Nuclear Physics B270 [FS16] (1986) 186–204
North-Holland, Amsterdam

OPERATOR CONTENT OF TWO-DIMENSIONAL CONFORMALLY INVARIANT THEORIES

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Received 22 November 1985
(Revised 3 January 1986)

It is shown how conformal invariance relates many numerically accessible properties of the transfer matrix of a critical system in a finite-width infinitely long strip to bulk universal quantities. Conversely, general properties of the transfer matrix imply constraints on the allowed operator content of the theory. We show that unitary theories with a finite number of primary operators must have a conformal anomaly number $c < 1$, and therefore must fall into the classification of Friedan, Qiu and Shenker. For such theories, we derive sum rules which constrain the numbers of operators with given scaling dimensions.

1. Introduction

The fact that a statistical system with short range interactions at a critical point should be conformally invariant has many interesting consequences, particularly in two dimensions [1]. A simple example is the mapping of the plane into a finite-width strip, from which the correlation functions [2] and other quantities accessible to numerical calculation may be determined. They are related to properties of the transfer matrix along the strip, which we shall denote by $e^{-a\hat{H}}$, where a is the lattice spacing. In the continuum limit, $\hat{H}$ may be thought of as the hamiltonian operator of a quantum field theory in (1 + 1) dimensions.

Two particularly useful results of this mapping, which have already been discussed elsewhere [3,2], relate to the eigenvalues E_n of $\hat{H}$: for a strip whose width $l \to \infty$, with periodic boundary conditions,

$$E_0 \sim fl - \frac{\pi c}{6l}, \tag{1.1}$$

$$E_n - E_0 \sim \frac{2\pi x_n}{l}. \tag{1.2}$$

Eq. (1.1) relates the finite size correction to the lowest eigenvalue E_0 (the ground state energy) to the value of the conformal anomaly number c, which plays a central

0550-3213/86/$03.50

role in the analysis of conformal invariance, and which may be used to label different universality classes [4, 5]. Eq. (1.2) relates the energy gaps of the excited states (which are the inverse correlation lengths in the strip) to the scaling dimensions x_n of the scaling operators of the theory, and gives a very accurate way of measuring them [6, 7].

In the first part of this paper, we show that conformal invariance predicts a great deal more about the structure of the transfer matrix. In particular, matrix elements of operators between eigenstates of $\hat{H}$ are related to the universal coefficients of the operator product expansion. We are also able to determine the form of the corrections to the results in eqs. (1.1),(1.2) and to demonstrate the existence of universal ratios of their amplitudes.

In the second part, we exploit the fact that the transfer matrix for an infinitely long strip of width l also yields the partition function for a rectangle with periodic boundary conditions (a torus) of dimensions $l \times l'$:

$$Z(l, l') = \mathrm{Tr}\, e^{-l'\hat{H}}. \tag{1.3}$$

A similar result holds for a parallelogram (see eq. (3.8)) if l'/l is generalized to a complex number. The condition that $Z(l, l') = Z(l', l)$ then implies nontrivial constraints on the eigenvalues of $\hat{H}$ and their degeneracy, and hence on the allowed number of independent operators with a given scaling dimension in the theory. In the general theory of conformal invariance, it is shown that to each scaling operator ϕ with scaling dimension x corresponds an infinite number of other operators $L_n\phi$, with dimensions $x - n$, which are generated in the short distance expansion of ϕ with the stress tensor T. Since the scaling dimensions cannot be negative in a unitary theory (for example, one in which $\hat{H}$ is hermitian,) there exist so-called *primary* operators for which $L_n\phi = 0$ for all $n > 0$. Belavin, Polyakov and Zamolodchikov [4] showed that if we parametrize c by

$$c = 1 - \frac{6}{m(m+1)} \tag{1.4}$$

and m is rational, the set of operators whose scaling dimensions $(h, \bar{h})$ (where $x = h + \bar{h}$) are given by the Kac formula [8] $h = h_{p,q}, \bar{h} = h_{\bar{p},\bar{q}}$, where

$$h_{p,q} = \frac{(p(m+1) - qm)^2 - 1}{4m(m+1)} \tag{1.5}$$

form a *finite* set of primary operators, in the sense that no more are generated in the operator product expansion. Friedan, Qiu and Shenker [5] showed that in a unitary theory with $c < 1$, m must be an integer > 2, and that the only possible scaling dimensions of the primary operators are given by the Kac formula with $1 \leqslant q \leqslant p \leqslant m-1$. Goddard, Kent and Olive [8a] showed that these conditions are also sufficient for unitarity. Our first result complements the results of these papers. We show that in a unitary theory with a finite number of primary operators, c must necessarily be less than one, and the theory must therefore fall into the classification of Friedan, Qiu and Shenker.

For such theories, it turns out that the eigenvalue structure of the transfer matrix may be completely determined. The essential results are contained in the character formulas for the appropriate representations of the Virasoro algebra, which were derived by Rocha-Caridi [9]. We show that the condition that $Z(l, l')$ be symmetric under the interchange $l \leftrightarrow l'$ may be satisfied as long as the quantities $\mathcal{N}(p, q; \bar{p}, \bar{q})$, defined as the number of operators with $h = h_{p,q}$ and $\bar{h} = h_{\bar{p},\bar{q}}$, satisfy certain sum rules.

This is of interest because although the Friedan, Qiu and Shenker [5] classification dictates the allowed values of the scaling dimensions of operators, it does not determine which operators may actually appear in a given theory. Indeed, there may be more than one possible set of operators. For example, both the universality classes of the 3-state Potts model [5,10] and that of a "generic" tetracritical point [11] have been identified with $m = 5$. However, not all scaling dimensions allowed by the Kac formula appear in the Potts model, and some of the others should be doubled. In the tetracritical model, on the other hand, it seems as though all values appear. We show that both these examples satisfy the sum rules, and are able to exhibit the complete set of primary operators in both cases. In fact, these are the *only* unitary models with $m = 5$. The general solution of the sum rules, for arbitrary m, has however eluded us.

2. Structure of the transfer matrix

We begin by recalling the form of the general two-point function in a strip of width l with periodic boundary conditions [2]. In the infinite plane, the two-point function of an operator ϕ with scaling dimensions $(h, \bar{h})$ is

$$\langle \phi(z, \bar{z}) \phi(z', \bar{z}') \rangle = (z - z')^{-2h} (\bar{z} - \bar{z}')^{-2\bar{h}}. \tag{2.1}$$

If ϕ is a primary operator, under the conformal mapping $w = f(z)$ the correlation function transforms according to

$$\langle \phi(z, \bar{z}) \phi(z', \bar{z}') \rangle = (f'(z))^{h} \left(\overline{f'(z)}\right)^{\bar{h}} (f'(z'))^{h} \left(\overline{f'(z')}\right)^{\bar{h}} \langle \phi(w, \bar{w}) \phi(w', \bar{w}') \rangle. \tag{2.2}$$

Choosing $f(z)=(l/2\pi)\ln z$ and using (2.1), we obtain the correlation function in the strip:

$$\langle\phi(w,\bar{w})\phi(w',\bar{w}')\rangle=\frac{(\pi/l)^{2x}}{(\sinh\pi(w-w')/l)^{2h}(\sinh\pi(\bar{w}-\bar{w}')/l)^{2\bar{h}}}. \qquad (2.3)$$

Putting $w=u+iv, w'=u'+iv'$, this has the expansion, for $u>u'$,

$$\left(\frac{2\pi}{l}\right)^{2x}\sum_{N,\bar{N}=0}^{\infty}a_N a_{\bar{N}}\exp[-2\pi(x+N+\bar{N})(u-u')/l]$$
$$\times\exp[2\pi i(s+N-\bar{N})(v-v')/l], \qquad (2.4)$$

where $x=h+\bar{h}$ is the scaling dimension of ϕ, $s=h-\bar{h}$ is its spin, and the coefficients a_N are given by

$$a_N=\frac{\Gamma(x+N)}{\Gamma(x)N!}. \qquad (2.5)$$

On the other hand, the correlation function in the strip may be evaluated using transfer matrix techniques. In that case the scaling "operators" $\phi(u,v)$ become true operators $\hat{\phi}(v)$ acting on the same Hilbert space as does the transfer matrix. The correlation function may be written

$$\langle\phi(u,v)\phi(u',v')\rangle=\sum_n\langle 0|\hat{\phi}(v)|n,k\rangle e^{-(E_n-E_0)(u-u')}\langle n,k|\hat{\phi}(v')|0\rangle, \qquad (2.6)$$

where $|n,k\rangle$ is a complete set of eigenstates of $\hat{H}$ of energy E_n and momentum k (quantized in units of $(2\pi/l)$, so that the matrix elements depend on v and v' as $e^{ik(v-v')}$. Comparing with (2.4), we see that to each primary operator of dimension x and spin s there correspond an infinite number of eigenstates of $\hat{H}$, labelled by $(N,\bar{N})$, with energy $E_0+2\pi(x+N+\bar{N})/l$ and momentum $2\pi(s+N+\bar{N})/l$. The lowest such state must be non-degenerate, and we denote it by $|\phi\rangle$. From (2.4) we see that

$$\langle 0|\hat{\phi}(v)|\phi\rangle=(2\pi/l)^x. \qquad (2.7)$$

Associated with each primary operator ϕ is an infinite number of other scaling operators in the conformal block [4, 5] of ϕ. The operators $L_{-n}\phi$ are defined by the short-distance expansion of ϕ with the (zz) component of the stress tensor T:

$$T(z)\phi(z_1,\bar z_1)+\sum_{n=0}^{\infty}(z-z_1)^{-2-n}L_{-n}\phi(z_1,\bar z_1). \tag{2.8}$$

In the same way, the operators $\bar L_{-n}\phi$ are defined via the short-distance expansion with $\bar T$. Further operators may be generated by repeated short-distance expansions with T and $\bar T$. The most general operator ψ at level $(N,\bar N)$ has the form

$$L_{-k_1}\cdots L_{-k_m}\bar L_{-k'_1}\cdots\bar L_{-k'_{m'}}\phi, \tag{2.9}$$

where $k_1\leqslant\cdots k_m$, $k'_1\leqslant\cdots\leqslant k'_{m'}$, and $\Sigma k_j=N$, $\Sigma k'_j=\bar N$. This operator has scaling dimensions $(h+N,\bar h+\bar N)$, and therefore corresponds to an eigenstate $|\psi\rangle$ of $\hat H$ of energy $x+N+\bar N$ and momentum $s+N-\bar N$, both measured in units of $(2\pi/l)$. Since the two-point function between ψ and the primary operator ϕ is non-vanishing, this eigenstate must be identified with one of those appearing in eq. (2.6). However, not all the operators at a given level are independent. Indeed, for those operators whose scaling dimensions are given by the Kac formula, there is considerable degeneracy. The number of independent operators at level $(N,\bar N)$ will be equal to the degeneracy of the appropriate eigenstate of $\hat H$.

The correlation function $\langle\phi\psi\rangle$ is given in the infinite plane in terms of differential operators [4] acting on $\langle\phi\phi\rangle$. This can be conformally transformed to the strip, yielding all the matrix elements of the form $\langle 0|\phi|\psi\rangle$. Only matrix elements involving the lowest states $|\phi\rangle$ in a given block have a very simple form, however. Further matrix elements of this type may be obtained by transforming the 3-point function to the strip. In the infinite plane the general 3-point function of primary operators has the form [12]

$$\left\langle\phi_i(z_1,\bar z_1)\phi_j(z_2,\bar z_2)\phi_k(z_3,\bar z_3)\right\rangle=c_{ijk}z_{12}^{-h_i-h_j+h_k}z_{23}^{-h_j-h_k+h_i}z_{31}^{-h_k-h_i+h_j}$$
$$\times\bar z_{12}^{-\bar h_i-\bar h_j+\bar h_k}\bar z_{23}^{-\bar h_j-\bar h_k+\bar h_i}\bar z_{31}^{-\bar h_k-\bar h_i+\bar h_j}, \tag{2.10}$$

where c_{ijk} is the operator product expansion coefficient of ϕ_k in the short-distance expansion of ϕ_i and ϕ_j. In the strip one finds

$$\left\langle\phi_i(u_1,v_1)\phi_j(u_2,v_2)\phi_k(u_3,v_3)\right\rangle=\left(\frac{2\pi}{l}\right)^{x_i+x_j+x_k}c_{ijk}\mathrm{e}^{-2\pi x_i(u_1-u_2)/l}\mathrm{e}^{-2\pi x_k(u_2-u_3)/l}$$
$$\times\mathrm{e}^{2\pi i s_i(v_1-v_2)/l}\mathrm{e}^{2\pi i s_k(v_2-v_3)/l} \tag{2.11}$$

for $u_1\gg u_2\gg u_3$. The same correlation function evaluated in the transfer matrix formalism is

$$\langle 0|\hat{\phi}_i(v_1)|\phi_i\rangle e^{-2\pi x_i(u_1-u_2)/l}\langle \phi_i|\hat{\phi}_j(v_2)|\phi_k\rangle e^{2\pi x_k(u_2-u_3)/l}\langle \phi_k|\hat{\phi}_k(v_3)|0\rangle . \quad (2.12)$$

Comparing these expressions, and using (2.7), one finds

$$\langle \phi_i|\hat{\phi}_j(v)|\phi_k\rangle = \left(\frac{2\pi}{l}\right)^{x_j} c_{ijk} e^{2\pi i(s_i-s_k)v/l} . \quad (2.13)$$

Thus the universal operator product expansion coefficients c_{ijk} are measurable in terms of matrix elements of operators between low-lying states. In practice, the operators $\hat{\phi}_j$ will not be normalized, in which case c_{ijk} may be obtained from

$$c_{ijk} = \frac{\langle \phi_i|\hat{\phi}_j(v)|\phi_k\rangle}{\langle \phi_j|\hat{\phi}_j(v)|0\rangle} e^{2\pi i s_k v/l} \quad (2.14)$$

2.1. CORRECTIONS TO FINITE-SIZE SCALING

One of the applications of the above result concerns the corrections to the result (1.2) for finite values of l. These occur because at a critical point the hamiltonian will differ from the fixed-point hamiltonian by terms involving irrelevant operators. If we assume that this departure is small, we can write the infinitesimal transfer matrix as

$$\hat{H} = \hat{H}^* + \sum_j a_j \int dv\, \hat{\phi}_j(v) , \quad (2.15)$$

where the a_j are unknown parameters. To first order in the perturbation,

$$E_n - E_0 = \frac{2\pi x_n}{l} + \sum_j a_j \int dv \,\langle \phi_n|\hat{\phi}_j(v)|\phi_n\rangle . \quad (2.16)$$

Using (2.13) this may be written

$$E_n - E_0 = \frac{2\pi x_n}{l}\left(1 + \sum_j a_j' c_{nnj}(2\pi/l)^{x_j-2} + \cdots\right) . \quad (2.17)$$

This shows the typical form of correction to scaling terms, since we may identify $2-x_j$ with the renormalization group eigenvalue of ϕ_j. For ϕ_j to be irrelevant, $x_j > 2$, and the correction term becomes negligible as as $l \to \infty$, as expected. Eq. (2.17) shows that ratios of correction to scaling amplitudes are universal, and related to ratios of operator product expansions coefficients.

The conformal block of the identity operator $\mathbb{1}$ is present in all theories. It contains the operators $L_{-2}\mathbb{1} \propto T$ and $\bar{L}_{-2}\mathbb{1} \propto \bar{T}$. These operators are not allowed to appear in $\hat{H}$ since they are not scalars, but $L_{-2}\bar{L}_{-2}\mathbb{1}$, which has $x = 4$, is allowed. We thus expect corrections to finite size scaling of order l^{-2} to be present in any theory. (These are often referred to as "analytic" corrections, but note that terms $O(l^{-1})$ are not allowed.) On a lattice, non-scalar operators are also allowed. For example, on a square lattice operators of spin ± 4 will appear. The most relevant operators with $s = 4$ are $L_{-4}\mathbb{1}$ and $L_{-2}^2\mathbb{1}$, which both have $x = 2$ also, and therefore lead to $O(l^{-2})$ corrections. For a self-dual Ising model, these will be no other "non-analytic" corrections, since the only other blocks are those of the energy density ε, which is odd under duality, and the magnetization σ, which is odd under spin reversal. In other models, there will of course be non-analytic corrections, although they may be hard to disentangle [13].

3. Operator content of unitary theories

In the last section, we defined a primary operator as one annihilated by the lowering operators L_n and $\bar{L}_n$ for all $n > 0$. In a unitary theory, in which scaling dimensions must be positive, it is possible, given a list of operators in the theory, to construct all the primary ones by repeatedly applying the lowering operators. We now show that if the number of primary operators so constructed is *finite*, then the conformal anomaly number c must be less than one.

We consider the partition function for a theory defined on an $l \times l'$ rectangle, with toroidal boundary conditions, in the limit that $l, l' \to \infty$ with $l'/l \equiv \delta$ fixed. From (1.3) and (1.1) this has the form

$$Z(l, l') = e^{-fA + \pi c\delta/6} \sum_n e^{-(E_n - E_0)l'}, \tag{3.1}$$

where A is the area. In the limit under consideration, only those energy gaps which scale like l^{-1} contribute. By eq. (1.2), they are given by the dimensions of all the independent scaling operators in the theory. The sum over n may be broken into a sum over conformal blocks, and a sum over the operators in each block. At level $(N, \bar{N})$ in a block, the general operator has the form (2.9). There are $P(N)P(\bar{N})$ such operators, where $P(N)$ is the number of partitions of N into positive integers, not necessarily distinct. In general , however, some of these operators may not be independent. In a unitary theory, each term in (3.1) is positive. Therefore an upper bound on the contribution to the sum in (3.1) from one conformal block, whose primary operator has scaling dimension x, is

$$e^{-2\pi x\delta} \sum_{N, \bar{N}} P(N)P(\bar{N}) e^{-2\pi(N + \bar{N})\delta}. \tag{3.2}$$

This is just the square of the generating function for $P(N)$. It is equal to $f(\delta)^{-2}$, where

$$f(\delta)=\prod_{n=1}^{\infty}(1-e^{-2\pi n\delta}). \tag{3.3}$$

We therefore have the upper bound on the partition function

$$Z(l,l')\leqslant e^{-fll'+\pi c\delta/6}f(\delta)^{-2}\sum_{\substack{\text{primary}\\ \text{operators}}} e^{-2\pi x_n\delta}. \tag{3.4}$$

Now consider the limit $\delta\to 0$. In appendix B we show that $f(\delta)$ satisfies the inversion relation

$$f(\delta)=\delta^{-1/2}e^{\pi(\delta-\delta^{-1})/12}f(\delta^{-1}). \tag{3.5}$$

As $\delta\to 0$, the fact that $Z(\delta)=Z(\delta^{-1})$ implies from (3.1) that $Z(\delta^{-1})\sim e^{-fA+\pi c/6\delta}$, and hence, comparing with (3.4) that

$$e^{-fA+\pi c/6\delta}\leqslant \delta\mathcal{N}e^{-fA+\pi/6\delta}, \tag{3.6}$$

where $\mathcal{N}$ is the number of primary operators. If $\mathcal{N}$ is finite, we see that c must be strictly less than one. In that case, every primary operator must be degenerate at some level, and hence its scaling dimensions are given by the Kac formula.

This result must be interpreted carefully. It is clearly possible to consider theories consisting of several decoupled models, each of which has $c<1$. Since c is additive, the resulting theory may well have $c>1$, yet may appear at first glance to have finite $\mathcal{N}$. This is not so, however. As an example consider the Ashkin-Teller model. This consists of two Ising models, each with $c=\frac{1}{2}$, with a four-spin coupling between them. Consider the decoupling point, where this vanishes. Within each Ising model, the magnetization operators $\sigma^{(1)}$ and $\sigma^{(2)}$ are primary, being annihilated by $L_n^{(1)}$ and $L_n^{(2)}$, for $n>0$, respectively. In the composite model, a primary operator is one annihilated by $L_n\equiv L_n^{(1)}+L_n^{(2)}$. It is easy to show that there is an infinity of such operators, for example

$$\sigma^{(1)}\left(L_{-1}^{(1)}-L_{-1}^{(2)}\right)^k\sigma^{(2)}\qquad (k=1,2,\ldots). \tag{3.7}$$

While this example may seem somewhat pedantic, it is important to realize that away from the decoupling point such primary scaling operators become non-trivial, although the number of them remains infinite, consistent with the fact that c remains at one.

Another case occurs if there is some additional symmetry which may relate the infinite number of primary operators back to a finite set. This happens, for example,

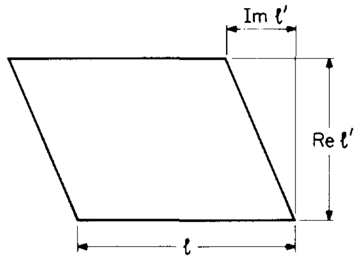

Fig. 1. Definition of l and l' for an arbitrary parallelogram. The partition function is invariant under the interchange $l \leftrightarrow l'$.

in supersymmetric theories [5, 14, 15], where in addition to the operators L_n we have their fermionic partners G_n. Associated with each primary operator is an infinite set of other primary operators formed by acting with the G_n. If we redefine the concept of primary to refer to operators annihilated by both the L_n and the G_n, then the above analysis is modified. Theories with a finite number of such primary operators must have $c < \frac{3}{2}$. This is consistent with the result of refs. [5, 14, 15]. (The relative factor of $\frac{1}{2}$ for the fermions can be traced to the fact that $G_n^2 = 0$.)

3.1. INVERSION SUM RULES

We now restrict ourselves to the case of unitary theories with a finite number of primary operators, so that they fall into the classification of Friedan, Qiu and Shenker [5]. It is first necessary to generalize (3.1) to the case of an arbitrarily shaped parallelogram. The shape may be specified by a single *complex* number $\delta = l'/l$, as is illustrated in fig. 1. Toroidal boundary conditions are once again assumed. The partition function in such a geometry is given in terms of the infinitesimal transfer matrix $\hat{H}$ and the momentum operator $\hat{k}$ of an infinitely long strip by

$$Z(l, l') = \mathrm{Tr}(\mathrm{e}^{-\hat{H}})^{\mathrm{Re}\, l'}(\mathrm{e}^{-i\hat{k}})^{\mathrm{Im}\, l'}, \tag{3.8}$$

where we have used the fact that $\hat{H}$ and $\hat{k}$ commute, and that $\mathrm{e}^{ia\hat{k}}$ translates through a distance a. Inserting a complete set of eigenstates of $\hat{H}$ and $\hat{k}$,

$$Z(l, l') = \mathrm{e}^{-fA + \pi c\, \mathrm{Re}\, \delta/6} \sum_n \exp[-E_n \mathrm{Re}\, l' - ik_n \mathrm{Im}\, l']. \tag{3.9}$$

Thus an operator with scaling dimensions $(h, \bar{h})$ will contribute to the sum in (3.9) a term $\mathrm{e}^{-2\pi(h\delta + \bar{h}\delta^*)}$. Now consider the contribution of one conformal block, whose primary operator has dimensions $(h, \bar{h})$, to the sum. This will be of the form $\chi(\delta)\bar{\chi}(\delta^*)$, where

$$\chi(\delta) = \mathrm{e}^{-2\pi h\delta} \sum_N d(N) \mathrm{e}^{-2\pi\delta}, \tag{3.10}$$

together with a similar expression for $\bar{\chi}(\delta^*)$. Here $d(N)$ is the degeneracy at level N of the operator in question.

Eq. (3.10) has precisely the form of the character formula for the appropriate representation of the Virasoro algebra. These formulas have been derived by Rocha-Caridi [9]. The result is that, for an operator corresponding to $h = h_{p,q}$ in the Kac formula, $\delta(\delta) = \delta_{p,q}(\delta)$, where

$$\chi_{p,q}(\delta) = f(\delta)^{-1} g_{p,q}(\delta), \tag{3.11}$$

$f(\delta)$ is as given in eq. (3.5), and

$$g_{p,q}(\delta) = \sum_{k=-\infty}^{\infty} \left(\exp\left(-\frac{2\pi\delta}{4m(m+1)} \times \left[(2m(m+1)k + (m+1)p - mq)^2 - 1 \right] \right) - \{ q \to -q \} \right) \tag{3.12}$$

If we denote the number of primary operators with $h = h_{p,q}$ and $\bar{h} = h_{\bar{p},\bar{q}}$ by $\mathcal{N}(p, q; \bar{p}, \bar{q})$, then (3.9) becomes

$$Z(l, l') = e^{-fA + \pi c \operatorname{Re} \delta/6} \sum_{p,q;\bar{p},\bar{q}} \mathcal{N}(p, q; \bar{p}, \bar{q}) \chi_{p,q}(\delta) \chi_{\bar{p},\bar{q}}(\delta^*). \tag{3.13}$$

The sum over (p, q) is over the values

$$1 \leqslant q \leqslant p \leqslant m - 1. \tag{3.14}$$

Note that the expansion in (3.12) is very rapidly convergent for $\operatorname{Re} \delta > 1$. However, it is actually convergent for all $\operatorname{Re} \delta > 0$. One of the remarkable features of (3.12) is that it allows the symmetry of $Z(l, l')$ under $l \to l'$ to be respected, provided the $\mathcal{N}(p, q; \bar{p}, \bar{q})$ satisfy certain constraints. These we now describe.

The first step is to apply the Poisson sum formula to (3.11). This has the effect of bringing δ into the denominator of the exponent. After a little algebra, one obtains

$$g_{p,q}(\delta) = \left(\frac{2}{m(m+1)\delta} \right)^{1/2} \exp\left(\frac{\pi(\delta - \delta^{-1})}{2m(m+1)} \right) \times \sum_{r=-\infty}^{\infty} \exp\left(\frac{-\pi(r^2 - 1)}{2\delta m(m+1)} \right) \sin\frac{r\pi p}{m} \sin\frac{r\pi q}{m+1}. \tag{3.15}$$

Substituting in (3.13), and using (3.5), (1.4),

$$Z(\delta)=\mathrm{e}^{-fA+\pi c\,\mathrm{Re}\,\delta^{-1}/6}\frac{2|f(\delta^{-1})|^{-2}}{m(m+1)}\sum_{p,q;\bar{p},\bar{q}}\mathcal{N}(p,q;\bar{p},\bar{q})$$

$$\times\left[\sum_{r=-\infty}^{\infty}\exp\left(-\frac{\pi(r^2-1)}{2\delta m(m+1)}\right)\sin\frac{r\pi p}{m}\sin\frac{r\pi q}{m+1}\right]$$

$$\times\left[\sum_{\bar{r}=-\infty}^{\infty}\exp\left(-\frac{\pi(\bar{r}^2-1)}{2\delta^* m(m+1)}\right)\sin\frac{\bar{r}\pi\bar{p}}{m}\sin\frac{\bar{r}\pi\bar{q}}{m+1}\right]. \tag{3.16}$$

This is to be equated to

$$Z(\delta^{-1})=\mathrm{e}^{-fA+\pi c\,\mathrm{Re}\,\delta^{-1}/6}|f(\delta^{-1})|^{-2}\sum_{p,q;\bar{p},\bar{q}}\mathcal{N}(p,q;\bar{p},\bar{q})$$

$$\times\left[\sum_{k=-\infty}^{\infty}\left(\exp\left(-\frac{\pi}{2\delta m(m+1)}\times\left[(2m(m+1)k+(m+1)p-mq)^2-1\right]\right)-\{q\to-q\}\right)\right]$$

$$\times\left[\sum_{\bar{k}=-\infty}^{\infty}\left(\exp\left(-\frac{\pi}{2\delta^* m(m+1)}\times\left[(2m(m+1)\bar{k}+(m+1)\bar{p}-m\bar{q})^2-1\right]\right)-\{\bar{q}\to-\bar{q}\}\right)\right]. \tag{3.17}$$

We now proceed to equate coefficients of powers of $\mathrm{e}^{1/\delta}$ and e^{1/δ^*}. The fact that the exponents of the leading terms agree is a check on the validity of the result in (1.1). The first sum rule comes from these terms, which correspond to $r, \bar{r}=\pm 1$ in (3.16), and $k=\bar{k}=0$, $p=q=\bar{p}=\bar{q}=1$ in (3.17). The result is

$$\sum_{p,q;\bar{p},\bar{q}}\mathcal{N}(p,q;\bar{p},\bar{q})\sin\frac{\pi p}{m}\sin\frac{\pi q}{m+1}\sin\frac{\pi\bar{q}}{m+1}=\frac{m(m+1)}{8}, \tag{3.18}$$

where we have set $\mathcal{N}(1,1:1,1)=1$, corresponding to the fact that the identity operator should appear just once.

It would appear that an infinite number of other constraints follow from equating the non-leading powers. It is part of the magic of (3.12) that this is not so. First note that values of r and $\bar{r}$ in (3.16) such that

$$r, \bar{r} \equiv \begin{cases} 0 & (\mathrm{mod}\, m), \quad \text{or} \\ 0 & (\mathrm{mod}\, m+1), \end{cases} \tag{3.19}$$

do not contribute to the sums in (3.16). Thus we may run these sums from zero to infinity, including a factor 4 on the right-hand side. Now we use the following

Lemma. The square of any integer r^2 such that r is not divisible by m or $m+1$ can be written uniquely as

$$r^2 = (2m(m+1)k' + (m+1)p' - mq')^2, \tag{3.20}$$

where the integers (k', p', q') satisfy $1 \leqslant |q'| \leqslant p' \leqslant m-1$.

We shall refrain from giving a general proof. The case $m = 3$ is illustrated in fig. 2. The lemma implies that the sums over r and $\bar{r}$ can be converted into sums over k', p', q' and $\bar{k}', \bar{p}', \bar{q}'$ of the form

$$\sum_{k'=-\infty}^{\infty} \Bigg(\exp\left(- \frac{\pi}{2\delta m(m+1)} \left[(2m(m+1)k' + (m+1)p' - mq')^2 - 1 \right] \right)$$
$$\times (-1)^{(p+1)(p'+q')} \sin\frac{\pi pp'}{m} \sin\frac{\pi qq'}{m+1} + \{q' \to -q'\} \Bigg), \tag{3.21}$$

with a similar expression for the sum over $\bar{r}$. This now has almost the form of (3.17). The sums over k are seen to be redundant, and we end up with a finite number of constraints which we call the *inversion sum rules*:

$$\sum_{p,q;\bar{p},\bar{q}} \mathfrak{N}(p,q:\bar{p},\bar{q})(-1)^{(p+q)(p'+q')+(\bar{p}+\bar{q})(\bar{p}'+\bar{q}')}$$
$$\times \sin\frac{\pi pp'}{m} \sin\frac{\pi qq'}{m+1} \sin\frac{\pi \bar{p}\bar{p}'}{m} \sin\frac{\pi \bar{q}\bar{q}'}{m+1}$$
$$= \tfrac{1}{8} m(m+1) \mathfrak{N}(p',q';\bar{p}',\bar{q}'). \tag{3.22}$$

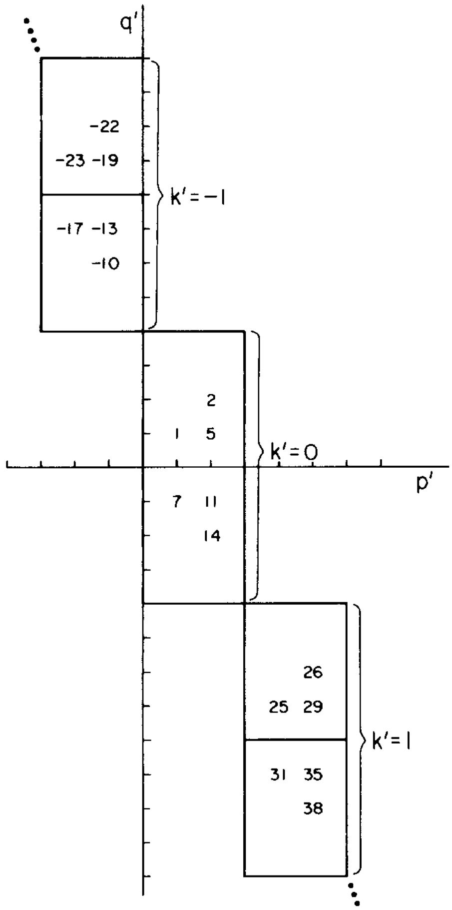

Fig. 2. Illustration of the lemma for $m=3$. The values of $2m(m+1)k'+(m+1)p'-mq'$ are shown. The blocks correspond to different values of k'. Every integer not divisible by m or $m+1$ appears, regardless of sign, just once in this table.

3.2. SOLUTIONS TO THE SUM RULES

Eq. (3.22) may be written in a matrix form

$$M\mathcal{N}=\mathcal{N}, \tag{3.23}$$

where M is a direct product of two $\frac{1}{2}m(m-1)\times\frac{1}{2}m(m-1)$ matrices. We are interested in eigenvectors of M with eigenvalue one. In general, this eigenspace is multidimensional, so there is a whole manifold of solutions. However, we require only those in which the $\mathcal{N}(p,q;\bar{p},\bar{q})$ are non-negative integers. This Diophantine nature makes the enumeration of the solutions difficult.

One solution which may always be found is

$$\mathcal{N}(p,q;\bar{p},\bar{q}) = \delta_{p,\bar{p}}\delta_{q,\bar{q}}. \tag{3.24}$$

This corresponds to the case of all possible scalar operators being present in the theory, and no others. Presumably such theories correspond to the generic multicritical points in the sequence of models of Andrews, Baxter and Forrester [16], analysed further by Huse [11]. He showed that each model could be associated with a value of m, and he identified exponents corresponding to all scalar operators with dimension $x < 1$. Presumably all the other scalar operators will then be generated in the operator product expansion. The first sum rule (3.18) shows that no other operators may then appear in the theory, because each term in (3.18) is non-negative.

To proceed further, we have examined low values of m. The analysis is simplified by the fact that, because of the periodic boundary conditions, only operators with integer spin may occur. That is, $\mathcal{N}(p,q;\bar{p},\bar{q}) = 0$ unless $h_{p,q} - h_{\bar{p},\bar{q}}$ is an integer. Also, for the partition function to be real,

$$\mathcal{N}(p,q;\bar{p},\bar{q}) = \mathcal{N}(\bar{p},\bar{q};p,q). \tag{3.25}$$

This considerably restricts the dimension of the space to be searched for solutions.

m = 3. In this case the allowed values of h and $\bar{h}$ are $0, \frac{1}{16}, \frac{1}{2}$, and there are no non-scalar integer spin operators. Truncated to the space of scalar operators, the matrix is

$$M = \frac{1}{4}\begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 0 \end{pmatrix}, \tag{3.26}$$

from which it follows trivially that the only solution is of the form (3.24). The three primary scalar operators are the unit operator, the energy density and the magnetization operators of the Ising model.

m = 4. Once again there are no non-scalar integer spin operators. The truncated matrix is

$$M = \frac{1}{5}\begin{pmatrix} t & 2t & t & 2t' & t' & t' \\ 2t & 0 & 2t & 0 & 2t' & 2t' \\ t & 2t & t & 2t' & t' & t' \\ 2t' & 0 & 2t' & 0 & 2t & 2t \\ t' & 2t' & t' & 2t & t & t \\ t' & 2t' & t' & 2t & t & t \end{pmatrix}, \tag{3.27}$$

where $t = \sin^2(\frac{1}{5}\pi) = \frac{1}{8}(5-\sqrt{5})$ and $t' = \sin^2(\frac{2}{5}\pi) = \frac{1}{8}(5+\sqrt{5})$. Because $\sqrt{5}$ is irrational, this leads to a system of 12 equations. It is straightforward to show that the only solution is once again of the form (3.24). This case has been identified with the universality class of the tricritical Ising model [5].

$m=5$. This case is more interesting because there exists the possibility of non-scalar operators, corresponding to $(p,q;\bar{p},\bar{q})=(2,1;3,1)$, $(3,1;2,1)$, $(1,1;4,1)$, $(4,1;1,1)$; that is $(h,\bar{h})=(\frac{2}{5},\frac{7}{5}),(\frac{7}{5},\frac{2}{5}),(0,3),(3,0)$ respectively. The resulting truncated 14×14 matrix has as its elements rational multiples of t or t'. This leads to a system of 28 equations, which we shall refrain from giving in detail. They simplify to

$$\begin{aligned}
\mathfrak{N}(1,1;1,1)=\mathfrak{N}(2,1;2,1)=\mathfrak{N}(3,1;3,1)&=\mathfrak{N}(4,1;4,1)\equiv a\,,\\
\mathfrak{N}(2,2;2,2)=\mathfrak{N}(3,2;3,2)=\mathfrak{N}(4,2;4,2)&=\mathfrak{N}(4,4;4,4)\equiv f\,,\\
\mathfrak{N}(3,3;3,3)&=\mathfrak{N}(4,3;4,3)\equiv i\,,\\
\mathfrak{N}(3,1;2,1)&=\mathfrak{N}(4,1:1,1)\equiv k\,,
\end{aligned}\qquad(3.28)$$

where

$$\begin{aligned}
a&=k+f\,,\\
5a&=3f+2i+k\,,\\
i&=a+k\,,\\
5k&=a-3f+2i\,.
\end{aligned}$$

Since a is normalized to 1, the crucial equation is (3.28). It has two solutions, leading to the two possibilities

$$a=f=i=1\,,\qquad k=0\,,\qquad(3.29)$$

or

$$a=k=1\,,\qquad i=2\,,\qquad f=0\,.\qquad(3.30)$$

The first possibility (3.29) corresponds to the solution (3.24). This model has been identified with a generic tetracritical point [11]. (In field theory this corresponds to a scalar field with a ϕ^8 interaction.) The second solution (3.30) is the universality class of the 3-state Potts model. This is so, because some of the known scaling dimensions have been already identified [5, 10], and it was observed that operators corresponding to q even are absent. This means $f=0$. Also, the magnetization operators were found to correspond to $q=3$. Since in the 3-state Potts model the order parameter has two components, we would expect to find $i=2$. The complete list of scaling dimensions of primary operators is then

$$\begin{aligned}
&(0,0),(\tfrac{2}{5},\tfrac{2}{5}),(\tfrac{7}{5},\tfrac{7}{5}),(3,3)\,, &&\text{energy;}\\
&(\tfrac{1}{15},\tfrac{1}{15})\times2,(\tfrac{2}{3},\tfrac{2}{3})\times2\,, &&\text{magnetization;}\\
&(\tfrac{2}{5},\tfrac{7}{5}),(\tfrac{7}{5},\tfrac{2}{5}),(0,3),(3,0)\,, &&\text{chiral.}
\end{aligned}\qquad(3.31)$$

All previously found operators [5] appear in this list.

4. Summary and further remarks

In the first part of this paper, we have completed the program begun in ref. [2]. We have shown how all important universal properties of conformally invariant two-dimensional theories, including critical exponents and operator product expansion coefficients, may be related to numerically accessible properties of the transfer matrix of a finite width strip. The value of these results will lie in the investigation of new models, rather than in reproducing already known results. The multicritical points in the models obtained by Andrews, Baxter and Forrester [16] whose exponents do not [11] appear to fit the Kac formula are of the first type.

Second, we showed that unitary models with a finite number of primary operators (in the narrow sense defined by Belavin, Polyakov and Zamolodchikov [4]) have $c < 1$. This result partially fills a gap in the line of reasoning which picks out those models in the Friedan, Qiu and Shenker [5] classification as being special. For these models, we showed how the character formulas of Rocha-Caridi [9] give the partition function in an arbitrarily shaped parallelogram, once the number of operators with given scaling dimensions are known. Exploiting the symmetry of the parallelogram, we then derived sum rules which must be satisfied by these numbers. It is remarkable how the scaling dimensions allowed in the models in the Friedan, Qiu and Shenker [5] classification enable this symmetry to be realized. An arbitrary list of scaling dimensions would not have this property. This is another argument pointing to the special role of degenerate theories. We note that the symmetry of the parallelogram, which corresponds to the invariance of $Z(\delta)$ under the modular group, has recently been exploited to limit the possible gauge groups in heterotic string theories [22].

Finally, we obtained all solutions of the sum rules for $m = 3, 4, 5$, and showed that only the models which have been previously identified (Ising, tricritical Ising, 3-state Potts, and generic tetracritical point,) are in fact allowed. We gave for the first time a complete list of primary operators for these models. Solution of the sum rules for larger values of m will require greater effort or sophistication. However, it would appear that the number of solutions should grow with m. This points to the existence of as yet unexplored models, even with $c > 1$. However, it is important to realize that existence of a solution to the sum rules does not imply existence of a corresponding model, since the sum rules are only a necessary condition for the model to be consistent.

The sum rules form a more severe constraint on a theory than closure of the operator product expansion and crossing symmetry, which in some cases does determine the operator product expansion coefficients [17]. For example, in the case $m = 3$, the operator product expansion closes with the operators $\mathbb{1}$ and ε, the energy density. However, the sum rules show that the magnetization σ must be included to get a consistent theory. Once the solution of the sum rules is obtained, the expression (3.16) gives the shape dependence of the free energy at criticality in an arbitrary parallelogram. It would be interesting to generalize this to other quantities such as the susceptibility.

This work was supported by the National Science Foundation under grant no. PHY83-13324.

Appendix A

Several interesting properties of a model in a finite width, infinitely long strip can be obtained approximately if the infinitesimal transfer matrix $\hat{H}$ is truncated to a finite set of low-lying states. The simplest case is to consider just two states, which will be a reasonable approximation for some quantities if the gap to the first excited state is small, followed by a larger gap to the second excited state. Such is the case for the Ising model, which has gaps of $\pi/4l, 2\pi/l$ to the lowest excited states [2]. In general, if the magnetization operator has scaling dimension x, in the truncated basis

$$\hat{H} = \frac{2\pi x}{l}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A.1}$$

where $\hat{H}$ is subtracted so that $E_0 = 0$. An external magnetic field h corresponds to adding a term

$$\hat{H}_1 = h\left(\frac{2\pi}{l}\right)^x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{A.2}$$

It is trivial to diagonalize the sum and obtain the h-dependent part of the free energy per unit area:

$$f \simeq \frac{\pi x}{l} - \left[\left(\frac{\pi x}{l}\right)^2 + \left(\frac{2\pi}{l}\right)^{2x} h^2\right]^{1/2}. \tag{A.3}$$

From this follow the susceptibilities $\chi^{(n)} \equiv \partial^n f/\partial h^n|_{h=0}$. In particular

$$\chi^{(2)} \simeq (2\pi/l)^{2x}(l/\pi x), \tag{A.4}$$

and the dimensionless coupling constant [18]

$$g \equiv \chi^{(4)}/l^2(\chi^{(2)})^2 \simeq -3/\pi x. \tag{A.5}$$

Using eq. (2.3) it is possible to show that the corrections to (A.4) are in fact $O(x^2)$.

Both $\chi^{(2)}$ and g have been measured [19, 20] for the Ising model, showing good agreement with the above crude estimates, if we take $x = \frac{1}{8}$.

Appendix B

We derive the inversion relation (3.5) for $f(\delta)$. This can be written, using the Euler pentagonal number theorem [21] as

$$f(\delta) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-\pi\delta(3n^2+n)-i\pi n}. \tag{B.1}$$

Applying the Poisson sum formula, one obtains

$$f(\delta) = (3\delta)^{-1/2}\mathrm{e}^{(\pi/12)(\delta-\delta^{-1})} \sum_{r=-\infty}^{\infty} \mathrm{e}^{-r(r+1)\pi/3\delta}\cos\tfrac{1}{6}(2r+1)\pi. \tag{B.2}$$

Since

$$\cos\tfrac{1}{2}(2r+1)\pi = \begin{cases} \tfrac{1}{2}\sqrt{3} & (-1)^n, \quad r = 3n \\ \tfrac{1}{2}\sqrt{3} & (-1)^n, \quad r = 3n-1 \\ 0, & \qquad\quad r = 3n+1, \end{cases} \tag{B.3}$$

the sum over r may be rewritten as a sum over n. After a little manipulation, this has the same form as the sum in (B.1), with δ replaced by δ^{-1}.

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