The Clifford Algebra:

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} \quad (\text{Euclidean space})$$

<u>construct</u> representation of Spin(N)

$$M^{ij} = \frac{i}{4}[\gamma^i, \gamma^j]$$

Lie algebra SO(N)

$$\gamma_1 = \sigma_1 \otimes 1 \cdots\cdots \otimes 1$$

$$\gamma_2 = \sigma_2 \otimes 1 \cdots\cdots \otimes 1$$

$$\gamma_3 = \sigma_3 \otimes \sigma_1 \cdots\cdots \otimes 1$$

$$\gamma_4 = \sigma_3 \otimes \sigma_2$$

$$\vdots$$

$$\gamma_{2N-1} = \sigma_3 \otimes \cdots\cdots \otimes \sigma_1$$

$$\gamma_{2N} = \sigma_3 \otimes \cdots\cdots \otimes \sigma_2$$

$$\gamma_{2N+1} = (-i)^N \gamma_1 \cdots\cdots \gamma_{2N}$$

conjugate representation for generator

T^a

$$\rho(T^a) \longrightarrow -\rho(T^a)^T$$

$$\rho(g) = \rho(e^{i\omega_a T^a}) = e^{i\omega_a \rho(T^a)}$$

$$= e^{-i\omega_a \rho(T^a)^T} = e^{-i\omega_a \rho(T^a)^*} = \rho^*(g)$$

$$\rho(T^a)^T = (\rho(T^{a\dagger}))^T = \rho(T^{a\dagger})$$

Real Representation:

$$\cancel{\rho(T^a) = (C^{-1})^T \rho(T^{*a})^T C^T}$$

$$\Rightarrow \cancel{\rho(T^a)^T = -(C^T)\rho(T^a)(C^{-1})^T}$$

$$\Rightarrow \cancel{\rho(T^a)} =$$

$$\rho(T^a)^T = -C\rho(T^a)C^{-1}$$

take conjugation:

$$\rho(T^a) = -(C^{-1})^T \rho(T^a)^T C^T$$

$$\rho(T^a)^T = (CC^{-1T})\,\rho(T^a)^T\,(CC^{-1T})^{-1}$$

$$\Rightarrow \left[(CC^{-1})^T, \rho(T^a)^T\right] = 0$$

$$\Rightarrow CC^{-1T} = c\mathbb{1}$$

$$\Rightarrow C = cC^T = c(cC^T)^T = c^2 C$$

$$\Rightarrow c^2 = \pm 1 \Rightarrow C \begin{cases} \nearrow \text{symmetric} \\ \searrow \text{anti-symmetric} \end{cases}$$

即电荷共轭矩阵，要么是对称的，要么是反对称的

unitary transformation: $\rho(T'^a) = U^\dagger \rho(T^a) U$

$$\Rightarrow \rho(T'^a)^T = U^\dagger \rho(T^a)^T U^* = -U^\dagger C \rho(T^a) C^{-1} U^*$$

$$= -(U^\dagger C U)\rho(T^a) U^\dagger C^{-1} U^* \Rightarrow C = U^\dagger C U$$

$C' = U^T C^T C$ → symmetry 不依赖于 basis 的选择

• 然而若能够找到一组 basis 使得所有的群元都是实的，则是 real；否则为 pseudoreal.

$A^T = T$ (complex) $\Rightarrow$ semi-definite

$\cancel{A\Psi_1 = \lambda_1\Psi_1 \Rightarrow A^T A \Psi_1 = A^T\lambda_1\Psi_1 = \lambda_1^2\Psi_1}$

$\langle \Psi_1 | A | \Psi_1 \rangle = \lambda_1 \langle \Psi_1 | \Psi_1 \rangle$

unitary: C and symmetric $\Rightarrow$ $\boxed{e^{i\theta}\,\mathbb{1}}$

• $\begin{cases} \text{Doublet rep of } SU(2) \\ \text{Fundamental rep of } Sp(n) \end{cases}$

那 i = odd 是对称的；i = even anti-symmetry

$$\begin{cases} C_1 = \gamma^1 \cdots\cdots \gamma^{2k-1} \\ C_2 = \gamma^2 \cdots\cdots \gamma^{2k} \end{cases} \quad + \text{ny}$$

(1): $k \equiv 0 \pmod 4$

Ex: $\underbrace{\sigma_1 \otimes 1 \otimes \cdots\cdots 1}_{8} \equiv \gamma_1$

$\cancel{\sigma_2 \otimes \sigma_1}$

$\sigma_3 \otimes \sigma_1 \otimes \overset{1}{\cancel{\sigma_1}} \otimes 1$

$\sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes 1$

$\Rightarrow$ $\sigma_3 \otimes \sigma_3 \otimes \sigma_3 \otimes 1$

$$\Rightarrow \begin{cases} C_1 = \underline{(-i\sigma_2) \otimes \sigma_1 \otimes (-i\sigma_2) \otimes \sigma_1} \\ C_2 = (i\sigma_1) \otimes \sigma_2 \cdots\cdots (i\sigma_1) \otimes \sigma_2 \end{cases}$$

$\{$ ~~C~~ ~~conjua~~

$$\begin{cases} \gamma_1 = \sigma_1 \\ \gamma_2 = \sigma_2 \end{cases} \Rightarrow \begin{cases} C_1 = \sigma_1 \\ C_2 = \sigma_2 \end{cases}$$

$\checkmark$ $\gamma_1 = \sigma_1 \otimes 1 \otimes 1 \otimes 1$

$\gamma_2 = \sigma_2 \otimes 1 \otimes 1 \otimes 1$

$\checkmark$ $\gamma_3 = \sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1$

$\gamma_4 = \sigma_3 \otimes \sigma_1 \otimes 1 \otimes 1$

$\vdots$

与手征矩阵为反对易的

$\gamma_9 = \sigma_3 \otimes \cdots\cdots$

$\gamma_{10} = \sigma_3 \otimes \cdots\cdots$

$1 \quad \sigma_1 \quad \sigma_3 \quad \sigma_3 \quad \sigma_3$

$\sigma_1 \quad \bar{\sigma}_3 \quad \bar{\sigma}_3 \quad \bar{\sigma}_3 \quad \bar{\sigma}_3 = \sigma_1$

$$\begin{cases} C_1 = \sigma_1 \otimes (-i\sigma_2) \cdots\cdots (-i\sigma_2) \otimes \sigma_1 \\ C_3 = \sigma_2 \otimes \cdots\cdots \otimes \sigma_2 \end{cases}$$

~~$k = 2 \pmod 2$~~

$$\begin{cases} C_1 \gamma^i C_1^{-1} = \gamma^{iT} \\ C_2 \gamma^i C_2^{-1} = -\gamma^{iT} \end{cases}$$

$$\Downarrow$$

电荷共轭阵都阵为~~对称的~~反对称

时间反演也是对称的

~~$\gamma_0 = 1 \otimes$~~ **time-reversal symmetry**

$\gamma_0 = \sigma_1 \otimes 1$ (symmetric) $\otimes$ 2k

$\gamma_2 = (i\sigma_2) \otimes \sigma_2$ (symmetric)

$\gamma_3 = (i\sigma_2) \otimes \sigma_1$ (anti-symmetric)

$\gamma_4 = (i\sigma_2) \otimes \sigma_3$ (anti-symmetric)

$$\begin{cases} \gamma_1\gamma_2 = -\sigma_3 \otimes \sigma_2 = \begin{pmatrix} \sigma_2 & \\ & -\sigma_2 \end{pmatrix} \\ \gamma_3\gamma_4 = -1 \otimes \sigma_2 = \begin{pmatrix} \cancel{\sigma_2} & \\ & i\sigma_2 \end{pmatrix} \end{cases}$$

$\boxed{\text{charge}}$ $\boxed{\text{conjugation}}$

✓

$$\Downarrow$$

time-reversal

$k \equiv 2 \pmod 4$

$$\begin{cases} \gamma_1 = \sigma_1 \otimes 1 \\ \gamma_2 = \sigma_2 \otimes 1 \\ \gamma_3 = \sigma_3 \otimes \sigma_1 \\ \gamma_4 = \sigma_3 \otimes \sigma_2 \end{cases}$$

$$\begin{cases} C_1 = (-i\sigma_2) \otimes \sigma_1 \quad (\text{anti-symmetric}) \equiv C \\ C_2 = (i\sigma_1) \otimes \sigma_2 \quad (\text{symmetric}) \equiv T \end{cases}$$

$$C_1 \gamma^i C_i^{-1} = -(\gamma^i)^T$$

$$\cancel{\gamma_1 \gamma_3 \gamma^1 \gamma_3^{-1} \gamma_1^{-1} = -\gamma_3 \gamma_3^{-1} \gamma_1^{-1} = = \gamma_1^{-1}}$$

$$= \cancel{= \gamma_1}$$

$$C_2 \gamma^i \cancel{\gamma} C_i^{-1} = -(\gamma^i)^T$$

<u>Representation</u>

学习如何寻找 charge conjugation 和 time reversal matrix: