hamiltonian is given by Eq. (11.4)

$$H\{\mathbf{S}, \mathbf{d}(\mathbf{n})\} = \tfrac{1}{2}\gamma^2\chi^{-1}S^2 - \gamma\mathbf{S}\cdot\mathbf{H}(t) + H_D \qquad (11.34)$$

where we want to keep for the moment the general expression (11.26) for H_D. Applying the commutation relations (11.29) and (11.30), we obtain the general equations of motion[36]:

$$\dot{\mathbf{S}} = \gamma\mathbf{S} \times \mathbf{H}(t) + \mathbf{R}_D \qquad (11.35)$$

$$\dot{\mathbf{d}}(\mathbf{n}) = \mathbf{d}(\mathbf{n}) \times \gamma\{\mathbf{H}(t) - \gamma\mathbf{S}/\chi\} \qquad (11.36)$$

where the "dipole torque" $\mathbf{R}_D$ is given by

$$\mathbf{R}_D \equiv -\int (d\Omega/4\pi)\{\mathbf{d}(\mathbf{n}) \times [\delta H_D/\delta\mathbf{d}(\mathbf{n})] + \text{c.c.}\}. \qquad (11.37)$$

Equations (11.35)–(11.37) have been derived under the assumption that the adiabatic approximation is a good description and that the effects of susceptibility anisotropy, if any, can be ignored. [For a more general set of equations which take the susceptibility anisotropy into account and a discussion of when it can be neglected, see Leggett (1974a), Sec. 6. It is to be emphasized that it can *always* be neglected in the limit $T \to T_c$.] To the extent that these assumptions are valid, Eqs. (11.35)–(11.37) should afford a description of the spin dynamics, linear or nonlinear, of any triplet superfluid, whatever the nature of the pair wave function. In the next section we shall consider some specific applications.

E. Some applications

Once we have Eqs. (11.35)–(11.37), their application to specific cases is essentially a matter of mathematics, so that in accordance with the spirit of this review I shall not go into it in much detail. In fact, since the application to the ABM and BW states has been discussed fairly extensively by Wheatley, I shall mainly concentrate on examining the extent to which the NMR evidence can be used to support these identifications for ³He-A and B, respectively.

Let us first consider the case of unsaturated c.w. resonance under the condition that the dipole energy is a minimum in the equilibrium configuration. (This is not necessarily always true, since as we saw in Sec. X there may be other, competing orientational effects; however it should certainly be true in the limit of "bulk" samples.) In that case the solution of Eqs. (11.35)–(11.37) can be carried out in a straightforward way by linearizing around the equilibrium configuration; the requisite algebra is given in (1974a), Sec. 5, but it is instructive for present purposes to derive the result in a slightly different way. We consider first the motion in zero magnetic field and write down the equations of motion (11.35)–(11.36) in quantum mechanical form:

$$\dot{S}_i = (i\hbar)^{-1}[S_i, H_D] \equiv R_{Di}, \qquad (11.38)$$

$$\dot{R}_{Di} = (i\hbar)^{-1}[R_{Di}, E(S)] = (i\hbar)^{-1}[R_{Di}, S_j]\gamma^2\chi^{-1}S_j, \qquad (11.39)$$

with summation over repeated indices implied and $E(S)$

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[36] Note that $\mathbf{H}(t)$ here is the total external field (including any rf contribution), in contrast to the notation of (1974a) but in agreement with that of Wheatley. Equation (4.14) of (1974a) has a sign wrong in the first term on the right-hand side (but subsequent formulae have the correct sign).

defined by (11.3). Combining these we find

$$\ddot{S}_i = -(\hbar^2)^{-1}[[S_i, H_D], S_j]\gamma^2\chi^{-1}S_j. \tag{11.40}$$

This equation is generally true. For small oscillations we can linearize it by taking the double commutator equal to its value in the equilibrium configuration, which we denote by the subscript "eq." In this way we obtain

$$\ddot{S}_i = -\hat{\Omega}_{ij}{}^2 S_j \tag{11.41}$$

where the elements of the tensor $\hat{\Omega}_{ij}{}^2$ are given by

$$\hat{\Omega}_{ij}{}^2 \equiv \gamma^2\chi^{-1}\hbar^{-2}[[S_i, H_D], S_j]_{\mathrm{eq}}. \tag{11.42}$$

This tensor has a simple physical interpretation: as we noted above, the operator $\exp(i\delta\boldsymbol{\omega}\cdot\hat{S}/\hbar)$ generates an infinitesimal rotation $\delta\boldsymbol{\omega}$ of the spin coordinates, and therefore the change in the dipole energy under such a rotation is

$$\begin{aligned}\delta H_D &\equiv H_D' - H_D \\ &= \exp(i\delta\boldsymbol{\omega}\cdot\mathbf{S}/\hbar)H_D\exp(-i\delta\boldsymbol{\omega}\cdot\mathbf{S}/\hbar) - H_D.\end{aligned} \tag{11.43}$$

If the dipole energy is to be a minimum in the equilibrium configuration (though not otherwise) the term linear in $\delta\boldsymbol{\omega}$ must vanish: the terms quadratic in $\delta\boldsymbol{\omega}$ give

$$\begin{aligned}\delta H_D &= \hbar^{-2}\{\delta\boldsymbol{\omega}\cdot\mathbf{S}H_D\delta\boldsymbol{\omega}\cdot\mathbf{S} - \tfrac{1}{2}(\delta\boldsymbol{\omega}\cdot\mathbf{S})^2H_D - \tfrac{1}{2}H_D(\delta\boldsymbol{\omega}\cdot\mathbf{S})^2\} \\ &= \tfrac{1}{2}\hbar^{-2}[[S_i, H_D], S_j]_{\mathrm{eq}}\delta\omega_i\delta\omega_j.\end{aligned} \tag{11.44}$$

Comparing this with Eq. (11.42) we see that we have

$$\hat{\Omega}_{ij}{}^2 = \gamma^2\chi^{-1}(\partial^2 H_D/\partial\omega_i\partial\omega_j), \tag{11.45}$$

where the second derivative means the change in H_D under an infinitesimal rotation $\partial\boldsymbol{\omega}$. ($\delta\omega_z$ is of course equivalent to $\delta\theta$, where θ is the angle used in Sec. XI.C.) The explicit expression for $\hat{\Omega}_{ij}{}^2$ is given in (1974a) and is somewhat cumbrous: in the most general case it is (in the present notation)

$$\begin{aligned}\hat{\Omega}_{ij}{}^2 = {}& 4\pi\gamma^4\chi^{-1}\Psi^2(T)\,\mathrm{Re}\int(d\Omega/4\pi)\int(d\Omega'/4\pi) \\ &\times[R^2(\mathbf{n}-\mathbf{n}')\{\hat{\mathbf{q}}\cdot\mathbf{d}^*(\mathbf{n})\hat{\mathbf{q}}\cdot\mathbf{d}(\mathbf{n}')\delta_{ij} - \hat{\mathbf{q}}\cdot\mathbf{d}^*(\mathbf{n})\hat{q}_j d_i(\mathbf{n}') \\ &-(\hat{\mathbf{q}}\times\mathbf{d}^*(\mathbf{n}))_i(\hat{\mathbf{q}}\times\mathbf{d}(\mathbf{n}'))_j\}]_{\mathrm{eq}}.\end{aligned} \tag{11.46}$$

However, for many purposes it is more convenient to represent $\boldsymbol{\Omega}$ in the form (11.45). We see that it represents a sort of matrix of restoring forces for $\mathbf{S}$. In fact, the solution of (11.40) in the general case gives three *linearly* polarized oscillations with frequencies given by the eigenvalues of the tensor $\boldsymbol{\Omega}$; the problem is in fact completely equivalent to that of a three-dimensional anisotropic harmonic oscillator.

Now consider what happens when we apply an external field $\mathbf{H}_0$. If the field and the dipole force dominate the orientation, as we have assumed, then one principal axis of $\boldsymbol{\Omega}$ can be chosen to lie along the field direction (say the z direction). We then find that the oscillation along this axis (i.e., the longitudinal or parallel resonance) has a frequency $\omega^2 = \Omega_z{}^2(\equiv\Omega_{zz}{}^2)$ which is independent of $\mathbf{H}_0$. However, the two oscillations in the xy plane are mixed by the precession in the field: in fact the equation of motion of the deviation of the spin from its equilibrium value, $\mathbf{S}'$, can be found by adding to Eq. (11.41) the standard term arising from precession:

$$\dot{\mathbf{S}}' + \boldsymbol{\omega}_L \times \dot{\mathbf{S}}' = -\hat{\Omega}^2\mathbf{S}' \qquad (\boldsymbol{\omega}_L \equiv \gamma\mathbf{H}_0). \tag{11.47}$$

Solution of this equation leads to the longitudinal oscillation, as above, and two elliptically polarized transverse (perpendicular) oscillations with frequencies given by

$$\begin{aligned}\omega^2 = {}& \tfrac{1}{2}[(\omega_L{}^2 + \Omega_x{}^2 + \Omega_y{}^2) \pm \{(\omega_L{}^2 + \Omega_x{}^2 + \Omega_y{}^2)^2 \\ &- 4\Omega_x{}^2\Omega_y{}^2\}^{1/2}],\end{aligned} \tag{11.48}$$

where $\Omega_x{}^2$, $\Omega_y{}^2$ are the eigenvalues of $\boldsymbol{\Omega}$ in the xy plane. If and only if one of these eigenvalues, say $\Omega_x{}^2$, is zero, then one of the oscillations has zero frequency and the other a frequency which for *all* values of H_0 is given by

$$\omega^2 = \gamma^2 H_0{}^2 + \Omega_y{}^2(T). \tag{11.49}$$

For details of the spectral weight of the resonances, etc., see (1974a). Notice carefully that even when a mode has zero frequency, it may absorb some of the spectral weight in the Kramers–Kronig relation, so that the observed weight in the finite-frequency mode, Eq. (11.49), may give an apparent susceptibility *less* than that measured by static methods (never, under the conditions we are at present considering, more—but see below).

The above results enable us to put some fairly tight constraints on the nature of ³He-A and B in the light of the observed NMR c.w. behavior alone. Consider first the observations in ³He-A. It is now fairly well established (see Wheatley) that Eq. (11.49) holds at least down to very low fields. This implies that one of the eigenvalues of $\boldsymbol{\Omega}^2$ in the xy plane is zero; and this in turn implies, via (11.45), that the dipole energy is invariant under (small) rotations of the spin coordinates, i.e., of the vector $\mathbf{d}(\mathbf{n})$, around a certain axis in the xy plane. Moreover, the experimentally very well established fact that the shift in the square of the transverse frequency [i.e., $\Omega_y{}^2(T)$] is numerically equal to the square of the longitudinal resonance frequency $\Omega_z{}^2(T)$ implies that small rotations around the y and z axes change the dipole energy in the same way. Clearly these constraints are satisfied by any state for which $\mathbf{d}(\mathbf{n})$ is of the general form

$$\mathbf{d}(\mathbf{n}) = \hat{\mathbf{x}}f(\mathbf{n}), \tag{11.50}$$

where $f(\mathbf{n})$ is an arbitrary function such that $|f(\mathbf{n})|^2$ is invariant under rotation around the x axis [and of course such that the dipole energy is actually a minimum rather than a maximum in the state (11.50)]. The simplest case of (11.50) is of course the ABM state (9.52): let us call the general state of the form (11.50) "ABM-like."[37] (It is conceivable, but not very probable, that there are also non-ABM-like states which satisfy the above constraints.) The experimental evidence on ³He-A, therefore, strongly

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[37] A more precise definition of an "ABM(-like)" state has been given by Takagi (1974b).

suggests that it is ABM-like, though not necessarily ABM (see also below).

Turning now to the data on ^{3}He-B (see Wheatley), we recall that no substantial transverse shift is observed in bulk (for the small shifts which are observed, see below) while there is a definite though somewhat broad longitudinal resonance, at least on the melting curve and close to T_c. This implies that $\Omega_x^2 = \Omega_y^2 = 0$, but that Ω_z^2 is finite. In other words the dipole energy is changed by a small rotation around the z axis but not by small rotations around the x or y axes. This would be expected if (as is the case for the BW phase, see Sec. X.A) the dipole energy in zero field is minimized by a rotation of given angle around an arbitrary axis $\hat{\omega}$, and a field tends to orient this axis parallel to itself: a further rotation around the z axis then shifts the angle of rotation θ from the "magic" value $\cos^{-1}(-\frac{1}{4})$ and so costs appreciable dipole energy, while a rotation of $\mathbf{d}(\mathbf{n})$ around the x or y axis is equivalent (to lowest order only) to reorientation of $\hat{\omega}$ without a change in θ, and therefore costs only the very small orienting energy (10.16). Consequently Ω_x^2 and Ω_y^2, and hence the transverse resonance shift, though not strictly zero, would be extremely small and proportional to H_0^2.[38] Thus, the experimental c.w. resonance data on ^{3}He-B are consistent with the hypothesis that it is a BW phase. However, there do exist other states which are not of the BW type which also have the property $\Omega_x^2 = \Omega_y^2 = 0$, $\Omega_z^2 \neq 0$ (see Barton and Moore, 1974a,b; Moore *et al.*, 1974), so that the B-phase c.w. data *alone* are perhaps not quite unambiguous evidence of identification.

If we assume that ^{3}He-A and B are indeed, respectively, the ABM and BW phases, we can easily obtain the ratio of the longitudinal frequencies in the two phases by comparing the way in which the dipole energy varies under a rotation around the z axis in the two cases [cf. Eq. (11.45)]. From Eqs. (10.11) and (10.14) we find

$$(\partial^2 H_D/\partial\omega_z^2)_{\mathrm{ABM}} = \tfrac{6}{5} g_D^{\mathrm{ABM}}(T), \tag{11.51}$$

$$(\partial^2 H_D/\partial\omega_z^2)_{\mathrm{BW}} = \tfrac{4}{5} g_D^{\mathrm{BW}}(T)(\partial^2/\partial\theta^2)\{\cos\theta + 2\cos^2\theta\}_{\cos\theta=-1/4} = 3 g_D^{\mathrm{BW}}(T), \tag{11.52}$$

where it should be remembered that according to (10.8) $g_D(T)$ is proportional to $\Psi^2(T)$ for the two phases. From Eqs. (11.51)–(11.52), (11.45), and (10.8) we get the ratio

$$(\Omega_{\mathrm{BW}}/\Omega_{\mathrm{ABM}})^2 = \tfrac{5}{2}(\chi_{\mathrm{ABM}}/\chi_{\mathrm{BW}})(\Psi_{\mathrm{BW}}^2/\Psi_{\mathrm{ABM}}^2). \tag{11.53}$$

In the GL region, the ratio of Ψ^2 can of course equally well be replaced by the ratio of Δ^2. As discussed by Wheatley, the ratio of the longitudinal resonance frequencies observed at the A–B transition at the melting curve agrees well with (11.53) if we assume $\Psi_{\mathrm{ABM}} = \Psi_{\mathrm{BW}}$ and use the experimental melting-curve values of the susceptibilities. However, near the PCP, Ψ^2 or Δ^2 is proportional to κ^{-1} [Eq. (9.28)] and κ should be the same for the two phases at the equilibrium transition (see Sec. IX.B); moreover near T_c we have $\chi_{\mathrm{BW}} \approx \chi_{\mathrm{ABM}} = \chi_n$. Hence the theory predicts quite unambiguously a value of $(5/2)^{1/2}$ for the ratio of the longitudinal resonance frequencies in the limit as we approach the PCP. The substantial discrepancy of the experimentally observed ratio from this value (see Wheatley) is perhaps the most serious objection, at present, to the identification of ^{3}He-B with the BW phase.

Let us now turn briefly to the nonlinear behavior. The nonlinear longitudinal resonance behavior in the ABM and BW phases has been studied theoretically by Maki and Tsuneto (1974a) and by Maki (1974a), respectively. We have already discussed the ^{3}He-A longitudinal resonance in Subsection C, above; here we just note that a unified treatment of both phases (and indeed a quite general phase) can be obtained if we write generally

$$H(S_z, \theta) = \tfrac{1}{2}\gamma^2\chi^{-1}S_z^2 - \gamma S_z H_z(t) - H_D(\theta) \tag{11.54}$$

$$[S_z, \theta] = -i\hbar, \tag{11.55}$$

where $H_D(\theta)$ is the dipole energy as a function of angle of rotation around the z axis, and is given explicitly for the ABM and BW phases by Eqs. (11.9) and (10.14), respectively, i.e.,

$$H_D^{(\mathrm{ABM})} = -\tfrac{3}{5} g_D(T)\cos^2\theta \tag{11.56}$$

$$H_D^{(\mathrm{BW})} = \tfrac{4}{5} g_D(T)(\cos\theta + 2\cos^2\theta). \tag{11.57}$$

Equation (11.55) may be derived from the more general Eqs. (11.30) for an arbitrary state.

If we consider an experiment of the type described by Wheatley; in which a part ΔH of the external field is suddenly turned off and the subsequent "ringing" behavior observed, we can use the conservation of energy to determine the value(s) of ΔH at which the ringing frequency drops to zero. Evidently, the criterion for this to happen is that the extra spin energy created by the turning-off should be just sufficient to carry θ from the initial value to a value corresponding to a *maximum* of $H_D(\theta)$ (cf. Wheatley's discussion of the "pendulum" analogy). In other words, the critical value(s) ΔH_c satisfy

$$\tfrac{1}{2}\chi(\Delta H_c)^2 = H_D^{\max} - H_D^{\min}. \tag{11.58}$$

Using Eqs. (11.58), (11.51), and (11.52), one may relate $\gamma\Delta H$ to the value of the linear longitudinal resonance frequency. We find

$$(\gamma\Delta H_c/\Omega)_{\mathrm{ABM}} = 1, \qquad (\gamma\Delta H_c/\Omega)_{\mathrm{BW}} = (\tfrac{3}{5})^{1/2}, (\tfrac{5}{3})^{1/2}. \tag{11.59}$$

(The two values in the BW case correspond to the fact that the function (11.57) has *two* maxima of different height as a function of θ.) For a discussion of the experimental situation, see Wheatley, and for a more detailed discussion of the ringing behavior, Maki and Tsuneto (1974a) and Maki (1974a).[39]

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[38] An effect corresponding to this has been seen in ^{3}He-B: see Osheroff, 1974.

[39] It should be stressed, however, that no theory of the nonlinear longitudinal resonance which incorporates relaxation effects as yet exists. Consequently, the significance of discrepancies between theory and experiment in this area is not yet entirely clear.

More general studies of the nonlinear dynamics of the BW phase have been carried out by Maki and Hu (1974) and by Brinkman (1974). In particular, the latter has derived the very useful form of Eqs. (11.35)–(11.37) quoted by Wheatley (8.9)–(8.11). [These equations may be obtained by starting from the equation $\mathbf{d}(\mathbf{n}) = \hat{R}\mathbf{n}$, expressing the matrix $\hat{R}$ explicitly in terms of $\hat{\omega}$ and θ, differentiating the result with respect to time, and comparing with Eqs. (11.35)–(11.37); in doing the last it may be helpful to take dot and cross products with $\mathbf{n}$ and integrate over the Fermi surface.]

There are a number of other interesting applications of Eqs. (11.35)–(11.37) which there is no space to discuss in detail. Brinkman *et al.* (1974) carried out a detailed analysis of the c.w. transverse NMR in the B phase, assuming it to be BW: they used their theory of textures in the B phases (cf. Sec. X.D) from which it follows that the characteristic axis $\hat{\omega}$ is slightly disoriented from the external field axis and hence the results quoted above must be modified. In fact, one predicts a range of small *transverse* resonance shifts, in agreement with the experiments of Osheroff and Brinkman (1974). Osheroff and Anderson (1974) have applied the theory of the c.w. longitudinal and transverse resonance in ^{3}He-A near the A transition (see Sec. XIII and Wheatley Sec. IX) and find excellent agreement with their experimental results. [For more details of the theory, see Takagi (1974b).] Finally Takagi (1974c) has studied the c.w. NMR behavior in situations in which the equilibrium configuration does not minimize the dipole energy by itself (cf. Sec. X.D). He finds that the behavior of the ABM phase under these conditions is very anomalous, with the possibility of more than one transverse resonance line, negative shifts, and in some cases an integrated absorption which gives an apparent susceptibility *greater* than that measured in a static experiment. It is conceivable that these results may be relevant to some of the anomalous experimental results obtained in restricted geometrics, which is precisely the case in which the dipole energy is most likely not to be a minimum in equilibrium (cf. Sec. X.D).

I would like to close this subsection with a general remark. The fact that the NMR behavior, both linear and nonlinear, of the A phase is in very good agreement with that predicted for an ABM phase, and that there is also overall agreement, though with some exceptions, between the B-phase experiments and the predictions for the BW phase, is often taken as conclusive evidence for the corresponding identifications of ^{3}He-A and B. However, it should be pointed out that what the NMR experiments actually explore is *the dependence of the dipole energy on rotations of the spin coordinates*; that is, the most general quantity they can in principle measure is the quantity $H_D\{\hat{R}\mathbf{d}(\mathbf{n})\}$, where $\hat{R}$ is an arbitrary rotation. Consequently, if we can find other phases which show the same dependence of the dipole energy on rotation, then at least from the point of view of the NMR experiments alone they will be equally good candidates for ^{3}He-A and B. We have in fact already pointed out that the A-phase data *alone* prove at most that the A phase is "ABM-like," not that it is ABM. I return to this point in the Conclusion.

F. Relaxation and linewidths

In the theory developed in this section we have so far relied entirely on the adiabatic approximation described in Subsection B: as was pointed out there, this assumed that all variables except the "quasi-conserved" ones [$\mathbf{S}$ and the orientation of $\mathbf{d}(\mathbf{n})$] take their "instantaneous" equilibrium values, i.e., that the relaxation to the state which for given $\mathbf{S}$ and $\mathbf{d}(\mathbf{n})$ is the equilibrium state is essentially instantaneous. Such an assumption clearly cannot give any irreversibility or relaxation. If we make an analogy with the propagation of (ordinary hydrodynamic) sound in a normal system, we see that the adiabatic approximation is analogous to the assumption that the particle distribution at any point in the liquid has its equilibrium value for the given local values of density and fluid velocity. Such an assumption is valid only in the limit $\omega\tau \rightarrow 0$, where ω is the sound-wave frequency and τ the characteristic time for relaxation of the particle distribution to its local equilibrium value. If we allow now for a finite value of τ, we get a damping which is of order $\omega^2\tau$. Correspondingly, we would expect that in the present case the damping in low fields would be of order $\omega_D{}^2\tau$, where ω_D is a typical dipole frequency—e.g., the longitudinal resonance frequency. On the other hand, for $\omega_D \ll \gamma H_0 \equiv \omega_L$ (where H_0 is the external field) we would expect, as argued in Subsection B, that the damping of the transverse resonance is of order *at most* $(\omega_D/\omega_L)^2\omega_L\tau$.

It has so far not proved possible to work out the corrections to the adiabatic approximation which are necessary to take proper account of the damping (cf., however, below). However, a calculation of the c.w. resonance phenomena which does so has been developed within the framework of a microscopic approach by Combescot and Ebisawa (1974). They use a kinetic-equation method similar to that developed by Betbeder-Matibet and Nozières (1969) for superconductors, and take relaxation into account by assigning to the normal quasiparticles a phenomenological temperature-dependent relaxation time $\tau(T)$. The result of their calculation is that the ABM longitudinal resonance has a finite linewidth $\Delta\omega_{||}$ given by

$$\Delta\omega_{||}{}^{(\mathrm{ABM})} = [\Omega_{\mathrm{ABM}}{}^2(T)\tau(T)/(1+\tfrac{1}{4}Z_0)]f(T), \qquad (11.60)$$

where $\Omega_{\mathrm{ABM}}{}^2$ is the longitudinal resonance frequency squared (or equivalently the shift of the squared transverse frequency, since as we saw above these are identical for the ABM phase). The function $f(T)$ is given by

$$f(T) \equiv \int \frac{d\Omega}{4\pi} \int_0^\infty \frac{\beta}{2} \left(\frac{\epsilon_k}{E_\mathrm{k}}\right)^2 \operatorname{sech}^2\left(\frac{\beta E_\mathrm{k}}{2}\right) d\epsilon_\mathrm{k} \qquad (11.61)$$

[note that it differs from the Yosida function, Eq. (6.14), by the factor of $(\epsilon_k/E_\mathrm{k})^2$ in the integral]. Evidently $f(T)$ tends to 1 as $T \rightarrow T_c$, and falls off with decreasing T faster than the Yosida function. For the transverse resonance Combescot and Ebisawa found a width

$$\Delta\omega_{\perp}{}^{(\mathrm{ABM})} = [\Omega_{\mathrm{ABM}}{}^2(T)\tau(T)/(1+\tfrac{1}{4}Z_0)] \times \{\Omega_{\mathrm{ABM}}{}^2(T)/[\Omega_{\mathrm{ABM}}{}^2(T) + (\gamma H_0)^2]\}f(T), \qquad (11.62)$$

where the denominator in the second factor is just the observed shifted resonance frequency. The expression (11.62) reduces to (11.60) as $H_0 \rightarrow 0$, as it clearly should; we notice that the damping for large H_0 is actually of order

$\omega_D{}^4\tau/(\gamma H_0)^2$. Finally, they found for the BW longitudinal resonance

$$\Delta\omega_{||}{}^{(\mathrm{BW})} = \frac{\Omega_{\mathrm{BW}}{}^2(T)\tau(T)[2f(T) + Y(T)]}{(2 + Y(T))\cdot[1 + \frac{1}{4}Z_0(\frac{2}{3} + \frac{1}{3}Y(T))]}, \tag{11.63}$$

where $Y(T)$ is the Yosida function, Eq. (6.14). In comparing these results with the experimental data one must remember that apart from this "intrinsic" linewidth there may also be an "extrinsic" apparent linewidth due to texture effects which orient different parts of the sample differently, and therefore give a range of frequency shifts which will add up to look like a broad line (see in particular Brinkman *et al.*, 1974).

A second microscopic calculation of the linewidth in the ABM phase has been given by Maki and Ebisawa (1974) using Green's-function techniques. Although they also incorporate relaxation by putting in a phenomenological quasiparticle lifetime, the results they obtain differ from those above in that $\Omega_{\mathrm{ABM}}{}^2(T)$ in Eq. (11.60) and the first factor of (11.62) is replaced by a factor which is only weakly temperature dependent near T_c. Hence whereas for instance Eq. (11.60) predicts that $\Delta\omega_{||}{}^{\mathrm{ABM}}$ is proportional to $(T_c - T)$ near T_c, the Maki–Ebisawa calculation predicts that it is nearly temperature-independent. It is not clear why the two methods give such different results.

I would like to conclude by speculating briefly on how the results found by microscopic methods relate to the adiabatic approximation developed in the rest of this section. Let us consider for definiteness the A-phase longitudinal resonance. At first sight it is tempting to incorporate the damping phenomenologically by putting in a relaxation term in the equation of motion of S_x and/or θ, directly. However, I do not believe this is likely to be correct; the direct relaxation of S_z should be essentially given by the ordinary longitudinal relaxation time T_1, which is extremely long (of the order of minutes), while the direct relaxation of θ is forbidden because of the coherent nature of the superfluid state (see Leggett, 1974a, Sec. 4). Rather I believe one should visualize the relaxation mechanism as follows: Although the equation of motion (11.13) is an equation for the total spin S_z, which is carried by both Cooper pairs and normal quasiparticles, it is actually only the Cooper pairs which "tunnel" between the up and down spin bands. Now when the pairs tunnel, the number of pairs in (say) the up spin band is thrown out of equilibrium *relative to the number of up-spin normal quasiparticles*, and the "instanteous" dependence of the spin polarization energy $H(S_z)$ on S_z is not given simply by the terms written down in (11.10), that is by $\gamma^2\chi^{-1}S_z{}^2 - \gamma S_z H_z(t)$, but by a somewhat different expression. However, after some time τ the number of up spin pairs will have come into equilibrium with the number of up spin normal quasiparticles, and thereafter Eq. (11.10) is justified. It seems very probable, then, that the irreversibility and hence the damping arises in the process of equilibration of the pairs and the normal component within a single spin band. If this picture is correct, it should be possible to make it quantitative and to formulate thereby a theory of damping which is applicable to nonlinear as well as linear phenomena. This I hope to do elsewhere.[40]

XII. COLLECTIVE EXCITATIONS

A very substantial fraction of recent theoretical work on the new phases of ^{3}He has been devoted to the study of collective oscillations[41] (Maki and Ebisawa 1973, 1974a,c; Wölfle 1973a,b, 1974b; Combescot 1974a,b; Brinkman and Smith, 1974; Maki 1974a,b; Maki and Tsuneto 1974b; Serene 1973; Saslow 1973; Putterman 1973: cf. also Anderson, 1973; de Gennes, 1973a,b). Unfortunately, the situation with respect to comparison of theory and experiment seems at present much less happy here than in the case of the NMR behavior: broadly speaking, those phenomena which are easiest to observe experimentally (such as the ultrasound absorption near T_c) seem most difficult to calculate theoretically, and vice versa. In this section we shall give a fairly brief and qualitative discussion of this subject, concentrating on the physical description of the various possible modes and obtaining quantitative results only when it is particularly straightforward to do so with the help of the concepts already discussed. A very useful general survey of this topic, at least as regards the spin-independent oscillations, is given by Wölfle (1974b), on the basis of kinetic-equation techniques.

At first sight it is possible to classify collective oscillations of an anisotropic superfluid into (a) "density" type oscillations in which the Cooper pairs flow as a whole without changing their spin or orbital configuration, (b) "spin waves" in which the pairs change their spin configuration but not their orbital state, and (c) "orbital waves" in which the orbital configuration of the pairs varies but not their spin. On closer inspection this classification is less clear-cut than it seems, for several reasons: (b) and (c) are coupled not only by the dipole forces but, in the BW state at least, by the peculiar property of the latter state that there is no physical distinction between spin and orbital rotation (see Brinkman and Smith, 1974, and below), while (a) and (c) are strictly separable, if at all, only in the long-wavelength limit. Nevertheless, the above classification makes some sense as a first approximation. I shall devote the first subsection to (a) and the second to (b): oscillations of class; (c) will not be discussed explicitly here (cf. Introduction).

A. Zero, first, second, and fourth sound

In normal liquid ^{3}He two kinds of density wave are known to exist: ordinary ("first") sound, in which the liquid is always in local thermodynamic equilibrium, and the peculiar mode known as "zero sound" (Landau, 1957) which is a "collisionless" collective mode corresponding to a pole of the density-fluctuation propagator (2.37) (which is of course calculated neglecting collisions). If the characteristic quasiparticle collision time is τ, then hydrodynamic (first) sound exists for $\omega\tau \ll 1$ and collisionless (zero) sound for $\omega\tau \gg 1$: in the intermediate region ($\omega\tau \sim 1$) the

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[40] See A. J. Leggett and S. Takagi submitted to Phys. Rev. Letters.

[41] By "collective excitations" in this section we always mean collective excitations with finite wavelength (i.e., propagating waves) as distinct, say, from the uniform longitudinal magnetic resonance.

behavior is complicated and we get strong absorption. [For a detailed discussion, see, e.g., Abrikosov and Khalatnikov (1959), Pines and Nozières (1966) or Baym and Pethick (1974)]. For real liquid ^{3}He, especially at high pressures, the repulsive quasiparticle interactions are so strong that the velocity of zero sound (and in fact the quasiparticle distribution) differs only by less than one per cent from that of first sound [cf. Eq. (2.39)]. The "zero sound" referred to so far is what is often called the "longitudinal" variety, which is the only kind which corresponds to a density fluctuation. At high pressure other kinds of zero sound, corresponding to fluctuations of the transverse component of current, are predicted theoretically to exist (see Baym and Pethick, 1974) but have not to date been observed experimentally.

If we now turn to the isotropic superfluid liquid ^{4}He, there are apart from ordinary first sound a number of special wave modes which can propagate in the *hydrodynamic* regime. (Some of them may also have analogs in the collisionless regime, but the theoretical situation here is not entirely clear.) If we put the liquid in a narrow tube (of dimension small compared to the viscous pentration depth) then the normal component is clamped by the walls, but an oscillation of the density and pressure in which only the superfluid component moves is still possible: this is the so-called *fourth sound* and its velocity is given by

$$c_4^2 = (\rho_s/\rho)c_1^2, \tag{12.1}$$

where $c_{4,1}$ are, respectively, the velocities of fourth and first sound. Secondly, it is possible for the superfluid and normal components to flow relative to one another in such a way that there is no mass current,[42] i.e., for all $\mathbf{r}$ and t we have

$$\mathbf{P}(\mathbf{r}t) = \rho_s v_s(\mathbf{r},t) + \rho_n \mathbf{v}_n(\mathbf{r},t) = 0. \tag{12.2}$$

Such a mode involves no density or pressure variation but does involve temperature oscillations: it is of course the so-called *second sound*. Its velocity is given by

$$c_2^2 = (\rho_s/\rho_n)(TS^2/\rho c_v), \tag{12.3}$$

where S is the entropy per unit volume and c_v the specific heat. Finally we have *third sound*, a wave which propagates on the surface of the superfluid. [For an extended discussion of these various modes and a derivation of their velocities, see, e.g., Khalatnikov (1965).]

The spin-independent oscillations of the superfluid phases of liquid ^{3}He are expected to combine the complexities of behavior of normal ^{3}He and of superfluid ^{4}He, as well as exhibiting a number of phenomena which have no analog in either. Let us first examine the *hydrodynamic* region $\omega\tau \ll 1$ (note that this region gets smaller as the temperature falls below T_c, since the normal-state collision time is expected to get longer). We expect first of all ***first sound***, with a velocity determined in the standard way by the static compressibility K_s

$$c_1^2 = N/mK_s \quad (\equiv \rho/m^2K_s). \tag{12.4}$$

In the normal phase the compressibility is given from Landau's Fermi-liquid theory by

$$K_s = (dn/d\epsilon)(1+F_0)^{-1}. \tag{12.5}$$

[See, e.g., Pines and Nozières (1966); this is an exact formula which does not imply neglect of the higher F_l's.] In the superfluid phase the compressibility is unchanged, at least to order $(\Delta/\epsilon_F)^2$; the argument for this is analogous to the argument (see Sec. VII.A) showing that the experimental susceptibility of an ESP phase is unchanged. Then using Eqs. (2.5) and (2.28) we find that below T_c as well as above the first-sound velocity is given by

$$c_1^2 = \tfrac{1}{3}v_F^2(1+F_0)(1+\tfrac{1}{3}F_1). \tag{12.6}$$

Next let us consider fourth sound. Clearly in this case we cannot take over expression (12.1) directly, since the superfluid density ρ_s for an anisotropic superfluid is a tensor quantity. To discuss this mode quantitatively we start from the continuity equation:

$$\partial\rho/\partial t + \nabla\cdot\mathbf{P} = 0, \tag{12.7}$$

where $\mathbf{P}$ is the mass current (momentum) density. Under hydrodynamic conditions we have from (6.20) generalized to the anisotropic case [cf. (6.25)]:

$$P_\alpha = \rho_{\alpha\beta}{}^n v_\beta{}^{(n)} + \rho_{\alpha\beta}{}^s v_\beta{}^{(s)} = \rho_{\alpha\beta}{}^s v_\beta{}^{(s)}, \tag{12.8}$$

where in writing the second equality we took into account that the normal component is clamped by the walls ($\mathbf{v}^{(n)} = 0$), and where as always summation over repeated indices is implied. Combination of (12.7) and (12.8) gives in the limit of small oscillations ($\mathbf{v}^{(s)} \to 0$)

$$\partial\rho/\partial t + \rho_{\alpha\beta}{}^s(\partial v_\beta{}^{(s)}/\partial x_\alpha) = 0. \tag{12.9}$$

To proceed further we need the equation of motion of the superfluid velocity $\mathbf{v}^{(s)} \equiv \mathbf{v}_s$. For present purposes we will *assume* that as in superconductors and in liquid ^{4}He it is determined by the gradient of the chemical potential μ:

$$\partial\mathbf{v}_s/\partial t = -m^{-1}\nabla\mu. \tag{12.10}$$

(Physical arguments leading to this conclusion can be given along the lines indicated by Khalatnikov (1965) for ^{4}He.) Finally we use the relation, valid in local thermodynamic equilibrium

$$\nabla\mu = (\partial\mu/\partial\rho)\nabla\rho = (mK_s)^{-1}\nabla\rho. \tag{12.11}$$

Combining Eqs (12.9)–(12.11) and noting that $\rho/m^2K_s = c_1^2$ [Eq. (12.4)] we finally obtain

$$\partial^2\rho/\partial t^2 - c_1^2(\rho_{\alpha\beta}{}^s/\rho)(\partial^2\rho/\partial x_\alpha\partial x_\beta) = 0. \tag{12.12}$$

If we assume a wave propagating in the direction specified by unit vector $\hat{\mathbf{q}}$, then according to (12.12) its velocity is given by

$$c_4^2(\hat{\mathbf{q}}) = c_1^2(\rho_{\alpha\beta}{}^s/\rho)\hat{q}_\alpha\hat{q}_\beta. \tag{12.13}$$

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[42] This is true to the extent that the coefficient of thermal expansion is negligible—an approximation which is very good for ^{4}He (and should also be good for ^{3}He).

For the BW phase ρ_s is isotropic and we simply recover (12.1). For the ABM phase, on the other hand, the fourth-sound velocity is anisotropic, depending on the angle θ between $\hat{\mathbf{q}}$ and the pair angular momentum vector $\mathbf{l}$: in fact near T_c we have from (9.54)

$$c_4^2(\hat{q}) = c_1^2(\bar{\rho}_s/\rho)\tfrac{3}{5}(2-\cos^2\theta), \qquad (12.14)$$

where $\bar{\rho}_s \equiv \frac{1}{3}\,\mathrm{Tr}\rho_s$ is the "average" value of the superfluid density. For lower temperatures c_4 may be more dramatically anisotropic because of the strong Fermi-liquid corrections (cf. Sec. VI.B). Whether or not it will be possible to see this anisotropy experimentally is somewhat doubtful, since in the narrow tubes necessary to observe fourth sound the vector $\mathbf{l}$ will tend to be anchored perpendicular to the walls and the fourth sound must of course be propagated parallel to them; one would therefore tend to expect to observe in practice the value of c_4 corresponding to $\theta = \pi/2$. For a discussion of the experimental situation we refer to Wheatley.

Turning now to second sound, one might intuitively expect that in the same way as Eq. (12.1) for fourth sound was generalized to (12.13), the formula (12.3) would similarly be valid provided that ρ_s/ρ_n was replaced by $(\rho_s/\rho_n)_{\alpha\beta}\hat{q}_\alpha\hat{q}_\beta$. It turns out, interestingly enough, that this is true only in the limit $T \rightarrow T_c$; according to the calculations of Wölfle (1973b) the quantity S should also be replaced by a quantity which depends in general on the direction of $\hat{\mathbf{q}}$. The physical reason for this is apparently that the hydrodynamic equation for the entropy flow in an isotropic superfluid, namely,

$$\mathbf{j}_S = TS\mathbf{v}_n \qquad (12.15)$$

must itself be replaced, in the anisotropic case, by a tensor equation. We refer to Wölfle (1973b) for further details. [Evidently, however, one still expects Eq. (12.3) to be valid for the BW state.] The order of magnitude of c_2 should be given correctly by (12.3): we note that it is $\sim 10^{-2}v_F$. Second sound may also be quite highly damped (Takagi, 1974d) so that it may not be an easy matter to see it experimentally.

We now ask briefly what happens in the "collisionless" limit $\omega\tau \gg 1$? As we saw, in the normal case we get zero sound in this limit. On the other hand, at absolute zero we expect the superfluid to behave like an ideal elastic fluid at *all* frequencies much less than the "gap frequency" $\Delta/\hbar$, so that it presumably sustains a sound wave (the analog of the "Anderson–Bogoliubov" mode of an isotropic neutral superfluid) whose velocity is given by Eq. (12.4) and hence by (12.6). (This result is derived for the BW state by Czerwonko (1967).) One expects that there will be some kind of smooth transition, in the region $0 < T < T_c$, between these two types of limiting behaviour. Wölfle (1974), (1973b) has studied the question of sound propagation in the collisionless regime in detail and obtains results which appear to confirm this expectation: his calculated values of the sound velocity for the ABM state drop back from the zero sound value c_0 towards the first sound value c_1 as the temperature is lowered below T_c, in agreement with the experimental behavior (see Wheatley, Sec. X). The relevant calculations are somewhat complex and I shall not attempt to summarize them here (see also Ebisawa and Maki, 1974; Serene, 1973). Still less shall I try to describe the theoretical work on the absorption of ultrasound near T_c, a subject which is complex in the extreme and where a comparison of theory with experiment has so far been hindered not only by its intrinsic difficulty but by uncertainty as to the probable orientation of many experimental samples, (see the above references).

B. Spin waves

Surprisingly enough, the theoretical situation with regard to spin waves—a kind of excitation which has no direct analog in other known superfluids—is in many ways rather clearer than with respect to oscillations involving the density (or superfluid density). One reason for this is that in many cases (though not all) it is possible to assume that the "spin" and "orbital" degrees of freedom decouple to a first approximation. A second reason is that under suitable conditions we may essentially forget about the normal components, so that spin waves are the analog of fourth sound rather than of second, first or zero sound. To see this, we divide the liquid into four "components"—the up-spin Cooper pairs ($S\uparrow$), the down-spin pairs ($S\downarrow$), the up-spin normal component ($N\uparrow$), and the down-spin normal component ($N\downarrow$). The propagation of a spin wave will involve, locally, the flow of spin currents, that is, counterflow of $S\uparrow$ and $N\uparrow$ against $S\downarrow$ and $N\downarrow$. Now, the flow of $S\uparrow$ and $S\downarrow$ is metastable for reasons of the kind discussed in Sec. VI.B; but the flow of $N\uparrow$ and $N\downarrow$ is not, and there is no reason why collisions between these two components cannot bring them both to rest (since, if they are moving in counterflow, they possess no net momentum, there is no conservation law inhibiting this process). Consequently, after a time of order of the quasiparticle collision time τ, the up and down normal currents will have effectively annihilated one another; and if the period of the oscillation is long compared to $\tau(\omega\tau \ll 1)$ then the normal component is effectively at rest and can be ignored. This argument would not work for oscillations involving a net mass current (or heat current) since then $N\uparrow$ and $N\downarrow$ would be flowing parallel and the conservation of momentum prevents their being brought to rest by collisions between them. For the moment, therefore, we shall simply ignore the normal component, assuming that $\omega\tau \ll 1$; corrections to the formulas derived are discussed briefly below, as is the "collisionless" limit $\omega\tau \gg 1$. For a more detailed discussion of the topic of spin waves we refer to Combescot (1974a,b), Maki and Tsuneto (1974b), Maki (1974b), Brinkman and Smith (1974), and Maki and Ebisawa (1974): cf. also Vuorio (1974).

If for the moment we set the external field equal to zero and ignore the dipole forces, we can give a discussion of spin waves which almost exactly parallels the one given above for fourth sound. We have, first, the conservation law (7.61) (valid to the extent that the system conserves total spin):

$$\partial S_i/\partial t + (\partial/\partial x_\alpha)J_{i\alpha}{}^{\mathrm{sp}} = 0 \qquad (12.16)$$

where $J_{i\alpha}{}^{\mathrm{sp}}$ is the spin current dyadic (we recall that Roman subscripts label spin coordinates and Greek subscripts spatial ones). As in Sec. VII.D, we introduce the "spin superfluid velocity dyadic" $\Omega_{i\alpha}$ defined by Eq. (7.60), i.e.,

$$\partial d_i(\mathbf{n})/\partial x_\alpha = (2m/\hbar)\epsilon_{ijk}d_j(\mathbf{n})\Omega_{k\alpha}. \qquad (12.17)$$

According to the definition of the "spin superfluid density," Eq. (7.62), we have under the condition that the normal component is stationary the relation

$$J_{i\alpha}{}^{\mathrm{sp}} = (\hbar/2m)\rho_{ij:\alpha\beta}{}^{\mathrm{spin}}\Omega_{j\beta}. \tag{12.18}$$

In the argument that follows we consider the limit of *small* oscillations, i.e., $\mathbf{S} \to 0$. In this limit all space and time gradients of $\mathbf{d}$, ρ^{spin} etc., are themselves proportional to S; hence, whenever we get such gradients multiplying S itself the resultant term is proportional to S^2 and can be neglected by comparison with terms of order S. This consideration is used several times in the ensuing argument.

Substituting (12.18) in (12.16), we find in the limit of small oscillations

$$\partial S_i/\partial t + (\hbar/2m)\rho_{ij:\alpha\beta}{}^{\mathrm{spin}}(\partial\Omega_{j\beta}/\partial x_\alpha) = 0. \tag{12.19}$$

We must now find an equation of motion for $\mathbf{\Omega}$. To do this, we interpret Eq. (11.36) as a *local* equation, i.e. we put (since $\mathbf{H} \equiv 0$)

$$\partial\mathbf{d}(\mathbf{n}:\mathbf{r}t)/\partial t = -(\gamma^2/\chi)\mathbf{d}(\mathbf{n}:\mathbf{r}t) \times \mathbf{S}(\mathbf{r}t) \tag{12.20}$$

and taking the spatial derivative of this equation (in the limit of small oscillations) we have

$$(\partial/\partial t)(\partial d_i/\partial x_\alpha) = -(\gamma^2/\chi)\epsilon_{ijk}d_j(\partial S_k/\partial x_\alpha). \tag{12.21}$$

Comparing this with the time derivative of (12.17) (again in the small oscillation limit) we find

$$\partial\Omega_{k\alpha}/\partial t = -(\gamma^2\hbar/2m\chi)(\partial S_k/\partial x_\alpha) \tag{12.22}$$

and finally, from (12.19) and (12.22)

$$\partial^2 S_i/\partial t^2 - [\gamma^2\hbar^2/\chi(2m)^2]\rho_{ij:\alpha\beta}{}^{\mathrm{spin}}(\partial^2 S_j/\partial x_\alpha\partial x_\beta) = 0. \tag{12.23}$$

Using Eq. (2.26) for the normal-state susceptibility χ_n and also Eq. (2.5), we can rewrite (12.23) in the form

$$\partial^2 S_i/\partial t^2 - \tfrac{1}{3}v_F{}^2(m^*/m)(1+\tfrac{1}{4}Z_0)(\chi_n/\chi) \times (\rho_{ij:\alpha\beta}{}^{\mathrm{spin}}/\rho)(\partial^2 S_j/\partial x_\alpha\partial x_\beta) = 0. \tag{12.24}$$

From the form of (12.24) and the form of the spin superfluid density $\rho_{ij:\alpha\beta}{}^{\mathrm{spin}}$ [Eq. (7.66)] it is obvious that in general for spin waves propagating in a given direction $\hat{\mathbf{q}}$ there is more than one eigenfrequency, corresponding to different directions of the spin polarization. For the ABM phase we find, by using Eq. (9.56), that the principal axes are the same for all $\hat{\mathbf{q}}$, namely an axis along $\mathbf{d}$ and two arbitrary axes perpendicular to it: for the axis parallel to $\mathbf{d}$ we find $\omega \equiv 0$ and in fact, we can easily convince ourselves that no spin wave can propagate with this polarization. Spin waves polarized in any direction in the plane perpendicular to $\mathbf{d}$ are degenerate and have frequency

$$\omega^2 = \tfrac{1}{3}v_F{}^2(1+\tfrac{1}{4}Z_0)(\delta_{\alpha\beta} - Y_{\alpha\beta}(T))\hat{q}_\alpha\hat{q}_\beta, \tag{12.25}$$

where $Y_{\alpha\beta}(T)$ is defined as in Eq. (6.27), (but with $|\Delta(\mathbf{n})|^2$ of course replaced by $|\mathbf{\Delta}(\mathbf{n})|^2$ as usual for a triplet state). [Equations (12.24)–(12.25) need some correction if the Landau parameter Z_1 is different from zero.] Near T_c (only) we can obtain a simple relation between the spin wave velocity $c_s(\hat{\mathbf{q}})$ and the fourth sound velocity $c_4(\hat{\mathbf{q}})$: from (12.6), (12.13), (9.54) and the Landau effective-mass relation (2.42) we have

$$c_s{}^2(\hat{\mathbf{q}}) = [(1+\tfrac{1}{4}Z_0)/(1+F_0)]c_4{}^2(\hat{\mathbf{q}}). \tag{12.26}$$

[One can actually show that this relation is not affected by a finite Z_1: cf. the comment on (7.68).] On the other hand, for $T \to 0$, we can use the fact that $\rho_s = \rho$ to obtain Eq. (12.26) but with the right hand side multiplied by m/m^*. This factor comes in the last resort from the fact that the spin current carried by a quasiparticle is proportional to m/m^* times its mass current [cf. Eq. (2.15)]. The result (12.25) agrees with that of Combescot (1974b), Eq. (14); it agrees with that of Maki and Tsuneto (1974b) [Eq. (15), neglecting the dipole term $\Omega_l{}^2$] only in the limit $T \to T_c$ or if $F_1 = 0$, $m = m^*$.

For the BW phase we have, according to Eq. (9.51), the following expression for the spin superfluid density in a system of axes (i, α') in which the orbital coordinates have been rotated relative to the spin ones through the rotation $\hat{R}$ occurring in its definition, Eq. (9.46)

$$\rho_{ij,\alpha'\beta'}{}^{\mathrm{spin}}/\rho = (m/m^*)[1 - Y(T)][\delta_{ij}\delta_{\alpha'\beta'} - \tfrac{1}{5}(\delta_{ij}\delta_{\alpha'\beta'} + \delta_{i\alpha'}\delta_{j\beta'} + \delta_{i\beta'}\delta_{j\alpha'})]. \tag{12.27}$$

By inserting this form into Eq. (12.24) we see that the spin wave eigenmodes are polarized parallel and perpendicular to $\hat{\mathbf{q}}' \equiv \hat{R}\hat{\mathbf{q}}$. For the velocities $c_{s||}$, $c_{s\perp}$ we find

$$c_{s||}{}^2 = \tfrac{1}{2}c_{s\perp}{}^2 = \tfrac{2}{15}v_F{}^2(1+\tfrac{1}{4}Z_0)(\chi_n/\chi)[1 - Y(T)]. \tag{12.28}$$

This agrees with the results of Combescot (1974b) [his Eqs. (19) and (20), with $\omega_0 = 0$] and of Maki (1974b). It disagrees with that of Brinkman and Smith (1974) [their Eqs. (13) and (14)[43]] by a factor of 2 even near T_c, and also by factors of the order of m/m^* at lower temperatures. [cf. the comment on Eq. (10.43), above]. According to Eqs. (9.49), (9.48), (12.6), and (12.13) we have the relation, valid at all temperatures (if $Z_1, Z_2 = 0$)

$$c_{s||}{}^2/c_4{}^2 = c_{s\perp}{}^2/2c_4{}^2 = \frac{2}{5}\,\frac{\{1+\tfrac{1}{3}F_1Y(T)\}\cdot\{1+\tfrac{1}{4}Z_0[\tfrac{2}{3}+\tfrac{1}{3}Y(T)]\}}{(1+F_0)(1+\tfrac{1}{3}F_1)[\tfrac{2}{3}+\tfrac{1}{3}Y(T)]}. \tag{12.29}$$

It is interesting to note that these "spin waves" can equally well be regarded as "orbital waves" (Brinkman and Smith, 1974) since, for the BW state, it is a meaningless question whether the particles have changed their spin motion while leaving their orbital motion unchanged, or vice versa.

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[43] Note that their definition of ρ_s includes a factor of m^{-1} as compared to ours.

When we introduce an external field and/or the dipole forces, the situation becomes more complicated. It can be handled simply by adding to Eq. (12.24) for $\partial^2 S_i/\partial t^2$ the appropriate terms from (11.35). Evidently, in the ABM state, one spin wave mode must tend as $q \to 0$ to the longitudinal resonance frequency Ω_l. This mode has in fact always a polarization along the external field and the simple dispersion relation

$$\omega^2 = \Omega_l^2 + c_s^2(\hat{\mathbf{q}})q^2. \tag{12.30}$$

There is a single transverse mode whose frequency is given simply by adding $(\gamma H_0)^2$ to the right-hand side of (12.30) (Combescot, 1974b): thus, as q tends to zero its frequency tends to the shifted transverse resonance frequency, as we might expect. For the BW phase the complete disperson relation, including both dipole and external field terms, has been discussed by Brinkman and Smith (1974), although they point out that an approach based on the above lines may not be entirely realistic in a field since the normal component motion is no longer completely diffusive (i.e., because the number of normal up-spin and down-spin particles is not equal, a spin current will be associated with a mass current which will be conserved).

Finally we briefly consider the question of corrections due to the diffusive flow of the normal component. These can be handled in a phenomenological way by generalizing Eq. (12.18) to include a "normal" spin current $\mathbf{J}^{(n)}$

$$J_{i\alpha}{}^{\mathrm{sp}} = (\hbar/2m)\rho_{ij:\alpha\beta}{}^{\mathrm{spin}}\Omega_{j\beta} + J_{i\alpha}{}^{(n)}. \tag{12.31}$$

The normal spin current may be taken to obey a diffusion type of equation:

$$J_{i\alpha}{}^{(n)} = -D(\partial S_i/\partial x_\alpha), \tag{12.32}$$

where D is the spin diffusion coefficient: to estimate orders of magnitude we may take this equal to its value in the normal phase at T_c. (Actually, in the superfluid phase D will be a fourth-rank tensor quantity, but for present purposes we can neglect this complication.) If we insert (12.32) in (12.31), the effect is to add to (12.24) a term of the form $-D(\partial/\partial t)\nabla^2 S_i$ (where ∇^2 operates in coordinate space). If therefore we assume a disturbance of the form $\exp(i\mathbf{q}\cdot\mathbf{r})$, its frequency is given by an equation of the general form

$$\omega^2 - c_s^2q^2 - iD\omega q^2 = 0, \tag{12.33}$$

where c_s is a typical spin wave velocity (which may of course be anisotropic in general). Evidently, a well defined propagating wave will exist only if the imaginary part of the solution of (12.33) is small compared to the real part: this requires the condition

$$q \ll c_s/D \quad \text{or} \quad \omega \ll c_s^2/D. \tag{12.34}$$

If we work near T_c, substitute for D the normal state value $\frac{1}{3}v_F^2(1+\frac{1}{4}Z_0)\tau$ and use (12.25) or (12.28) we find the criterion for spin waves to be well defined to be roughly

$$\omega\tau \ll 1 - Y(T) \sim (m^*/m)(\rho_s/\rho), \tag{12.35}$$

where τ is the normal state relaxation time. On the other hand, the neglect of dipole terms in Eq. (12.33) is in general only legitimate provided $\omega \gg \omega_D(T)$ where $\omega_D(T)$ is a typical dipole frequency [cf. (12.30)]. Consequently, we normally[44] have well defined spin waves with approximately linear dispersion only in the frequency "window" given by

$$\omega_D(T) \ll \omega = c_s q \ll \tau^{-1}[1 - Y(T)].$$

This restriction may make it difficult (though not necessarily impossible) to see spin waves experimentally; the trouble is that since $\omega_D \sim (T_c - T)^{1/2}$ but $[1 - Y(T)] \sim (T_c - T)$, we cannot go too close to T_c, and if we do not then we have to excite waves of a wave vector q such that q^{-1} is generally rather small compared to a typical sample size. [See however the discussion in Maki and Tsuneto (1974b).]

One may also ask whether spin waves can be propagated in the opposite, "collisionless" limit, $\omega\tau \gg 1$. Since (unlike zero sound) a "collisionless" spin wave does not exist in the normal phase, one would intuitively guess that no such wave could be propagated until the normal component has become negligibly small compared to the superfluid component. The question of spin waves in the collisionless limit has been studied by Combescot (1974a) using kinetic equation techniques analogous to those of Wölfle (1974b) for density oscillations; his formulas for the spin wave velocities seem to contain imaginary parts which are large unless the normal component is small, which would confirm the above guess.

XIII. THE A TRANSITION IN A MAGNETIC FIELD

The transition from the A phase to the normal phase (hereafter called simply the A transition), and in particular its behavior in a magnetic field, at the moment appears to be one of the best-understood features of the behavior of ³He in the new phases.[45] Since the theory is rather straightforward and has been discussed and compared with the experiments by Wheatley (Sec. IX), I shall only comment on it rather briefly. General references on this subject include Ambegaokar and Mermin (1973), Mermin and Ambegaokar (1973), Brinkman and Anderson (1973), and Takagi (1974e). Subjects which will not be discussed here include the possible effect of critical fluctuations (Patton, 1974) and the behavior of the kinetic coefficients (Seiden, 1973, Soda and Fujiki, 1974, Shahzamanian 1974) (cf. Shumeiko, 1972).

A. General considerations and thermodynamics

Let us consider a weakly interacting Fermi gas which (in zero field) will go superfluid with angular momentum l at some critical temperature T_c and let us for the moment suppose that for some unknown reason it will form an ESP state rather than (say) the BW state. If now we apply a magnetic field H to the system in the normal phase, the effect is to enlarge the up-spin Fermi sea at the expense of

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[44] In certain special cases, however, we may be able to neglect the dipole forces for the mode in question, in which case the lower limit does not apply.

[45] With the exception, to some extent, of the ultrasonic attenuation: see references in last section.

the down-spin one. Although this does not affect the possibility of forming parallel-spin Cooper pairs (provided the ESP z axis coincides with the axis of the field, of course), it does nevertheless alter the situation because the density of states at the up-spin Fermi surface is increased and that at the down-spin Fermi surface correspondingly decreased. Now we saw in Sec. VII.A that as a first approximation the up and down spins could be regarded as two completely independent and noninteracting systems (cf. below), and therefore they should have independent transition temperatures which depend on the density of states in the standard way:

$$T_{c\uparrow} = 1.14\epsilon_c \exp - 1/\lambda_{l\uparrow}, \qquad T_{c\downarrow} = 1.14\epsilon_c \exp - 1/\lambda_{l\downarrow}, \tag{13.1}$$

where

$$\lambda_{l\uparrow} \equiv \tfrac{1}{2}(dn/d\epsilon)_{k_F\uparrow} | V_l | [= N_\uparrow(0) \mid V_l \mid] \tag{13.2}$$

and $(dn/d\epsilon)_{k_F\uparrow}$ is the density of states of *both* spins which would correspond to a Fermi surface at the position of the up-spin Fermi surface, etc. In writing down Eqs. (13.1) and (13.2) (which we shall use for qualitative purposes only) we assumed for simplicity that ϵ_c and V_l are not affected by the shift in the Fermi surface. We see from (13.1)–(13.2) that the up spins actually form Cooper pairs at a higher temperature than the down spin pairs, so that between $T_{c\uparrow}$ and $T_{c\downarrow}$ we have only the up-spin particles paired while the down-spin particles remain normal. It is therefore extremely natural to interpret the experimentally observed A_1 and A_2 transitions (see Wheatley) as corresponding to the onset of pairing of up- and down-spins, respectively.

One might at this point object that we already know that for some purposes (e.g., the nuclear magnetic resonance, see Sec. XI) it is quite illegitimate to treat the up- and down-spin particles as forming two noninteracting system: in fact the phase coherence between the $S_z = 1$ and $S_z = -1$ Cooper pairs is quite essential. This is true, but the energy involved in the phase coherence is the dipole energy, which is small compared to the other energies associated with the $A_1 - A_2$ splitting except in extremely low fields ($\lesssim 10$ G; S. Takagi, unpublished). Hence the dipole forces cannot play a significant role in determining the amplitude of the formation of up- and down-spin Cooper pairs: all they can do is to orient them once they are formed, just as in the usual A and B phases.

If the above interpretation of the A_1 and A_2 transitions is correct, then we expect that in the "A_1" phase (between A_1 and A_2) we have the up-spin gap $\Delta_\uparrow$ (cf. Sec. VII.A) finite while $\Delta_\downarrow$ is zero. Below A_2, $\Delta_\downarrow$ grows continuously from zero and at large distances from the transition ($T_c - T \gg T_{c1} - T_{c2}$) the two gaps are nearly equal and we can effectively apply the theory of the A phase as developed in the rest of this paper. Just below the A_2 transition, however, we have $\Delta_\uparrow \neq \Delta_\downarrow$. Thus, both in the A_1 phase and immediately below A_2, we are dealing with *nonunitary* ESP phases. It is natural to suppose that the orbital wave function of these phases is the same as in the A phase proper, that is (assuming the latter is "ABM-like")

$$\mathbf{d}(\mathbf{n}) = \mathbf{d}f(\mathbf{n}). \tag{13.3}$$

We already know that the gaps $\Delta_\uparrow(\mathbf{n})$, $\Delta_\downarrow(\mathbf{n})$ are different: according to Eqs. (13.3) and (7.33) [putting $\hat{Q} = \hat{\Delta}(\mathbf{n})$ and using the fact that $\mathbf{\Delta}(\mathbf{n})$ is just proportional to $\mathbf{d}(\mathbf{n})$] we find

$$\Delta_\uparrow(\mathbf{n}) = \Delta_\uparrow f(\mathbf{n}), \qquad \Delta_\downarrow(\mathbf{n}) = \Delta_\downarrow \exp(i\Delta\phi) f(\mathbf{n}), \tag{13.4}$$

where $\Delta\phi$ is an arbitrary phase corresponding to the possibility of rotating $\mathbf{d}$ in the xy plane [this symmetry will of course be broken by the dipole forces as in the ordinary A phase: cf. Sec. XI.C]; so that the state is characterized by two real quantities $\Delta_\uparrow$, $\Delta_\downarrow$ which are in general unequal. If we have a P-wave ABM state, then the orbital wave function is of course given by (e.g.)

$$f(\mathbf{n}) = (3/2)^{1/2}(n_y + in_z). \tag{13.5}$$

The fact that $\Delta_\uparrow \neq \Delta_\downarrow$ means that the vector $\mathbf{d}$ in Eq. (13.3) is now intrinsically *complex* (as we expect for a nonunitary state, cf. Sec. VII.C). In particular in the A_1 phase ($\Delta_\downarrow = 0$) we must have according to (7.33), $d_x + id_y \equiv 0$, so apart from a phase factor

$$\mathbf{d} = (1/\sqrt{2})(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \tag{13.6}$$

(where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ are unit vectors in spin space). It is interesting to notice that for a phase of this type the dipole energy [which for complex $\mathbf{d}$ is proportional to $-|(\mathbf{d}\cdot\mathbf{l})|^2$, cf. Eq. (10.10)] is minimized if $\mathbf{l}$ lies in *any* direction in the xy plane, so that we have degeneracy with respect to rotation of $\mathbf{d}$ (or $\mathbf{l}$) around the z axis. We can immediately conclude that no longitudinal magnetic resonance can exist in the A_1 phase. (This conclusion would also follow from the Josephson analogy: to get a Josephson effect *both* bulk metals have to be in the superconducting phase.) This observation may also have something to do with the very dramatic difference in the anisotropy of zero-sound attenuation at A_1 and A_2 recently observed at Cornell (D. M. Lee, private communication).

The fact that the up-spins go superfluid before the down-spins has a rather interesting effect on the spin susceptibility. If we start with the system in zero field in the normal state and apply a field H, then just enough down-spin particles will migrate into up-spin states to make the chemical potentials $\mu_\uparrow$, $\mu_\downarrow$ equal, and as a result we will get a magnetization $M = \chi H$, where χ is the usual Pauli susceptibility given by Eq. (2.26). Now, when we reach T_{c1} (i.e., $T_{c\uparrow}$) and up-spin pairs begin to form, the up-spin chemical potential $\mu_\uparrow$ is reduced, relative to its normal-state value, and some *extra* down-spin pairs migrate into the up-spin band. This process stops when we reach T_{c2}, since after that the two chemical potentials $\mu_\uparrow$, $\mu_\downarrow$ decrease together; however it is not reversed and the net result is that for fixed external field the magnetization (and hence the susceptibility), though temperature independent, is slightly greater than the temperature-independent normal-phase value. (For further details, see Takagi, 1974e.)

Indeed, one can see on thermodynamic grounds alone that there must be a jump in susceptibility at T_c. For any second-order phase transition we have the magnetic analogue of the Ehrenfest relations (since magnetization M

and entropy S show no discontinuity across the transition).

$$\Delta\chi = -(\partial T/\partial H)_{\mathrm{coex}}\Delta(\partial M/\partial T),$$
$$\Delta(\partial M/\partial T) = -(\partial T/\partial H)_{\mathrm{coex}}(\Delta c_v/T), \tag{13.7}$$

where the subscript indicates that the derivative is to be taken along the coexistence curve and Δ indicates the discontinuity of quantities across the curve. Combining these equations, we get

$$\Delta\chi = (\partial T/\partial H)^2_{\mathrm{coex}}(\Delta c_v/T). \tag{13.8}$$

Applying this relation to the A_1 and A_2 transitions separately and using the experimental fact that the specific heat jump has the usual sign across each separately, we see that χ must increase on going from the normal to the A phase. One can check that the formulae derived by Takagi (1974e) from the Landau–Ginzburg free energy (see next subsection) satisfy (13.8) (although some care is needed in defining χ in the A_1 phase).

B. Landau–Ginzburg free energy: spin fluctuation effects

As we saw above [Eq. (13.4)], the up- and down-spin gaps in a state of the type described will have the simple form

$$\Delta_\uparrow(\mathbf{n}) = \Delta_\uparrow f(\mathbf{n}), \qquad \Delta_\downarrow(\mathbf{n}) = \Delta_\downarrow(\mathbf{n})\exp(i\Delta\phi),$$
$$(\Delta_\uparrow \neq \Delta_\downarrow) \tag{13.9}$$

where $\Delta\phi$ is an arbitrary phase. If we were to substitute this form in the BCS free energy obtained from (5.72) by the prescription (7.12) we would get

$$F\{\Delta_\uparrow, \Delta_\downarrow\} = -\tfrac{1}{4}(dn/d\epsilon)(1 - T/T_c)(\Delta_\uparrow^2 + \Delta_\downarrow^2)$$
$$+ (21/160)\zeta(3)(\pi k_B T_c)^{-2}(dn/d\epsilon)(\Delta_\uparrow^4 + \Delta_\downarrow^4). \tag{13.10}$$

However, if there is a magnetic field present then, as we have seen, both $(dn/d\epsilon)$ and T_c are slightly different for the up and down spins, the shifts being symmetric around the zero-field values and proportional to H. If therefore we interpret T_c in Eq. (13.10) as the zero-field transition temperature, and neglect terms of order $H\Delta^4$ which would give higher-order effects, we must add to (13.10) a term of the general form

$$F' = -\mathrm{const.}\, H(\Delta_\uparrow^2 - \Delta_\downarrow^2). \tag{13.11}$$

Note that the resultant expression [(13.10) plus (13.11)] does not couple the up and down spins at all, as of course it should not.

However, we already know (Sec. IX) that the BCS form of the free energy is not likely to be a good approximation for ³He. Let us therefore look for the most general free energy as a function of $\Delta_\uparrow$ and $\Delta_\downarrow$ which contains terms up to quartic and has the correct symmetry properties. Such an expression is[46]

$$F\{\Delta_\uparrow, \Delta_\downarrow\} = \tfrac{1}{2}N(0)\{-(1 - T/T_c)(\Delta_\uparrow^2 + \Delta_\downarrow^2)$$
$$- \eta\cdot\tfrac{1}{2}\gamma\hbar(k_BT_c)^{-1}H(\Delta_\uparrow^2 - \Delta_\downarrow^2) + \tfrac{1}{2}\beta(k_BT_c)^{-2}$$
$$\times(\Delta_\uparrow^4 + \Delta_\downarrow^4) - \beta\delta(k_BT_c)^{-2}\Delta_\uparrow^2\Delta_\downarrow^2\}, \tag{13.12}$$

where we have chosen notation to agree as closely as possible with Wheatley's. Here the dimensionless quantities η, β, and δ are phenomenological parameters (but see below). Notice that the last term couples the magnitudes of the up and down gaps (but not, of course, their phases, since in the absence of spin-nonconserving forces gauge invariance with respect to transformation of up- and down-spin wave functions separately forbids this—cf. Sec. VII.A). For the form and temperature dependence of $\Delta_\uparrow$ and $\Delta_\downarrow$ which minimizes this free energy, and the resultant expression for physical properties such as specific heat and susceptibility jump, we refer to Wheatley (Sec. IX). The upshot of the calculations is that η, β, and δ can all be obtained[47] from experiment, δ in particular being measurable with great precision from NMR expreiments (see Osheroff and Anderson, 1974, and for the theory also Takagi, 1974b).

Now, the interesting thing about this result is that β and δ (though not η) can be related to the parameters a_i occurring in the general expression for the free energy, Eq. (9.16) (provided of course, that we assume that ³He-A is indeed a P-wave ABM state). In this way we can measure a new combination of the a_i's and obtain *inter alia*, a check on the validity of the spin-fluctuation theory. Since the derivation does not appear in the literature as far as I know, I will outline it briefly.

Since $\mathbf{d}(\mathbf{n})$ is just the normalized version of the gap $\mathbf{\Delta}(\mathbf{n})$, we have from (13.3) and (7.33)

$$d_x(\mathbf{n}) = (2\Delta)^{-1}(\Delta_\downarrow - \Delta_\uparrow)f(\mathbf{n}),$$
$$d_y(\mathbf{n}) = (-i/2\Delta)(\Delta_\uparrow + \Delta_\downarrow)f(\mathbf{n}), \qquad d_z(\mathbf{n}) = 0$$
$$[\Delta^2 \equiv (\Delta_\uparrow^2 + \Delta_\downarrow^2)/2]. \tag{13.13}$$

Furthermore we have for the ABM state (e.g.)

$$f(\mathbf{n}) = (3/2)^{1/2}(n_y + in_z). \tag{13.14}$$

Consequently we find for the averages of products of d_i's and their complex conjugates over the Fermi surface:

$$\langle d_i d_j\rangle_{\mathrm{av}} \equiv 0$$
$$\langle d_i^* d_j\rangle_{\mathrm{av}} = (4\Delta^2)^{-1}\{\delta_{ix}\delta_{jx}(\Delta_\uparrow - \Delta_\downarrow)^2 + \delta_{iy}\delta_{jy}(\Delta_\uparrow + \Delta_\downarrow)^2$$
$$+ i(\delta_{ix}\delta_{jy} - \delta_{jx}\delta_{iy})(\Delta_\uparrow^2 - \Delta_\downarrow^2)\}. \tag{13.15}$$

Now using (9.15) we find for the five invariants (recall that

---

[46] The β introduced in Eq. (13.11), which is used only in this section, should not be confused with the β used in Secs. V–IX for the coefficient of Ψ^4: the two differ by a constant.

[47] Note, however, that the *sign* of η cannot be obtained from any simple thermodynamic experiment. For the BCS model it is positive, but one cannot be sure that this will be so more generally.

in their definitions summation over repeated indices is implied!)

$$
\begin{aligned}
I_1 &\equiv (|\langle \mathbf{d}^2\rangle_{\mathrm{av}}|^2)^2 = 0,\\
I_2 &\equiv \langle d_i^* d_j^*\rangle_{\mathrm{av}} \cdot \langle d_i d_j\rangle_{\mathrm{av}} = 0,\\
I_3 &\equiv \langle d_i^* d_j\rangle_{\mathrm{av}} \cdot \langle d_i^* d_j\rangle_{\mathrm{av}}\\
&= (8\Delta^4)^{-1}\{(\Delta_\uparrow^4 + \Delta_\downarrow^4 + 6\Delta_\uparrow^2\Delta_\downarrow^2) - (\Delta_\uparrow^2 - \Delta_\downarrow^2)^2\},\\
I_4 &\equiv \langle|\mathbf{d}|^2\rangle_{\mathrm{av}} \equiv 1,\\
I_5 &\equiv \langle d_i^* d_j\rangle_{\mathrm{av}} \cdot \langle d_j^* d_i\rangle_{\mathrm{av}}\\
&= (8\Delta^4)^{-1}\{(\Delta_\uparrow^4 + \Delta_\downarrow^4 + 6\Delta_\uparrow^2\Delta_\downarrow^2) + (\Delta_\uparrow^2 - \Delta_\downarrow^2)^2\}.
\end{aligned}
\tag{13.16}
$$

Hence, using (9.16) (written in terms of Δ rather than Ψ) we have for the fourth-order terms in the free energy

$$
\begin{aligned}
F^{(4)} &= \tfrac{1}{2}(dn/d\epsilon) \cdot \tfrac{1}{2}\bar{\beta}\Delta^4 \sum_{i=1}^{5} a_i I_i\\
&= \tfrac{1}{2}(dn/d\epsilon)\tfrac{1}{2}\bar{\beta}\{\tfrac{1}{4}(a_4 + a_5)(\Delta_\uparrow^4 + \Delta_\downarrow^4)\\
&\quad + \tfrac{1}{2}(a_4 + a_5 + 2a_3)\Delta_\uparrow^2\Delta_\downarrow^2\},
\end{aligned}
\tag{13.17}
$$

where $\bar{\beta}$ is defined by (9.10a). Comparing (13.17) with (13.12) and using the fact that $N(0) \equiv \frac{1}{2}(dn/d\epsilon)$, we find the correspondence

$$
\begin{aligned}
\beta &= \tfrac{1}{2}(7/8\pi^2)\zeta(3)(a_4 + a_5),\\
\delta &= -\{1 + [2a_3/(a_4 + a_5)]\}.
\end{aligned}
\tag{13.18}
$$

The total specific heat jump from the normal phase to the A_2 phase measures the combination $2/\beta(1 - \delta)$ (see Wheatley); from (13.17), this is just proportional to $(a_3 + a_4 + a_5)^{-1}$, in agreement with the results of Sec. IX [cf. Eqs. (9.29), (9.27) and Table I]. However, the ratio of specific heat jumps across A_1 and A_2 separately, or (better) the NMR shifts, enable us to determine δ (see Wheatley). For the BCS model of course have $\delta = 0$; for the "naive" spin-fluctuation model of Sec. IX.A we find from Eqs. (9.20) and (9.21) the result

$$
\delta = \delta'/2,
\tag{13.19}
$$

where δ' is the parameter introduced in (9.21), which also has the significance of the spin-fluctuation contribution to the free energy of the BW state relative to the difference in BCS free energy between BW and ABM (see Table I). (Brinkman and Anderson (1973) call this quantity δ and, *assuming* the naive spin-fluctuation model, use it directly in the free energy (13.11); note that it differs from our δ by a factor of 2).

Let us finally consider what happens if we have a more sophisticated theory (whether of the spin-fluctuation type or any other) which produces arbitrary values of the a_i's. We can then define a quantity δ'' by Eq. (9.44):

$$
\delta'' \equiv \Delta F_{\mathrm{sf}}(\mathrm{BW})/[F_{\mathrm{BCS}}(\mathrm{BW}) - F_{\mathrm{BCS}}(\mathrm{ABM})].
\tag{13.20}
$$

This is the definition of their δ used by Brinkman, Serene, and Anderson (1974). Note carefully that it differs by a sign from the definition used by Brinkman and Anderson (1973), which is equivalent to our definition of δ' from (9.21); since in the BSA theory $\Delta F_{\mathrm{sf}}(\mathrm{BW})$ is negative, δ'' as defined by (13.19) is positive. If we define a quantity s which measures the BCS free energy as in (9.20), and write the spin-fluctuation contribution $a_{i(\mathrm{sf})}$ in the form

$$
a_{i(\mathrm{sf})} \equiv s\alpha_i
\tag{13.21}
$$

then from (13.18), (13.20), and Table I we find

$$
\begin{aligned}
\delta &= -[(2\alpha_3 + \alpha_4 + \alpha_5)/(4 + \alpha_4 + \alpha_5)],\\
\delta'' &= -[3(\alpha_1 + \alpha_4) + \alpha_2 + \alpha_3 + \alpha_5].
\end{aligned}
\tag{13.22}
$$

We see, therefore, that while in the naive spin-fluctuation model [with values of the α_i's given from (9.21)] we have the relation $|\delta''| = 2|\delta|$ $(= |\delta'|)$ this relation is not general. However, if one takes the BSA values of the ratios of the α_i's from (9.43) and fixes the overall magnitude so as to give δ the experimental value 0.25 (see Wheatley) then it turns out that the relation $\delta'' = 2\delta$ is still obeyed to a good approximation. In fact if one inserts the BSA values from (9.43) one finds the general relation between δ and δ'' to be

$$
\begin{aligned}
1 - \delta &= [1 - (21/40)\delta'']/[1 - (7/80)\delta''],\\
\delta'' &= 80\delta/(35 + 7\delta)
\end{aligned}
\tag{13.23}
$$

[cf. Wheatley, Eq. (9.12) and Osheroff and Anderson, 1974]. Hence for $\delta = 0.25$ we find $\delta'' = 16/29$. Note however that we have implicitly assumed that the BSA values, which were calculated on the assumption of unitarity, are also valid for nonunitary states [cf. Sec. IX.C].

XIV. CONCLUSION

In this section I want to consider briefly how well the generally accepted theories of ³He-A and B fit the existing experimental data, and then to comment on some open problems and possible further lines of investigation. In the first part, however, I shall not attempt to duplicate the very full discussion of Wheatley, but just concentrate on what seem to me to be the key points of confrontation between theory and experiment.

A. How firm is the identification of ³He-A and B?

The currently "orthodox" belief about the new phases of liquid ³He is that (a) both new phases are "anisotropic superfluids" that is, phases whose pair wave functions are essentially of the type described in Secs. V–VII of this paper, (b) both correspond to pairing with angular momentum l equal to 1, (c) ³He-A is the ABM phase and ³He-B the BW phase. Let us first ask whether this set of beliefs as a whole is compatible with the experimental data.

The first point to make is that the data on ³He-A taken by themselves appear, at least qualitatively, entirely compatible with the hypothesis that it is the ABM phase. Such a hypothesis is in excellent agreement with the c.w. NMR data, and the agreement of the nonlinear ringing behavior with the predictions of Maki and Tsuneto (1974a; see Sec. XI.C and Wheatley, Sec. VIII) though not perfect, is still impressive; the remaining discrepancies may well be due to the relaxation effects which are not accounted for in the theory. Again, the data on the superfluid density appear to agree at least roughly with Eq. (9.54), and the discrepancies may be due to the effects of geometry which at present are not well understood. Moreover, while the origin of the anomalous fluctuations in NMR absorption, ultrasound absorption etc. in ^{3}He-A (see the end of Wheatley's Sec. X) is at present unclear, it would be natural to associate them with the variations of the anisotropy axes, i.e., of the vector **l**, which as we have seen is pinned in bulk only by currents and hence presumably is somewhat susceptible to large fluctuations, (cf. Anderson, 1973). A further satisfying piece of evidence is the excellent agreement of theory and experiment on the A transition in a magnetic field, (see Sec. XIII). The theoretical predictions for kinetic coefficients and ultrasound absorption (see references in Secs. XII and XIII) also appear to be at least qualitatively compatible with the ABM hypothesis.

Turning now to ^{3}He-B, one can say that most of the data on this phase taken by itself is compatible with the hypothesis that it is the BW state. The main difficulty seems to lie in the static susceptibility, which as discussed in detail by Wheatley (Sec. III) appears to fit the theory well on the melting curve but not so well at lower pressures. The quantitative discrepancy at $T = 0$ is particularly worrying since at $T = 0$ the theoretical formula (9.49) is not sensitive (directly at least) to spin fluctuation effects [$Y(T) \rightarrow 0$ independently of such effects]: in fact, it requires only that the "Fermi-liquid-unrenormalized" susceptibility of the BW phase is $\frac{2}{3}\chi_n$ (which is a direct consequence of the symmetry of this phase), that the Landau parameter Z_2 is not unduly large [cf. Eq. (7.57)] and that the parameter Z_0 is not different in the superfluid phase from its normal-phase value. It is conceivable that this last assumption needs re-examining in the light of the unexpectedly large effects of spin fluctuations (cf. Tewordt *et al.*, 1974). Leaving the susceptibility aside, the c.w. NMR data are in excellent agreement with the theory for the BW phase, and while the failure so far to observe the predicted double dip in the nonlinear ringing frequency (see section XI.E, and Wheatley section IX) is a little disappointing, it is perhaps too early to become seriously worried, since here again the effects of relaxation are large and ill-understood. The same remark applies to the failure to observe a well-defined ringing behavior at lower pressures except very close to T_c. The superfluid density, spin susceptibility and specific heat appear to be in rough though not perfect agreement with Eq. (9.50). Finally the data on kinetic coefficients and ultrasound absorption, though not well understood, are at any rate not obviously incompatible with the identification of ^{3}He-B as the BW phase.

One should of course also remark that the quantitative success of the spin fluctuation model in predicting various experimental quantities on the assumption that ^{3}He-A is ABM and ^{3}He-B BW (see next subsection) may itself be regarded as strong evidence for the correctness of this assumption. However, it is perhaps relevant to point out that in this model it is the spin structure of the Cooper pair wave function, rather than its orbital structure, which is essential [cf. Eq. (9.15)] so that it is not entirely obvious that (say) $l = 3$ states with spin structures like those of the ABM and BW phases might not give very similar results.

A fairly direct and severe test of whether the B phase is indeed BW would be a measurement of the low temperature specific heat: this should be given accurately by the same formula as for a BCS S state, since the gap is isotropic, whereas all other anisotropic superfluid states, even the "BW-like" ones with equal numbers of $S_z = +1, 0, -1$ pairs, have anisotropic gaps (although not necessarily with nodes, cf. Sec. VII.B) and hence a specific heat which tends to zero slower than the BCS expression.

Undoubtedly, the principal difficulty in the orthodox identification is the failure (Wheatley, Sec. IX) to observe the predicted ratio, namely $(5/2)^{1/2}$, of the linear parallel (longitudinal) ringing frequencies in the B and A phases near the PCP. (As discussed by Wheatley, the c.w. experiments on the melting curve do give good agreement with the theory.) Unlike most of the other discrepancies which at present exist between experiment and theory, this one occurs in an area where neither theory nor experiment appears to have much room for maneuver: the theoretical prediction essentially relies only on the symmetry of the pair wave functions of the ABM and BW phases, while it seems difficult to explain away the experimental result in terms of misorientation (cf., Wheatley, Sec. IX). At the time of writing this discrepancy remains a major puzzle.

Let us therefore briefly ask how plausible are alternative identifications of ^{3}He-A and B. The first constraint is that in view of the nature of the phase diagram (Wheatley, Sec. IV) it is difficult to avoid the conclusion that the A and B phases result from the same instability of the normal phase. However, even if we assume that the A phase is a conventional anisotropic superfluid with given (odd) l, it does not *automatically* follow that the B phase is also a conventional anisotropic superfluid with the same l value but a different form of pair wave function: it could conceivably, for instance, be a more sophisticated type of condensed phase in which the Cooper pairs underwent a further type of correlation not allowed for in BCS-type theories, (e.g., so as to form some kind of quadruple). At first sight the linear and nonlinear NMR behavior would appear to settle the matter conclusively in favor of the anisotropic superfluid[48]; however, it is quite conceivable (and indeed in my opinion probable) that even for more sophisticated types of condensed state this behavior will be a function only of the dependence of the dipole energy on relative rotation of the spin and orbital coordinates (cf. Sec. XI). In that case, any given anisotropic superfluid state may be only a particular case of a much more general class of condensed states having the same NMR properties. It is clearly inappropriate to pursue this speculation here, since very little concrete theoretical work has been done along these lines, but it may be worth bearing in mind if it should turn out to be increasingly difficult to fit the B-phase data to the anisotropic superfluid model.

If then we confine ourselves to conventional anisotropic superfluid states, what are the alternatives to the orthodox identifications? This question has been considered by Barton and Moore (1974a,b), Mermin and Stare (1974), and Moore *et al.* (1974). The strongest constraints are imposed by the NMR data. It turns out that once one assumes that the l-value is the same for the A and B phases, it is perfectly pssible to find alternative models for the two

---

[48] The observation of a parallel ringing frequency of $2\gamma\Delta H$ in the limit $\Delta H \rightarrow \infty$ (cf. Sec. XI.E and Wheatley, Sec. VIII) would indeed exclude certain types of hypothesis about (e.g.) quadruple formation.

phases separately, but the results for quantities which relate them, e.g., the ratio of longitudinal ringing frequencies, are in no better and often worse agreement with experiment than those arising from the orthodox identifications, (unless one assumes $l \geq 5$, which is conceivable but aesthetically unappealing). At the moment, therefore, one must conclude that the case for the orthodox identifications, though not quite cast-iron, is at least very strong. Most theorists will probably hope that the annoying NMR anomaly somehow goes away!

B. How good is the spin fluctuation model?

Undoubtedly, the qualitative idea of spin fluctuation feedback has been a major advance in our understanding of ^{3}He-A and B—indeed, one can say it is the only important qualitative departure from BCS theory which has so far emerged from study of the new phases. Let us however briefly ask: assuming that they are indeed the ABM and BW phases, how rigorously can we test the model with existing and obtainable experimental data?

It is probably helpful to frame the ensuing discussion in terms not of a single parameter δ' or δ'' but of the contributions of the spin fluctuation feedback process to the coefficients a_i of the five fourth-order invariants, normalized in terms of the BCS contributions—that is, the α_i's defined in Eq. (13.21). Theoretical calculations (see Sec. IX.C) in general aim to produce accurate ratios of the α_i's but usually take their overall magnitude as a phenomenological parameter. They will, of course, be functions of pressure and also temperature: in the ensuing discussion, however, we shall consider only regions close to the second order transition, so that the α_i's can be treated as depending on pressure only. The discussion will then be applicable to *any* calculation of the α_i's, whether or not it uses the spin fluctuation model. We do not write out the pressure dependence explicitly in what follows.

At present, then, there are three quantities[49] which are in principle measurable for T near T_c which will give information on the α_i's: the specific heat jumps across the A–N and B–N transitions (including transitions from superheated phases), and the parameter δ which describes the splitting of the A_1–A_2 transitions (see Sec. XIII). Note that both the first two can be deduced for *all* pressures from thermodynamic and magnetic measurements (see Wheatley Sec. IV), though the values obtained above the PCP pressure are presumably somewhat less reliable than those below. Using Eq.s (9.29), (9.27), (13.21) and Table I we have

$$\Delta C_A/1.42C_n = \kappa_{\mathrm{ABM}}{}^{-1} = [\tfrac{6}{5} + \tfrac{3}{5}(\alpha_3 + \alpha_4 + \alpha_5)]^{-1}, \tag{14.1}$$

$$\begin{aligned}\Delta C_B/1.42C_n &= \kappa_{\mathrm{BW}}{}^{-1} \\ &= [1 + \tfrac{3}{5}(\alpha_1 + \alpha_4) + \tfrac{1}{5}(\alpha_2 + \alpha_3 + \alpha_5)]^{-1},\end{aligned} \tag{14.2}$$

$$\delta = -[(2\alpha_3 + \alpha_4 + \alpha_5)/(4 + \alpha_4 + \alpha_5)], \tag{14.3}$$

where the last equation is just (13.22) rewritten.

If for instance we take the results of Brinkman *et al.* (1974) for the α_i, Eq. (9.43) and insert them in (14.1)–(14.3) then if we put $\alpha_1 \equiv -\gamma$ we get

$$\Delta C_A/1.42C_n = (1.2 - 6.3\gamma)^{-1}, \tag{14.4}$$

$$\Delta C_B/1.42C_n = (1 - 2\gamma)^{-1}, \tag{14.5}$$

$$\delta = 17.5\gamma/(4 - 3.5\gamma). \tag{14.6}$$

These predictions may be compared with the experimental data on ΔC_A and ΔC_B (cf. Wheatley's Figs. 5 and 16) and on δ (Wheatley, Sec. IX, Osheroff and Anderson 1974: note again that the δ of the latter authors is what we have called δ'', and see Sec. XIII for the relation between the two parameters). It should be noted that the *difference* in specific heat between the two phases is extremely sensitive to small errors in δ in the region $\delta \sim \frac{1}{4}$.

C. General comments

If we assume that the new phases of liquid ^{3}He are indeed anisotropic superfluids, as is by now almost universally done, then what are the main unsolved problems? It seems to me that one of the most urgent concerns the question of the existence and magnitude of a macroscopic angular momentum (in the A phase). As was pointed out in Sec. VI, it is not a simple matter to deduce this from the relative angular momentum of the Cooper pairs; and since the feasibility and meaningfulness of a whole class of experiments will depend on the answer, the question is an urgent one. To answer it one may well have to take into account the effect of boundaries in a much more sophisticated way than is usually done. This question is in a sense one aspect of a more general problem in the theory of anisotropic superfluidity (and indeed superconductivity) namely: What is the relation between Cooper pairs and ordinary diatomic molecules? Can one make a smooth transition between them by varying the pairing potential in a suitable way? Does it make sense to distinguish between the process of formation of Cooper pairs and the process of their "Bose condensation"? Although some aspects of these questions can be said to be understood, the general situation is still very far from clear.

Turning to more specific matters, one would clearly like to know whether the complicated nature of the anisotropic superfluid will produce new kinds of quantization rules, and what effect these will have on structures such as vortices, disgyrations, and perhaps other more complicated singularities. Will there be other kinds of transport than the usual two-fluid type known in ^{4}He? (cf. Graham, 1974). Will the fact that the relation between heat current and normal-fluid velocity is a tensor relation lead to anomalous types of thermal behavior?

At a more specific level still, a theory of NMR relaxation which will cover nonlinear phenomena as well as linear ones is urgently needed. In another direction, there is plenty to be done in the area of microscopic feedback theories: for instance, it should be possible to estimate quantitatively the effect of feedback mechanisms not associated with spin fluctuations, and also to give an unambiguous answer to the

---

[49] It is of course also possible to consider the slope of the A–B transition curve (see Brinkman *et al.*, 1974; Kuroda, 1974b; Osheroff and Anderson, 1974). However, to the extent that one takes seriously Wheatley's Eq. (4.2) for the thermal differences of the two phases, this will be entirely determined by the difference in the specific heat jumps at T_c (the coefficient α is pressure independent and hence presumably arises only from the BCS terms). This appears to be also true for the model of Brinkman *et al.* (1974), at least if the effect of the sixth order terms is neglected.

question, how far the Landau parameters are changed in the superfluid phase by feedback effects. Many questions concerning spin and "orbit" waves, and more generally the propagation of disturbances involving the orientation of the Cooper pairs, remain unclear although progress in this direction seems likely to continue to be rapid. The same goes for studies of the NMR behavior under anomalous conditions. A possibly less attractive area to theorists, but one which is nevertheless interesting and experimentally important, is the complex of phenomena associated with superheating, supercooling, and "memory" effects (Osheroff, 1973, Wheatley, Sec. XI). The list could no doubt be extended almost indefinitely.

One might also ask whether, from our understanding of anisotropic superfluidity in ^{3}He, we can predict it elsewhere? One obvious candidate is dilute ^{3}He–^{4}He solutions, although it is at least equally possible that if superfluidity occurs in this system, it will be in an S-wave state (see, e.g., Østgaard, 1974). Another more speculative possibility is a monolayer of ^{3}He floating on liquid ^{4}He (C. W. Woo, private communication). However, perhaps the most promising type of candidate is a strongly paramagnetic metal such as Pd; one might hope that spin fluctuation exchange would be at least as important a mechanism of attraction here as in ^{3}He and that consequently "triplet superconductivity" (i.e., superconductivity with $S = 1$, odd l pairing) might occur. However, there is an experimental difficulty here: anisotropic superconductivity is predicted theoretically (Balian and Werthamer, 1963) to be very much more sensitive to nonmagnetic impurities than the isotropic variety, and the lower T_c for the pure metal the smaller the impurity concentration which will destroy it. It is a challenging problem to try to predict T_c and hence estimate the required purity. One should also mention that the subject of anisotropic superfluidity has also been discussed in the context of astrophysics with a view to its possible occurrence in neutron stars: however, comparison with experiment is of course somewhat more difficult in this case, and there are substantial differences from the ^{3}He problem.

What is superfluid ^{3}He good for? Of course, it combines the complexities of liquid-crystal anisotropy and superfluid behavior, but the whole is not necessarily greater than the sum of its parts and it is not yet clear whether this combination in itself will give fundamental new insights. It is also our first magnetic superfluid, and this (as stressed by Wheatley) may give interesting applications. To give just one example, if one passes ^{3}He through a superleak when it is in the A_1 phase (where only one spin species is paired) the emerging liquid should have its spins totally polarized! Another conceivable, though probably remote, possibility is that, because the orienting energies, especially for the B phase, are so fantastically tiny, it might be possible to arrange by a suitably ingenious geometry to reduce them right down to the thermal energy k_BT. In that case one might expect to see a sort of macroscopic Brownian motion as the Cooper pairs axes undergo thermal fluctuations. But perhaps the most fascinating possibility is that just as superconductors can be used to test rather fundamental aspects of the laws of physics (e.g., the "physical reality of the vector potential", see Mercereau, 1969), so the spin degree of freedom in superfluid ^{3}He would allow us further tests of the basic tenets of quantum mechanics. This however at the moment must be highly speculative.

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APPENDIX A. PROOF OF EQ. (7.74)

Consider a general quantity of the form

$$M^{(2)} = \tfrac{1}{2} \sum_{ij:kl} g_{ij}(\mathbf{r}_k - \mathbf{r}_l)\sigma_i^{(k)}\sigma_j^{(l)}, \qquad (k \neq l) \tag{A1}$$

where k, l label different particles and i, j Cartesian components of spin in units of $\hbar$, and where the function g_{ij} obeys the relation

$$g_{ij}(\mathbf{r}_k - \mathbf{r}_l) = g_{ji}(\mathbf{r}_k - \mathbf{r}_l). \tag{A2}$$

We want to show that the contribution to the expectation value of $M^{(2)}$ from the formation of Cooper pairs can be written

$$\begin{aligned}\langle M^{(2)} \rangle = \tfrac{1}{4} \int d\mathbf{r} \sum_{ij} g_{ij}(\mathbf{r}) \{ \delta_{ij} \mid \mathbf{F}(\mathbf{r}) \mid^2 \\ - 2\, \mathrm{Re} F_i^*(\mathbf{r}) F_j(\mathbf{r}) \},\end{aligned} \tag{A3}$$

where $\mathbf{F}(\mathbf{r})$ is given in terms of the anomalous averages $\langle a_{-\mathbf{k}\alpha} a_{\mathbf{k}\beta} \rangle$ or $\langle \psi_\alpha(\mathbf{r}) \psi_\beta(\mathbf{r}') \rangle$ by (7.22), (7.18), and (7.31):

$$\begin{aligned}\mathbf{F}(\mathbf{r}) &\equiv -\tfrac{1}{2} i \sum_{\alpha\beta} \sum_{\mathbf{k}} (\sigma_2 \boldsymbol{\sigma})_{\alpha\beta} \langle a_{-\mathbf{k}\alpha} a_{\mathbf{k}\beta} \rangle \exp(i\mathbf{k}\cdot\mathbf{r}) \\ &\equiv -\tfrac{1}{2} i \sum_{\alpha\beta} (\sigma_2 \boldsymbol{\sigma})_{\alpha\beta} \langle \psi_\alpha(\mathbf{R} - \mathbf{r}/2) \psi_\beta(\mathbf{R} + \mathbf{r}/2) \rangle,\end{aligned} \tag{A4}$$

where we assume for simplicity that we deal with a homogeneous state of the system where the second expectation value is not a function of $\mathbf{R}$.

We start by introducing the spin density

$$\sigma_i(\mathbf{r}) \equiv \sum_k \sigma_i^{(k)} \delta(\mathbf{r} - \mathbf{r}_k). \tag{A5}$$

In terms of this $M^{(2)}$ can be written

$$M^{(2)} = \tfrac{1}{2} \int d\mathbf{r} \int d\mathbf{r}' \sum_{ij} g_{ij}(\mathbf{r} - \mathbf{r}')\sigma_i(\mathbf{r})\sigma_j(\mathbf{r}') - K_0, \qquad \text{(A6)}$$

where K_0, which we do not write out explicitly, subtracts off the "unwanted" terms which would add terms with $k = l$ to (A1). Now we second-quantize $\sigma_i(\mathbf{r})$:

$$\sigma_i(\mathbf{r}) = \tfrac{1}{2} \sum_{\alpha\beta} \psi_\alpha^\dagger(\mathbf{r})\sigma_{\alpha\beta}{}^i\psi_\beta(\mathbf{r}), \qquad \text{(A7)}$$

where the $\sigma_{\alpha\beta}{}^i$ are elements of Pauli (not spin) matrices. Substituting in (A6) and commuting the terms so that the creation operators stand in front, we find as usual that the terms arising from the commutation cancel K_0 and we are left with

$$M^{(2)} = \tfrac{1}{8} \sum_{ij} \sum_{\alpha\beta} \sum_{\gamma\delta} \int d\mathbf{r} \int d\mathbf{r}' g_{ij}(\mathbf{r} - \mathbf{r}')$$
$$\times\, \sigma_{\alpha\beta}{}^i\sigma_{\gamma\delta}{}^j\psi_\alpha^\dagger(\mathbf{r})\psi_\gamma^\dagger(\mathbf{r}')\psi_\delta(\mathbf{r}')\psi_\beta(\mathbf{r}). \qquad \text{(A8)}$$

By a change of variables this can be written (from now on summation over repeated indices is implied)

$$M^{(2)} = \tfrac{1}{8} \int d\mathbf{R} \int d\mathbf{r} g_{ij}(\mathbf{r})\sigma_{\alpha\beta}{}^i\sigma_{\gamma\delta}{}^j$$
$$\times\, \psi_\alpha^\dagger(\mathbf{R}+\mathbf{r}/2)\psi_\gamma^\dagger(\mathbf{R}-\mathbf{r}/2)\psi_\delta(\mathbf{R}-\mathbf{r}/2)\psi_\beta(\mathbf{R}+\mathbf{r}/2).$$

We now define [cf. (6.44)]

$$F_{\alpha\beta}(\mathbf{r}) \equiv \langle \psi_\alpha(\mathbf{R} - \mathbf{r}/2)\psi_\beta(\mathbf{R} + \mathbf{r}/2)\rangle, \qquad \text{(A9)}$$

where we assume, as always, that the anomalous average does not depend on $\mathbf{R}$. Then if we work in unit volume as in the text, we find that the contribution to $M^{(2)}$ from formation of Cooper pairs is

$$\langle M^{(2)}\rangle_{\text{pairs}} = \tfrac{1}{8} \int d\mathbf{r} g_{ij}(\mathbf{r})\sigma_{\alpha\beta}{}^i\sigma_{\gamma\delta}{}^j F_{\gamma\alpha}{}^*(\mathbf{r})F_{\delta\beta}(\mathbf{r}). \qquad \text{(A10)}$$

According to (A4) and (A9) we have

$$\mathbf{F}(\mathbf{r}) \equiv -\tfrac{1}{2}i(\sigma_2\boldsymbol{\sigma})_{\alpha\beta}F_{\alpha\beta}(\mathbf{r}) \qquad \text{(A11)}$$

and by the prescription (7.32) of the text this can be inverted to give

$$F_{\alpha\beta}(\mathbf{r}) = i(\sigma_k\sigma_2)_{\beta\alpha}F_k(\mathbf{r}). \qquad \text{(A12)}$$

Consequently (A10) can be written

$$\langle M^{(2)}\rangle_{\text{pairs}} = \tfrac{1}{8} \int d\mathbf{r} g_{ij}(\mathbf{r})\sigma_{\alpha\beta}{}^i\sigma_{\gamma\delta}{}^j(\sigma_k\sigma_2)_{\alpha\gamma}{}^*(\sigma_l\sigma_2)_{\beta\delta}$$
$$\times\, F_k{}^*(\mathbf{r})F_l(\mathbf{r}). \qquad \text{(A13)}$$

Consider now the quantity

$$\Lambda_{ijkl} = \sigma_{\alpha\beta}{}^i\sigma_{\gamma\delta}{}^j(\sigma_k\sigma_2)_{\alpha\gamma}{}^*(\sigma_l\sigma_2)_{\beta\delta}, \qquad \text{(A14)}$$

where summation over the Greek indices is of course implied. Now for any Pauli matrix σ_k (with $k = 1, 2, 3$) we have

$$(\sigma_k\sigma_2)_{\alpha\gamma}{}^* \equiv (\sigma_2\sigma_k)_{\gamma\alpha}. \qquad \text{(A15)}$$

Hence, writing $\tilde{\sigma}$ for the transpose of the matrix σ, we have

$$\Lambda_{ijkl} = \sigma_{\alpha\beta}{}^i\tilde{\sigma}_{\delta\gamma}{}^j(\sigma_2\sigma_k)_{\gamma\alpha}(\sigma_l\sigma_2)_{\beta\delta} \equiv \operatorname{Tr}\sigma_i\sigma_l\sigma_2\tilde{\sigma}_j\sigma_2\sigma_k. \qquad \text{(A16)}$$

But we also have if $j = 1, 2$, or 3

$$\sigma_2\tilde{\sigma}_j\sigma_2 = -\sigma_j \qquad \text{(A17)}$$

and hence

$$\Lambda_{ijkl} = -\operatorname{Tr}\sigma_i\sigma_l\sigma_j\sigma_k = 2(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \qquad \text{(A18)}$$

Substituting this in (A13) we find

$$\langle M^{(2)}\rangle_{\text{pairs}} \equiv \tfrac{1}{8} \int d\mathbf{r} g_{ij}(\mathbf{r})\Lambda_{ijkl}F_k{}^*(\mathbf{r})F_l(\mathbf{r})$$
$$= \tfrac{1}{4} \int d\mathbf{r} g_{ij}(\mathbf{r})\{\delta_{ij}\,|\,\mathbf{F}(\mathbf{r})\,|^2 - 2\operatorname{Re}F_i{}^*(\mathbf{r})F_j(\mathbf{r})\}, \qquad \text{(A19)}$$

where we used the condition (A2). This agrees with (A3), Q.E.D.

APPENDIX B. THE DIPOLE ENERGY

A case of particular interest is the dipole energy (see Sec. X.A). To obtain this, we substitute

$$g_{ij}(\mathbf{r}) = (\gamma^2\hbar^2/r^3)(\delta_{ij} - 3\hat{r}_i\hat{r}_j) \qquad (\hat{\mathbf{r}} \equiv \mathbf{r}/r). \qquad \text{(B1)}$$

Then we find from (A19) the contribution of the Cooper pairs to the dipole energy:

$$H_D = -\tfrac{1}{2}\gamma^2\hbar^2 \int d\mathbf{r} r^{-3}\{|\,\mathbf{F}(\mathbf{r})\,|^2 - 3\,|\,\hat{\mathbf{r}}\cdot\mathbf{F}(\mathbf{r})\,|^2\}. \qquad \text{(B2)}$$

We take Fourier transforms (in unit volume as usual). Since the pair wave function falls off exponentially as $\mathbf{r} \to \infty$, we need not worry about boundary effects and can immediately write down the result [cf. (7.18)]

$$H_D = -\tfrac{1}{2}\gamma^2\hbar^2 \sum_{ij} \sum_{\mathbf{k}\mathbf{k}'} F_{\mathbf{k}i}{}^*F_{\mathbf{k}'j}K_{ij}(\mathbf{k} - \mathbf{k}')$$
$$K_{ij}(\mathbf{k} - \mathbf{k}') \equiv K_{ij}(\mathbf{q}) \equiv \int d\mathbf{r} \exp(i\mathbf{q}\cdot\mathbf{r})r^{-3}(\delta_{ij} - 3\hat{r}_i\hat{r}_j). \qquad \text{(B3)}$$

As we shall see below, K_{ij} depends only on the direction of $\mathbf{q}$, not on its magnitude, and we can therefore immediately perform the sum over the magnitudes of $\mathbf{k}$ and $\mathbf{k}'$: using (7.81) and (7.46) we find

$$H_D = -\tfrac{1}{2}\gamma^2\hbar^2\Psi^2 \int (d\Omega/4\pi) \int (d\Omega'/4\pi)K_{ij}(\hat{\mathbf{q}})\; d_i{}^*(\mathbf{n})\; d_j(\mathbf{n}')$$
$$[\hat{q} \equiv (\mathbf{n} - \mathbf{n}')/\,|\,\mathbf{n} - \mathbf{n}'\,|\,]. \qquad \text{(B4)}$$

To evaluate $K_{ij}(\mathbf{q})$ we choose axes so that the z axis is along $\mathbf{q}$: then by symmetry all off-diagonal components are zero and we have (since $\operatorname{Tr}K \equiv 0$)

$$K_{xx} = K_{yy} = -\tfrac{1}{2}K_{zz} \equiv -\tfrac{1}{2}I_0, \qquad \text{(B5)}$$
$$I_0 \equiv \int d\Omega \int r^{-1}\, dr \exp(iqr\cos\theta)(1 - 3\cos^2\theta),$$
$$= 8\pi \int_0^\infty (dz/z)\{3\sin z/z^3 - (3/z^2)\cos z - \sin z/z\}$$
$$= +\,(8\pi/3) \qquad \text{(B6)}$$

(the integral is convergent and with suitable caution can be integrated by parts). Generalizing (B5) to an arbitrary set of axes, we obtain

$$K_{ij}(\hat{\mathbf{q}}) = -(4\pi/3)(\delta_{ij} - 3\hat{q}_i\hat{q}_j) \tag{B7}$$

and substituting this in (B4) we finally obtain

$$H_D = +(2\pi/3)\gamma^2\hbar^2\Psi^2 \int (d\Omega/4\pi) \int (d\Omega'/4\pi)\{\mathbf{d}^*(\mathbf{n})\cdot\mathbf{d}(\mathbf{n}') - 3\hat{\mathbf{q}}\cdot\mathbf{d}^*(\mathbf{n})\hat{\mathbf{q}}\cdot\mathbf{d}(\mathbf{n}')\} \tag{B8}$$

in agreement with (10.6) of the text.

Equation (B8) is quite general, except that, as discussed in the text, the integrand may have to be multiplied by a factor $R^2(\mathbf{q})$ to take into account quasiparticle renormalization effects.

In the case of pure-l pairing (i.e., when $\mathbf{d}(\mathbf{n})$ is a combination of spherical harmonics Y_{lm} belonging to a single value of l) the double integral can be reduced to a single one:

$$H_D = +[\pi\gamma^2\hbar^2/l(l+1)]\langle R^2\rangle_{\text{av}}\Psi^2 \int (d\Omega/4\pi) \times \{3\,|\,\mathbf{n}\cdot\mathbf{d}(\mathbf{n})\,|^2 - |\,\mathbf{d}(\mathbf{n})\,|^2\} \tag{B9}$$

(note the change in sign), where $\langle R^2\rangle_{\text{av}}$ is a suitably weighted average of the renormalization factor. For the case $l = 1$ this is proved in the appendix of Leggett, 1974a: a more general proof has been given by Takagi (1973),[50] who shows that $\langle R^2\rangle_{\text{av}}$ is given explicitly by

$$\langle R^2\rangle_{\text{av}} = [l(l+1)/2\pi] \int d\mathbf{r}\bar{R}(r)r^{-3}\{j_l(k_F r)\}^2, \tag{B10}$$

where

$$R^2(q) = (3/4\pi) \int d\mathbf{r}\bar{R}(r)r^{-3}j_2(qr). \tag{B11}$$

j_l here is just the usual function occurring in the expansion of a plane wave in angular momentum components, i.e., the spherical Bessel function (Messiah, 1965, p. 488).

LIST OF SYMBOLS[51]

Symbol	Meaning	Definition or first use
$A(T)$	coefficient in expansion of $\Psi(\mathbf{n})$ in terms of $\Delta(\mathbf{n})$	5.57
a_i	coefficient of invariant I_i in fourth order GL free energy	9.16
$a_{i(sf)}$	contribution to above from spin fluctuation feedback	IX.C
$B(T)$	coefficient in expansion of $\Psi(\mathbf{n})$ in terms of $\Delta(\mathbf{n})$	5.57
C_n or $C_n(T_c)$	normal state specific heat at T_c	6.4
c_1	speed of ordinary ("first") sound	12.4, 12.6
c_2	speed of second sound	12.3
c_4	speed of fourth sound	12.1, 12.13
D	spin diffusion coefficient	12.31, 12.32
$\mathbf{d}(\mathbf{n})$	normalized gap function or order parameter	7.35
$d_{\alpha i}$	elements of expansion of $\mathbf{d}(\mathbf{n})$	9.12
$\mathbf{d}$	unit vector along $\mathbf{d}(\mathbf{n})$ for ABM state	9.53
$\bar{d}_{\alpha i}$	$\equiv \Delta d_{\alpha i}$	10.30
$dn/d\epsilon$	density of states (of both spins) at Fermi surface	2.5
$E_{\mathbf{k}}$	energy of broken-pair state relative to ground pair (quasiparticle excitation energy).	5.14, 5.39, 7.47b
$E_{\mathbf{k}\uparrow}$	excitation energy of spin-up quasiparticle	7.6
$\hat{E}_{\mathbf{k}}$	quasiparticle excitation energy matrix	7.17
F	free energy	
F_0, F_1	spin-independent Landau parameters ($\equiv F_0^s$, F_1^s)	2.13
$F_{\mathbf{k}}$	(anomalous average)	5.21, 5.28, 5.43, 5.44
$F(\mathbf{r})$	Fourier transform of above ("Cooper pair wave function")	6.42, 6.45

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[50] For the case $R^2 \equiv 1$ the proof is also given in Takagi, 1974a.

[51] This list does not include standard symbols nor those that are used only very locally. Where several equations are quoted, it generally means that the definition is progressively extended.

Symbol	Meaning	Definition or first use
$\hat{F}_{\mathbf{k}}$ $\hat{F}(\mathbf{r})$	matrix forms of above	7.18, 7.22 7.18
$\mathbf{F}_k$ $\mathbf{F}(\mathbf{r})$	vector representation of above [see (7.31)]	7.31
$f(\mathbf{n})$	normalized gap or order parameter for singlet states	5.65
$\hat{f}(\mathbf{n})$	matrix version of above for triplet states	7.28
$g_D(T)$	dipole coupling constant	10.8
$\hat{H}$ or H	Hamiltonian	11.4
$\mathbf{H}(t)$	external magnetic field	11.3
$I_i(i = .1, .5)$	five P-state fourth order invariants	9.14, 9.15
$\mathbf{J}^{\mathrm{sp}}$ or $J_{i\alpha}{}^{\mathrm{sp}}$	spin current dyadic	7.61
k_F	Fermi wave vector	2.1
K_s	static compressibility	2.39
K_L, K_T	coefficients of gradient free energies	10.33, 10.34
$\mathbf{l}$	unit vector along direction of angular momentum in ABM state	9.53
m^*	quasiparticle effective mass	2.4
$N(0)$	$\equiv \frac{1}{2}(dn/d\epsilon)$	13.11
$\mathbf{n}$	direction of quasiparticle wave vector $\mathbf{k}$ on Fermi surface ($\equiv \mathbf{k}/\|k\|$)	V.C
$n_c(T)$	"number of condensed pairs"	6.48
p_F	Fermi momentum	2.1
$\hat{R}$	rotation matrix for BW state	9.46
$\langle R^2\rangle_{\mathrm{av}}$	renormalization factor	10.8
R_c	"healing length"	10.45
$\mathbf{R}_D$	dipole torque	11.37
$\mathbf{S}$	total spin angular momentum	2.14
$\mathbf{S}(\mathbf{n})$	differential spin polarization of point $\mathbf{n}$ on Fermi surface	7.49a
s	measure of a_i's in BCS theory ($\equiv 3/5$) $\equiv \omega/qv_F$ (Sec. II only)	9.20 2.35
T_c	critical (transition) temperature	3.24
u_k	coefficient in BCS wave function	5.8
$V_{\mathbf{kk}'}$ or $V(\mathbf{k}, \mathbf{k}')$	quasiparticle pairing interaction	4.1
V_l	coefficient of expansion of above in spherical harmonics	4.2
$\mathbf{v}_{\mathrm{sp}}$	spin superfluid velocity	7.58
v_F	Fermi velocity	2.5
$\mathbf{v}_s$	superfluid velocity	6.22
$\mathbf{v}_n$	normal velocity	6.23
v_k	coefficient in BCS wave function	5.8
$Y(T)$	(generalized) Yosida function	6.14
$Y(\mathbf{n}, T)$	differential Yosida function	6.14, 7.50

Symbol	Meaning	Definition or first use
$\hat{Y}(T)$ or $\hat{Y}_{ij}(T)$	matrix of weighted averages of above	6.27
Z_0, Z_1	spin-dependent Landau parameters ($Z_0 = -4\bar{I}$ in "paramagnon" language)	2.13
α	coefficient of Ψ^2 in GL free energy	5.67, 5.68, 7.11
α_i	spin fluctuation term in a_i relative to BCS terms ($\equiv 5/3a_{i(sf)}$)	13.20
β	$\equiv 1/k_BT$	3.22
	coefficient of Ψ^4 in GL free energy	5.67, 5.69, 7.11, 7.25
	(in Ch. 13 only) coefficient of $\Delta_\uparrow{}^4 + \Delta_\downarrow{}^4$	13.12
$\bar{\beta}$	$\equiv \frac{7}{8}\zeta(3)(\pi k_BT_c)^{-2}$	9.10a
β_c	$\equiv 1/k_BT_c$	
γ_0	coefficient of gradient term in GL free energy	10.27
$\gamma_i (i = 1, 2, 3)$	coefficient of gradient term in GL free energy (for anisotropic case)	10.32
γ	gyromagnetic ratio	2.16
	(Sec. X only): coefficient as above	10.26
	(Sec. XIV only): $\equiv -\alpha_1$ in BCS theory	14.4
δ	coefficient of $\Delta_\uparrow{}^2\Delta_\downarrow{}^2$ in GL free energy	13.12
δ'	measure of spin fluctuation contribution to a_i's in "naive" theory	9.21
δ''	measure of spin fluctuation contribution to a_i's in BSA theory	9.44, 13.20
$\delta\chi_{ij}$	change of χ_{ij} from normal-state value	9.3
$\Delta_{\mathbf{k}}$ or $\Delta(\mathbf{n})$	gap parameter	5.13, 5.38, 5.51
$\hat{\Delta}_k$ or $\hat{\Delta}(\mathbf{n})$	matrix form of above for triplet states	7.16
$\boldsymbol{\Delta}(\mathbf{n})$	vector representation of above	7.31
$\lvert \Delta(\mathbf{n}: T) \rvert$	magnitude of gap for unitary triplet state	7.27, 7.47b
Δ or $\Delta(T)$	overall magnitude of gap parameter	5.53, 7.78, 9.12, 13.13
ΔF_{sp}	contribution in free energy from spin fluctuation feedback	9.35
δa_i	change of a_i due to spin fluctuation feedback	9.21
$\Delta\varphi$	phase relation between up and down-spin Cooper pairs	11.13
ΔC_v or ΔC	specific heat jump across normal–superfluid transition	6.5
$\Delta C_{A,B}$	values of above for A and B phases	14.1, 14.2
ϵ_{ijk}	Levi-Civita symbol	7.60
ϵ_k	energy of single (Landau quasi-) particle plane wave state	3.9, 5.15
ϵ_c	cutoff energy for model potential	3.13
ζ	$\equiv 1.14$	
$\zeta(3)$, $\zeta(4)$	Riemann zeta-function	
ζ_0	$\equiv (dn/d\epsilon)^{-1}Z_0$	4.10
η	coefficient in GL free energy in magnetic field	13.12
θ	angle of $\mathbf{d}$	11.6
$\hat{\theta}$ or θ_{ij}	see (7.53)	7.53
κ	parameter proportional to fourth order GL free energy	5.73, 7.28, 7.42, 9.10a, 9.27
λ_l	dimensionless coupling constant	3.24

Symbol	Meaning	Definition or first use
μ	chemical potential also, in (6a) only $\equiv \gamma\hbar$	
ξ_0	$\equiv \hbar v_F/\pi k_B T_c$ ($\sim$ pair radius)	6.47
$\xi(T)$	temperature-dependent correlation length	10.29
ξ_T, ξ_L	temperature-dependent correlation length (for anisotropic case)	10.41
ρ	total mass density	6.20
$\rho_{ij}{}^n$ or $\hat{\rho}_n$	normal density tensor	6.25, 6.30
$\rho_{ij}{}^s$ or $\hat{\rho}_s$	superfluid density tensor	6.33
$\rho_{ij,\alpha\beta}{}^{\mathrm{spin}}$	"spin superfluid density tensor"	7.62
$\boldsymbol{\sigma}$ or σ_i	quasiparticle spin in units of $\hbar$ (also used, with comment, in Sec. VII and Appendix only, for Pauli matrix)	
$\sigma_{\alpha\beta}{}^i$	components of Pauli matrix (see above)	7.31
τ	quasiparticle relaxation time	(II.D), (XI.B)
$\varphi(\mathbf{r}_1 - \mathbf{r}_2, \sigma_1\sigma_2)$	"pseudomolecular" wave function	5.1
χ	static magnetic susceptibility	2.25
χ_{ij}	static magnetic susceptibility tensor	7.52
χ^0 $\chi_{ij}{}^0$	values of above uncorrected for Fermi liquid effects	2.24 9.36
$\chi(\mathbf{r}' - \mathbf{r}, t' - t)$	dynamic spin susceptibility in units of $\hbar^2$	4.8
$\chi(\mathbf{q}\omega)$	Fourier transform of above	4.13
$\chi_{\mathrm{sp}}(\mathbf{q}, \omega)$	dynamic magnetic susceptibility, i.e., above $\times\gamma^2\hbar^2$ (Section II only)	2.38
Ψ	total wave function of system	5.2
$\Psi(\mathbf{n}: T)$	order parameter ($\equiv \sum_{\lvert k\rvert} F_{\mathbf{k}}$)	5.56
$\hat{\Psi}(\mathbf{n})$	matrix form of above	7.18
$\Psi(T)$ or Ψ	overall magnitude of above	5.65, 7.35
ω_L	Larmor frequency ($\equiv \gamma H_0$)	(XI.A)
ω_D	characteristic dipole frequency	(XI.B)
$\boldsymbol{\Omega}$ or $\Omega_{k\alpha}$	spin superfluid velocity dyadic	7.60
$\hat{\boldsymbol{\Omega}}^2$ or $\hat{\Omega}_{ij}{}^2$	NMR tensor	11.42, 11.46
ESP	see Sec. VII.A	
unitarity, unitary state	see (7.26)	
BW state	see (7.30), (IX.D)	
ABM state	see (9.9), (IX.D)	
ESP axes	see Section (VII.B) and (VII.C)	

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