Chapter 4

Fermi Liquid

4.1 Quasi-particles and Landau interaction parameters

4.2 Renormalization to physical properties

Let's consider a simple classical example. The object is connected by string . The other end of string is fixed on wall.

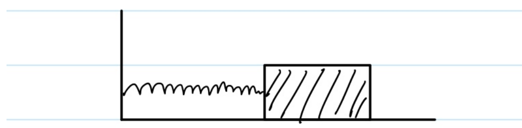

Fig 4.1

The input of system is force F and response is displacement s. The susceptibility is defined as

$$x = -\frac{x}{F} = \frac{1}{k} \tag{4.1}$$

The energy of sytem is described as

$$E = E_{\text{Ela}} - kx = \frac{1}{2}kx^2 - Fx \rightarrow E = -\frac{1}{2}\chi F^2 \tag{4.2}$$

The susceptibility also can be defined from energy

$$\chi = -\frac{\partial^2 E}{\partial F^2} \tag{4.3}$$

Example 4.2.1 (.)
We consider magnetism system . where the external field H will response to magnetization M . The energy increment is gien by

$$dE = HdM \tag{4.4}$$

Hence , the total energy is given by

$$E = E_M - HM \to \chi = -\frac{\partial^2 E}{\partial H^2} \tag{4.5}$$

Now we consider a more complex system, where object is connected with two springs . By the same way , the susceptibility is given by

$$\chi = \frac{1}{k_0 + k'} = \frac{\chi_0}{1 + \frac{k'}{k_0}} \tag{4.6}$$

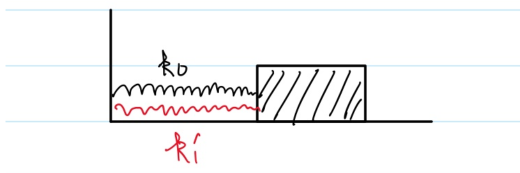

Fig 4.2

We understood this process with close loop process as shown on ().

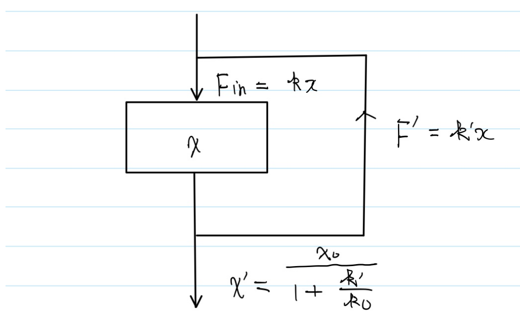

Fig 4.3

The feedback process will change the susceptibility. We expand the renoramlized susceptibility into Taylor series

$$\chi' = \chi_0 + \chi_1 + \chi_2 \cdots \text{where} \quad \chi_n = \chi_0 \left(-\frac{k'}{k}\right)^n \tag{4.7}$$

Question 1: .

If the k' is negative, what interesting things will be happend?

4.2.1 Magnetic susceptibility

We consider energy fnctional with second order . The spin index is polarized at z axis. Hnece, the desnsity variation are diagonal.

$$f^a(p,p')\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta}\delta n_{\beta\alpha}(p)\cdot\delta n_{\delta\gamma}(p')\to f^a(p,p')\vec{\sigma}\cdot\vec{\sigma}\delta n_{p\sigma}\delta n_{p'\sigma'} \tag{4.8}$$

Hence, we can derive that

$$\delta\varepsilon^{(2)}=\frac{1}{2N_0V}F_0^a\sum\sigma\cdot\sigma'\delta n_{p\sigma}\delta_{p'\sigma'}=\frac{V}{2N_0}F_0^a(S_z)^2 \tag{4.9}$$

We introduce molecular field h_{mol} , which induces energy increment ΔV

$$\Delta V=--V\int h_{\text{mol}}\cdot dS_z\implies h_{\text{mol}}=-\frac{1}{V}\frac{\partial E}{\partial S}=-\frac{1}{N_0}F_0^aS_z \tag{4.10}$$

The total magnetic field is given by

$$h_{\text{tot}}=h_{\text{ex}}+h_{\text{mol}} \tag{4.11}$$

The total magnetization S_z and total field can be related with susceptibility χ_0

$$S_z=\chi_0h_{\text{tot}}=\chi(0)\left(h_{ex}-N_0^{-1}F_0^aS_z\right)\implies\chi=\frac{\chi_0}{1+\chi_0N_0^{-1}F_0^a} \tag{4.12}$$

4.2.2 Compressiblity

In this subsection, we will discuss another quantity , namly compressibility.

Note:-

The defnition of compressibility is

$$\chi_{\text{comp}}=-\frac{1}{V}\frac{\partial V}{\partial p} \tag{4.13}$$

We can use observale quantity n and μ to express the compressiblity.

$$V=\frac{N}{n}\qquad PV=N\mu \tag{4.14}$$

The compressiblity can be written as

$$\chi=-\frac{1}{N}\frac{d}{d\mu}\left(\frac{N}{n}\right)=\frac{1}{n^2}\frac{dn}{d\mu} \tag{4.15}$$

The density variation functional could be written as

$$\delta\varepsilon^{(2)}=\frac{1}{2N_0V}F_0^s\sum_{p,p'}\delta n_p\delta n_{p'}=\frac{V}{2N_0}F_0^s\sum(\delta n)^2 \tag{4.16}$$

By the same way, we can define molecular field

$$h_{\text{mol}}=-\frac{F_0^s\delta n}{N_0} \tag{4.17}$$

which means that

$$\chi=\frac{\chi_0}{1+F_0^s} \tag{4.18}$$

4.2.3 Effective mass

The effective mass is renormalized with p wave channel . We will derive the explicit form below.The density on the real space could be written as

$$n(r,t) = \sum_{\sigma} \int \frac{d^3p}{(2\pi)^3} n_p(r,t) \tag{4.19}$$

The current is given by

$$\vec{j}(r,t) = \sum_{\sigma} \int \frac{d^3p}{(2\pi)^3} \nabla_p \varepsilon(p,\sigma) n_{p\sigma}(r,t) \tag{4.20}$$

We take linear order approximation

$$\begin{cases} \varepsilon_{p\sigma}(r,t) = \varepsilon_p^0 + \int \frac{d^3p'}{(2\pi)^3} f_{\sigma\sigma'}(p,p')\delta n_{p'\sigma'} \\ n_{p\sigma} = n_{p,\sigma}^0 + \delta n_{p\sigma}(r,t) \end{cases} \tag{4.21}$$

We substitute the EQ(4.21) into (4.22)

$$\vec{j}(r,t) = \sum_{\sigma} \int \frac{d^3p}{(2\pi)^3} \nabla_p \left(\varepsilon_p^0 + \int \frac{d^3p'}{(2\pi)^3} f_{\sigma\sigma'}(p,p')\delta n_{p'\sigma'} \right) \left(n_{p,\sigma}^0 + \delta n_{p\sigma}(r,t) \right) \tag{4.22}$$

We remove the background current, then we have

$$\begin{aligned} \vec{j}(r,t) &= \sum_{\sigma} \int \frac{d^3p}{(2\pi)^3} \left[\nabla_p \varepsilon_p^0 \delta n_{p\sigma}(r,t) + \nabla_p \int \frac{d^3p'}{(2\pi)^3} f_{\sigma\sigma'}(p,p')\delta n_{p'\sigma'} n_{p,\sigma}^0 \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \left[\nabla_p \varepsilon_p^0 \delta n_{p\sigma}(r,t) - \int \frac{d^3p'}{(2\pi)^3} f_{\sigma\sigma'}(p,p')\delta n_{p'\sigma'} \nabla_p n_{p,\sigma}^0 \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \nabla_p \varepsilon_p^0 \delta n_{p\sigma}(r,t) - \int \frac{d^3p}{(2\pi)^3} \frac{\partial n_{p,\sigma}^0}{\partial \varepsilon} \vec{v}_F \int \frac{d^3p'}{(2\pi)^3} f_{\sigma\sigma'}(p,p')\delta n_{p'\sigma'} \\ &= \int \frac{d^3p}{(2\pi)^3} \nabla_p \varepsilon_p^0 \delta n_{p\sigma}(r,t) - \int \frac{d^3p}{(2\pi)^3} \frac{\partial n_{p,\sigma}^0}{\partial \varepsilon} v_F \frac{4\pi}{2l+1} \sum_l P_l(\cos\theta) P_l(\cos\theta') \int \frac{d^3p'}{(2\pi)^3} \delta n_{p'\sigma'} \\ &= \int \frac{d^3p}{(2\pi)^3} \vec{v}_F \left(1 + \frac{F_0^s}{3} \right) \implies \frac{1}{m^*} = \frac{1}{m} \int \frac{d^3p}{(2\pi)^3} \end{aligned} \tag{4.23}$$

4.2.4 Arbitrary channel constribution

We define the radial density by integrate out momentum p

$$\delta n = V \int \frac{d^3p}{(2\pi)^3} \delta n(p) = V \int \frac{p^2 dp}{(2\pi)^3} \delta n(p) \int d\Omega = V \int d\Omega \delta n(\Omega) \tag{4.24}$$

The angular density distribution could be expand into normal modes

$$\delta n(\Omega) = \sum_{l,m} n_{l,m} Y_{l,m}(\Omega) \tag{4.25}$$

The kinetic increment could be decomposed into normal modes

$$
\begin{aligned}
\Delta E^2 &= \frac{1}{2V}\int \frac{dp^3}{(2\pi)^3} f_{\sigma,\sigma'}(p,p')\delta n_{p\sigma}\delta n_{p'\sigma'} \\
&= \frac{V}{2}\int d\Omega_p d\Omega_{p'} f_{\sigma,\sigma'}(p,p')\delta n(\Omega_p)\delta n(\Omega_p') \\
&= \frac{V}{2}N^{-1}(0)\int d\Omega_p d\Omega_{p'}\left(\sum_{l_1}\sum_{m_1=-l_1}^{l_1} F_l^s Y_{l_1m_1}(\Omega_p)\bar{Y}_{l_1m_1}(\Omega_p')\right)\left(\sum_{l_2}\sum_{m_2=-l_2}^{l_2} n_{l_2m_2}Y_{l_2m_2}(\Omega_p)\right) \\
&\left(\sum_{l_2}\sum_{m_3=-l_3}^{l_3} n_{l_3m_3}Y_{l_3m_3}(\Omega_p')\right) + (s\leftrightarrow a) \\
&= \frac{V}{2}N^{-1}(0)\sum_{l}^{\infty}\frac{4\pi}{2l+1}\sum_{m=-l}^{l} F_l^s \mid \delta n_{lm}\mid^2 + (s\leftrightarrow a)
\end{aligned}
\tag{4.26}
$$

We consider the first order increment

$$
\delta E^{(0)} = \sum_p \varepsilon_p \delta n_p = \int \frac{d^3p}{(2\pi)^3}\varepsilon_p\delta n_p \tag{4.27}
$$

4.2.5 Pomeranchuk instability

4.3 The Boltzmann equation and zero sound

4.3.1 Boltzmann equation

We start from Boltzmann equation. The particle density on the phase space is described by distribution function $f(\mathbf{r},\mathbf{p},t)$. In the other words ,

$$
f(\mathbf{r},\mathbf{p},t)d\mathbf{r}^3 d\mathbf{p}^3 \tag{4.28}
$$

Due to occurence of collisions, the particle number lying on the phase space will change . We consider the particle variation on the phase space. The Liouville theorem tells us phase space volumme conservation.

$$
\Delta N_{\text{Collision}} = (f(\mathbf{r}+\Delta\mathbf{r},\mathbf{p}+\Delta p,t) - f(\mathbf{r},\mathbf{p},t))\,d\mathbf{r}^3 d\mathbf{p}^3 \tag{4.29}
$$

The total differential of f is

$$
df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}\frac{\partial p}{\partial t}dt + \frac{\partial f}{\partial p}\frac{\partial p}{\partial t}dt = (\frac{\partial f}{\partial t} + \vec{v}\cdot\nabla_{\mathbf{r}}f + \vec{F}\cdot\nabla_{\mathbf{p}}f)dt \tag{4.30}
$$

We can see that the particle number flow comprise real space flow and momentum space flow. The total flow is equal to collison section. The collison section consists of forward process and reverse process.

$$
I = \int d^3\mathbf{r}d^3\mathbf{p}I(\Omega)(f(\mathbf{r}_1',\mathbf{p}_1',t)f(\mathbf{r}_1',\mathbf{p}_1',t) - f(\mathbf{r}_1,\mathbf{p}_1,t)f(\mathbf{r}_1,\mathbf{p}_1,t)) \tag{4.31}
$$

The $I(g,\Omega)$ is scattering section , which can be determined by Fermin golden rule. We consider the fermion system, the density distribution has very strong limit in virtue of Pauli principle.

$$
I = \frac{1}{V^2}\sum \mid \langle 3,4\mid V\mid 1,2\rangle\mid^2 \delta_{p_1+p_2=p_3+p_4}\delta_{\sigma_1+\sigma_2=\sigma_3+\sigma_4}\delta_{\varepsilon_1+\varepsilon_2=\varepsilon_3+\varepsilon_4}\left(n_1n_2(1-n_3)(n_4) - (1-n_2)(1-n_2)n_3n_4\right) \tag{4.32}
$$

Note:-

If we consider random approximation , then the collision section will turns into

$$\langle I\rangle = -\frac{\delta N}{\tau} \tag{4.33}$$

Hence, the solution will recover into equibrilium distrition gradually.

$$N(T) = N_0(1 - e^{-\frac{T}{\tau}}) \tag{4.34}$$

4.4 Zero sound

We consider collisonless cases . Hence, the Boltzman equation could be written as

$$\frac{\partial n(\mathbf{r},\mathbf{p},t)}{\partial t} + \nabla_{\mathbf{p}}\varepsilon(\mathbf{p},t)\nabla_{\mathbf{r}}n(\mathbf{r},\mathbf{p},t) - \nabla_{\mathbf{r}}\varepsilon(\mathbf{p},t)\nabla_{\mathbf{p}}n(\mathbf{r},\mathbf{p},t) = 0 \tag{4.35}$$

It's self-evident that Eq(4.35) is a nonlinear equation . We expand the density distribution and energy functional.

$$\begin{cases} \varepsilon(\mathbf{r},\mathbf{p},t) = \varepsilon_0(p) + \dfrac{1}{V}\displaystyle\sum_{p'} f^s(\mathbf{p},\mathbf{p}')\delta n_{\mathbf{p}'}(\mathbf{r},t) \\ n(\mathbf{r},\mathbf{p}) = n_0(\mathbf{r},\mathbf{p}) + \delta n(\mathbf{r},\mathbf{p},t) \end{cases} \tag{4.36}$$

At the equibrillium state, the density distrbution and energy functional don't rely on space position $\mathbf{r}$. The Boltzmann equation can be expanded into first order in relative with $\delta(\mathbf{r},\mathbf{p},t)$.

$$\frac{\partial \delta n(\mathbf{r},\mathbf{p},t)}{\partial t} + \vec{v}_{\mathbf{p}}\cdot\nabla_{\mathbf{r}}\delta n(\mathbf{r},\mathbf{p},t) - \nabla_{\mathbf{p}}\varepsilon\cdot\frac{1}{V}\sum_{p'} f^s(p.p')\nabla_{\mathbf{r}}\delta n(\mathbf{r},\mathbf{p}',t) = 0 \tag{4.37}$$

We insert relation $\nabla_{\mathbf{p}}\varepsilon = \frac{\partial\varepsilon}{\partial\varepsilon}\nabla_{\mathbf{p}}\varepsilon$ into Eq(**??**). The Boltzmann equation will reduce into

$$\frac{\partial \delta n(\mathbf{r},\mathbf{p},t)}{\partial t} + \vec{v}_{\mathbf{p}}\left(\cdot\nabla_{\mathbf{r}}\delta n(\mathbf{r},\mathbf{p},t) - \frac{\partial n}{\partial\varepsilon}\cdot\frac{1}{V}\sum_{p'} f^s(p.p')\nabla_{\mathbf{r}}\delta n(\mathbf{r},\mathbf{p}',t)\right) = 0 \tag{4.38}$$

We expand the density fluactuation into Fourier modes, namly

$$\delta n(\mathbf{r},\mathbf{p},t) = \sum_q \delta n(\mathbf{p})e^{\mathrm{i}(\vec{q}\cdot\vec{\mathbf{r}}-\omega t)} \tag{4.39}$$

where wavevector $\vec{q}$ is small . We insert Fourier mode into Eq(4.39)

$$(\omega - \vec{q}\cdot\vec{v}_F)\delta n(\mathbf{p}) + (\vec{v}_F\cdot\vec{q})\frac{\partial n}{\partial\varepsilon}\frac{1}{V}\sum_{p'} f^s(\mathbf{p},\mathbf{p}')\delta n(\mathbf{p}) = 0 \tag{4.40}$$

We define the dimensionless quantty $s = \frac{\omega}{qv_F}$ and choose direction of $\vec{v}_F$ as z axis. Now we integrate out the radial part by $\int \frac{p^2dp}{2\pi}$

$$(s-\cos\theta)\delta\hat{n}(\Omega) - \cos\theta \underbrace{\int \frac{p^2 dp}{2\pi}\frac{\partial n}{\partial \varepsilon}}_{\frac{N(0)}{4\pi}} \frac{1}{V}\sum_{p'} f^s(\mathbf{p},\mathbf{p}')\delta n(\mathbf{p}) = 0 \tag{4.41}$$

The osccilation on the Fermi surface is tesor wave. The Fermi surface oscilation is transfered by Landau ineraction. Hence, we consider decomposing the desity $\hat{n}(\Omega)$ into $SO(3)$ irreucible tensors.

$$\frac{1}{V}\sum_{p'} f^s(\mathbf{p},\mathbf{p}')\delta n(\mathbf{p}) = \int d\Omega' \left(F_l^s \sum_l \sum_{m=-l}^{l} \frac{4\pi}{2l+1} Y_{lm}(\Omega)\bar{Y}_{lm}(\Omega')\right)\left(\sum_{l'}\sum_{m=-l'}^{l'} u_l' Y_{l'm'}(\Omega')\right) \tag{4.42}$$

$$= \sum_{l'} \frac{4\pi}{2l'+1} u_{l'} F_{l'}^s Y_{l'm}(\Omega) \tag{4.43}$$

where Ω is the solid angle expanded by momentum p and Ω' expanded by Ω'. We insert Eq(4.43) into Eq(4.41) to derive such identity

$$\sum_{l'}\sum_{m=-l'}^{l'} n_{l'} Y_{l'm} - \sum_{l'}\sum_{m=-l'}^{l'} \frac{1}{2l'+1}\frac{\cos\theta}{s-\cos\theta} u_{l'} F_{l'}^s Y_{l'm}(\Omega) = 0 \tag{4.44}$$

We can see from Eq(4.44) that the angular momentum m can be viewed as internal gauge. Hence, We fix m to zero. Now, we can derive from Eq(4.44)

$$\frac{u_l}{\sqrt{(2l+1)}} - \sum_{l'} \frac{1}{\sqrt{(2l'+1)(2l+1)}} \int d\Omega \frac{\cos\theta}{s-\cos\theta} Y_{l0}(\Omega) Y_{l'0}(\Omega) F_{l'}^s \frac{u_{l'}}{\sqrt{2l'+1}} = 0 \tag{4.45}$$

We define the integral $Omega_{ll'}$ as below.

$$\Omega_{ll'} = -\frac{1}{\sqrt{(2l'+1)(2l+1)}} \int d\Omega \frac{\cos\theta}{s-\cos\theta} Y_{l0}(\Omega) Y_{l'0}(\Omega) F_{l'}^s \tag{4.46}$$

We consider the zero order of Eq(4.45).

$$u_0 + \Omega_{00} F_0^s = 0 \implies \frac{1}{F_o^s} = -\Omega_{00} \tag{4.47}$$

Note:-

We give some details about calculation of Ω_{00}

$$\begin{aligned}
\Omega_{00} &= -\int \frac{d\Omega}{4\pi}\frac{\cos\theta}{s-\cos\theta} = -\frac{1}{2}\int_{-1}^{1}\frac{x dx}{s-x} = -\frac{1}{2}\int_{-1}^{1}\left[x\mathcal{P}(\frac{1}{s-x}) + \mathrm{i}\pi\delta(s-x)\right]dx \\
&= -\frac{1}{2}\int_{-1}^{1}\left[\frac{s}{s-x} - 1 + \mathrm{i}\pi\delta(s-x)\right]dx \\
&= -\frac{s}{2}\log\frac{s+1}{s-1} + 1 - \mathrm{i}\frac{\pi}{2}\Theta(\mid s^2-1 \mid)
\end{aligned} \tag{4.48}$$

The solution of the Eq(4.47) could be depicted in Fig (4.4)

Note:-

We consider two limits

- $s \to 1^+$

$$-\frac{s}{2}\log\left|\frac{s+1}{s-1}\right| + 1 \approx 1 + \log\left|\frac{s-1}{2}\right| = \frac{!}{F_o^s} \implies s \approx 1 + e^{-\frac{2}{F_o^s}} \tag{4.49}$$

- $s \to \infty$

$$\frac{s}{2}\log\left|\frac{s-1}{s+1}\right| = \frac{s}{2}\left(-\frac{1}{s} - \frac{1}{2s^2} - \frac{1}{3s^3} - \frac{1}{s} + \frac{s^2}{2} - \frac{1}{3s^3}\right) + 1 = -\frac{1}{3s^2} \implies s = \sqrt{\frac{F_0^s}{3}} \tag{4.50}$$

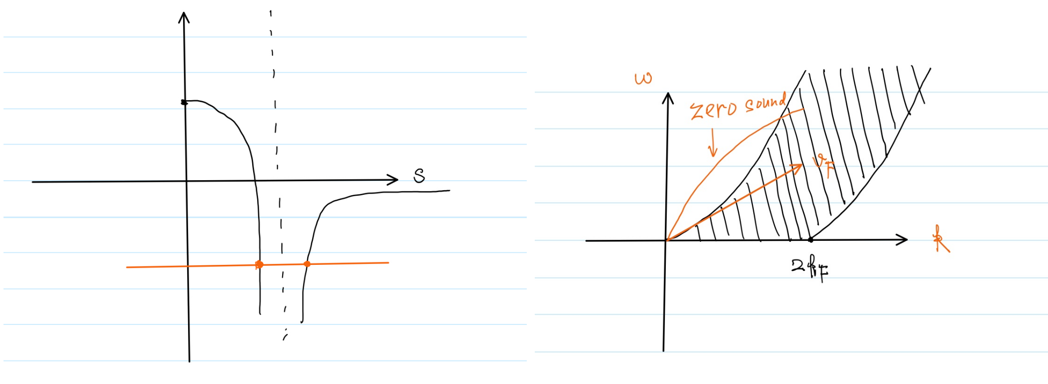

Fig 4.4

The real physcial solution lies on region $s > 1$. Otherwise, it will collapse into particle-hole continnum.