Chapter 5

Superconductivity

In this chapter, we will focus on microscopic theory of superconductivity.

5.1 BCS theory

5.1.1 Cooper problem

Cooper consider two body problem with attractive interaction. Fermi gas has stableFfermi surface. The exsitence of Fermi surface will exert strong restriction to electron scattering. We consider the two electrons scattering process. This process demands momentum conservation , namly

$$\vec{k}_1' + \vec{k}_2' = \vec{k}_1 + \vec{k}_2 \tag{5.1}$$

In virtue of Pauli principle , the process shown on Fig(13.1) is nor permitted . However, we the electrons on the Fermi surface tend to form process shown on Fig(13.1).

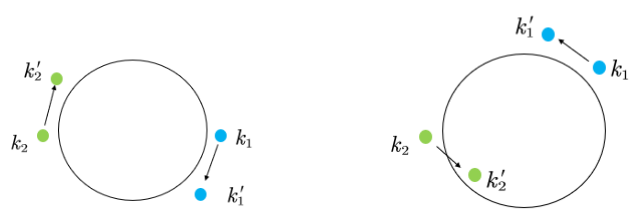

Figure 5.1: The first pannel shows the scattering of two electron with opposite momentum. The second pannel shows the general scattering process.

The two electrons with opposite momentum has more scattering space. Hence, the system could be descried as

$$H = H_0 + H_I = \sum_{k,\sigma} \varepsilon_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma} - \frac{g}{V} \sum_{k,k'} c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger c_{-k'\downarrow} c_{k'\uparrow} \tag{5.2}$$

We consider the wavefunction could be expressed as

$$|\,\psi\rangle = \sum_k a(k) c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \,|\,\mathrm{FS}\rangle \tag{5.3}$$

Combing with Eq(5.21) and (5.3), we could be write down the eigenequation

$$H \mid \psi\rangle = E \mid \psi\rangle \implies (2\varepsilon_k + E_0)a(k) - \frac{g}{V}\sum_{k'} a(k') = Ea(k) \tag{5.4}$$

We can slove the Δ from the consistent equation (5.4)

$$\Delta E = -2\hbar\omega_D \exp\left(-\frac{2}{N(0)g}\right) \tag{5.5}$$

Note:-

We make consistent equation for Eq(5.4)

$$\frac{a(k)}{\sum_{k'} a(k')} = \frac{g/V}{2\varepsilon_k + E_0 - E} \tag{5.6}$$

We make summation for k

$$1 = \sum_k \frac{g/V}{2\varepsilon_k + E_0 - E} \simeq \frac{g}{N(0)}\int_0^{\hbar\omega_D} d\varepsilon \frac{1}{2\varepsilon - \Delta E} = \frac{g}{2}N(0)\log\left(\frac{2\hbar\omega_D - \Delta E}{-\Delta E}\right) \tag{5.7}$$

We could summarize from the result (5.5) that it will form bounded state with lower energy than oringin fermi surface if we conisder two electron on the Fermi surface with attractive interaction. This phenomena is also call *Cooper instablity.*

5.1.2 BCS wavefunction

Schrieffer generalize the single Cooper pair to magny body wavefunction. Let's consider N Coopr pair, the single Cooper pair wavefunction could be described by $\psi(r_1, r_2, \sigma_2, \sigma_2)$. The BCS wavefunction could be written into

$$\Psi_{\text{BCS}} = \mathcal{A}\left(\psi(r_1, r_2, \sigma_1, \sigma_2)\cdots\psi(r_{2N-1}, r_{2N}; \sigma_{2n-1}, \sigma_{2n})\right) \tag{5.8}$$

where $\mathcal{A}$ is the anti-symmetric operation. We write down the single Cooper pair function

$$\begin{aligned}
\psi(r_1, r_2; \sigma_1, \sigma_2) &= \phi(\mid r_1 - r_2 \mid) \otimes \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \\
&= \sum_k \chi(k) e^{\mathrm{i}k(r_1 - r_2)} \otimes \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \\
&= \sum_k \chi(k) c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} \mid \text{Vac}\rangle
\end{aligned} \tag{5.9}$$

Hence, the many body wavefunction could be written into

$$\mid \Psi_{\text{BCS}}\rangle = \mathcal{N}^{-\frac{1}{2}}\left((\chi(k)c_{k\uparrow})c_{-k\downarrow}\right)^{\frac{N}{2}} \mid \text{vac}\rangle \tag{5.10}$$

The wavefunction (8.9) is written at canonical ensemble . We generalize the wavefunction into grand canonical ensemble

$$\mid \psi_{\text{BCS}}\rangle = \exp\left(\chi(k)c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}\right) \mid \text{vac}\rangle = \prod_k (1 + \chi(k)c^\dagger_{k\uparrow}c^\dagger_{-k\downarrow}) \mid \text{vac}\rangle \tag{5.11}$$

Hence, the many body wavefunction is decomposed into single particel wavefunction product state. Furthermore , we introduce the varitional parameter u_k, v_k to write the wavefunction into

$$| \psi_{BCS}(\phi)\rangle = \prod_k (| u_k | + | v_k | e^{\mathrm{i}\phi} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow}) | \mathrm{vac}\rangle \tag{5.12}$$

The varitional parameter u_k and v_k control the component of superconductivity. The BCS wavefunction is also coherent state , which means coherence of wavefunction phase.

We could project out the N particle wavefunction from Eq(5.12)

$$\psi_{BCS}(N) = \int_0^{2\pi} e^{-\mathrm{i}\frac{N}{2}\phi} \psi_{\mathrm{BCS}}(\phi) \tag{5.13}$$

5.1.3 BCS wavefunction varitation

We have write down the BCS wavefunction on the previous section . The next step is to optimize BCS wavefunction . We substitute the BCS wavefunction into hamiltonian (5.21)

$$E = \langle \psi_{BCS} | H | \psi_{BCS}\rangle = 2\sum_k \varepsilon_k | v_k |^2 - \frac{g}{V} \sum_{k_1,k_2} g(k_1,k_2) u^*_{k_1} v_{k_1} u_{k_1} v^*_{k_2} \tag{5.14}$$

The Eq(**??**) shows that the varitional parameter on channel k_1, k_2 should be matched with phase to gaurantee real energy. Now we can introduce the θ_k to describe variational parameter, namly $u_k = \cos\theta_k, v_k = \sin\theta_k$.

$$E = 2\sum_k \varepsilon \sin^2\theta_k - \frac{1}{V}\sum_{k_1,k_2} g(k_1,k_2) \sin\theta_1 \cos\theta_1 \sin\theta_2 \cos\theta_2 \tag{5.15}$$

We make variation for parameter θ_k

$$2\varepsilon_k \sin 2\theta_k - \frac{1}{V}\sum_{k'} g(k_1,k_2) \sin\theta_{k'} \cos\theta_{k'} \cos\theta_k = 0 \tag{5.16}$$

We solve the consitent equation (5.16) to derive gap function at zero temperature

$$\Delta = \hbar\omega \exp\left(-\frac{1}{gN(0)}\right) \tag{5.17}$$

Note:-

We construct consistent equation from Eq(5.16)

$$\tan 2\theta_k = \frac{\Delta}{\varepsilon_k} \tag{5.18}$$

where $\Delta(k_1) = \frac{1}{2V}\sum g(k_1,k_2)\sin 2\theta_{k_2}$. We sustitute the relation (8.17) into Δ_k

$$\Delta_k = \frac{1}{V}\sum_k \frac{\Delta_k}{2\xi_k} \simeq gN(0) \int_{-\hbar\omega_D}^{\hbar\Omega_D} \frac{1}{\sqrt{\Delta^2+\varepsilon^2}} d\varepsilon = 2\int_0^{\frac{\hbar\omega_D}{\Delta}} \frac{1}{\sqrt{1+x^2}} dx \tag{5.19}$$

5.1.4 Mean field theory

We use mean field theory to deal with hamiltonian (5.21) . We define supercondutor order parameter as $\Delta = \frac{g}{V}\sum_k \langle c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger \rangle$

$$\mathcal{O}_1\mathcal{O}_2 = \mathcal{O}_1\left(\mathcal{O}_2 - \langle\mathcal{O}_2\rangle + \langle\mathcal{O}_2\rangle\right) = \mathcal{O}_1\langle\mathcal{O}_2\rangle + \left(\mathcal{O}_1 - \langle\mathcal{O}_1\rangle + \langle\mathcal{O}_1\rangle\right)\left(\mathcal{O}_2 - \langle\mathcal{O}_2\rangle\right) \approx \mathcal{O}_1\langle\mathcal{O}_2\rangle + \langle\mathcal{O}_1\rangle\mathcal{O}_2 - \langle\mathcal{O}_1\rangle\langle\mathcal{O}_2\rangle \tag{5.20}$$

Hence, we derive effective mean field hamiltonian

$$H = \sum_{k,\sigma} \varepsilon_{k\sigma} c_{k\sigma}^\dagger c_{k\sigma} - \sum_k \Delta c_{-k\downarrow} c_{k\uparrow} - \sum c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger + \frac{g}{V}\sum_k \mid \Delta_k \mid^2 \tag{5.21}$$

The last term is called condensation energy. We introduce Numbu spinor $\psi = (c_{k\uparrow}, c_{-k\downarrow}^\dagger)^{\mathbf{T}}$

$$H = \sum_k (c_{k\uparrow}, c_{-k\downarrow}^\dagger)\begin{pmatrix} \varepsilon_k & -\Delta_k \\ -\Delta_k^* & -\varepsilon_k \end{pmatrix}\begin{pmatrix} c_{k\uparrow} \\ c_{-k\downarrow}^\dagger \end{pmatrix} + E_0 + \frac{g}{V}\sum_k \mid \Delta_k \mid^2 \tag{5.22}$$

where $E_0 = \sum_k \varepsilon_k$. We use Bogliubov transformation to diagonalize hamiltonian (5.22) . This part is easy to do. I leave it for somple task.

$$\begin{pmatrix} \beta_{k\uparrow} \\ \beta_{-k\downarrow}^\dagger \end{pmatrix} = \begin{pmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{pmatrix}\begin{pmatrix} c_{k\uparrow} \\ c_{k\downarrow}^\dagger \end{pmatrix} \qquad \tanh 2\theta_k = \frac{\Delta}{\varepsilon_k} \qquad \cos^2\theta_k = \frac{1}{2}\left(1 + \frac{\varepsilon_k}{E_k}\right) \qquad \sin^2\theta_k = \frac{1}{2}\left(1 - \frac{\varepsilon_k}{E_k}\right) \tag{5.23}$$

The hamiltonian turns into

$$H = \sum_k E_k\left(\beta_{k\uparrow}^\dagger\beta_{k\uparrow} + \beta_{-k\downarrow}^\dagger\beta_{-k\downarrow} - 1\right) \qquad E_k = \sqrt{\varepsilon_k^2 + \Delta^2} \tag{5.24}$$

Hence, we could written the gap function into Bogliubov quasiparticle

$$\begin{aligned}
\Delta = \frac{g}{V}\sum_k c_{k\uparrow}^\dagger c_{-k\downarrow}^\dagger &= \frac{g}{V}\sum_k (\cos\theta_k\beta_{k\uparrow} + \sin\theta_k\beta_{-k\downarrow})(\cos\theta_k\beta_{-k\downarrow} - \sin\theta_k\beta_{-k\uparrow}) \\
&= \frac{g}{V}\sum_k \sin\theta_k\cos\theta_k\langle\beta_{-k\downarrow}\beta_{-k\downarrow}^\dagger - \beta_{k\uparrow}^\dagger\beta_{k\uparrow}\rangle \\
&= \frac{gN(0)\Delta}{2}\int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{\tanh\frac{\beta\varepsilon_k}{2}}{\varepsilon}d\varepsilon \\
&= gN(0)\Delta\int_0^{\frac{\hbar\omega_D}{2k_BT_C}} \frac{\tanh x}{x}dx \\
&= gN(0)\Delta\left(\log\frac{\hbar\omega_D}{2k_BT_c}\tanh\frac{\hbar\omega_D}{2k_BT_c} - \int_0^{\frac{\hbar\omega_D}{2k_BT_c}} \log x\,\mathrm{sech}^2 x dx\right) \\
&= gN(0)\Delta\log\frac{2e^\gamma\hbar\omega_D}{\pi k_BT_c}
\end{aligned} \tag{5.25}$$

At the critical temperature, the gap is vanishing. We have that $k_BT_c = 1.13h\omega_D\exp\left(-\frac{1}{gN(o)}\right)$. We can derive the relation. between critical temperation and superconductor gap . We use the integral identiy (5.28) on the last step . We could derive the relation between critical temperature and gap function

$$\frac{\Delta}{k_K T_c} \approx 1.76 \tag{5.26}$$

The Eq(5.25) also tells us that the gap function is

$$\Delta = gV \sum_k \frac{\Delta}{2E_k} \tanh \frac{\beta E_k}{2} \tag{5.27}$$

Claim 5.1 .

$$\int_0^\infty \log x \mathrm{sech}^2 x dx = \log \frac{\pi}{4} - \gamma \tag{5.28}$$

We consider the integral below

Proof.

$$\int_0^\infty x^a \mathrm{sech}^2 x dx = \int_0^{+\infty} dx \frac{4e^{-2x}}{(1+e^{-2x})^2} x^a = 4 \sum_{n=0}^{\infty} \int_0^\infty x^a (-1)^{n-1} e^{-2nx} dx = \frac{2\Gamma(a+1)}{2^a} \eta(a) \tag{5.29}$$

The integral (5.28) is equal to

$$\frac{d}{da} \left(\frac{2\Gamma(a+1)}{2^a} \eta(a) \right) \bigg|_{a=0} = (\Gamma'(1)\eta(0) + \eta'(0) - \log 2\eta(0)) = \log \frac{\pi}{4} - \gamma$$

□

5.2 Thermodynamic quantity

5.2.1 Condensation energy

We could find from hamiltonian (5.21) that

$$\begin{aligned} H &= \sum_{k,\sigma} \varepsilon_{k\sigma} c^\dagger_{k\sigma} c_{k\sigma} - \sum_k \Delta c_{-k\downarrow} c_{k\uparrow} - \sum c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + \frac{g}{V} \sum_k \mid \Delta_k \mid^2 \\ &= \sum_k \xi_k \left(\beta^\dagger_k \beta_k + \beta^\dagger_{-k} \beta_{-k} \right) + (\varepsilon_k - \xi_k) + \frac{g}{V} \sum_k \mid \Delta_k \mid^2 \end{aligned} \tag{5.30}$$

The constant term on the (5.30) is just condensation energy

$$\begin{aligned} E_{\text{cond}} &= \sum_k (\varepsilon_k - \xi_k) + \frac{g}{V} \mid \Delta_k \mid^2 = \sum_k (\varepsilon_k - \sqrt{\varepsilon_k^2 + \Delta^2}) + \frac{\Delta^2}{2\xi_k} \\ &= V \frac{1}{V} \sum_k \left(\varepsilon_k - \frac{\varepsilon_k^2 + \Delta^2/2}{\sqrt{\varepsilon_k^2 + \Delta^2}} \right) \\ &= VN(0) \int_{-\hbar\omega_D}^{\hbar\omega_D} d\varepsilon \left(\varepsilon - \frac{\varepsilon^2 + \Delta^2/2}{\sqrt{\varepsilon^2 + \Delta^2}} \right) \\ &= -VN(o)\Delta^2 \end{aligned} \tag{5.31}$$

We insert the gap function (5.19) on the first line of (5.31), the details about step are given below . The condensation energy relies on density states and superconductor gap .

Note:-

$$\int_{-\hbar\omega_D}^{\hbar\omega_D} \frac{\varepsilon^2+\Delta^2/2}{\sqrt{\varepsilon^2+\Delta^2}} d\varepsilon = \Delta^2 \int_{-\frac{\hbar\omega_D}{\Delta}}^{\frac{\hbar\omega_D}{\Delta}} \frac{x^2+\frac{1}{2}}{\sqrt{1+x^2}} dx = \Delta^2 \int_0^{\frac{\hbar\omega}{\Delta}} \frac{2x^2+1}{\sqrt{1+x^2}} dx = \Delta^2 \frac{\hbar\omega}{\Delta} \cdot \sqrt{1+\left(\frac{\hbar\Omega_D}{\Delta}\right)}$$
$$\approx 2(\hbar\omega_D)^2 \left(1+\frac{1}{2}\left(\frac{\Delta}{\hbar\omega_D}\right)^2\right) \tag{5.32}$$

5.2.2 Specfic heat

Firstly , Let us review the statistial mechanics. The free energy for fermionic system is given by

$$F = -\sum_k \frac{1}{\beta} \log(1+e^{-\beta\varepsilon_k}) \tag{5.33}$$

The entropy could be obatained from free energy as

$$\begin{aligned} S = -\frac{\partial F}{\partial T} &= \sum_k \frac{\partial}{\partial T}\left(k_B T \log(1+e^{-\beta\varepsilon_k})\right) = -\sum_k \frac{\partial}{\partial T}\left(k_B T\log(1-f_k)\right) \\ &= -\sum_k k_B\left((1-f_k)\log(1-f_k) + f_k \log(1-f_k) - T\frac{1}{1-f_k}\frac{\partial f_k}{\partial T}\right) \\ &= -\sum_k k_B\left((1-f_k)\log(1-f_k) + f_k\log f_k\right) \end{aligned} \tag{5.34}$$

The capcity could be derived from entropy

$$\begin{aligned} C_V = T\frac{\partial S}{\partial T} = -\beta\frac{\partial S}{\partial \beta} &= -2\beta\left(\log\frac{f_k}{1-f_k}\frac{\partial f_k}{\partial\beta}\right) = -2\beta^2\sum_k \xi_k \frac{\partial f_k}{\partial \beta} \\ &= -2\beta^2 \sum_k \xi_k \frac{\partial f_k}{\partial(\beta\xi_k)}\left(\xi_k + \frac{\partial \xi_k}{\partial\beta}\right) \end{aligned} \tag{5.35}$$

We consider the sin freedom, then the entropy need to multiply by factor 2. The specific heat on the (5.35) consists of two part. The first part is just specific heat of normal metal when the temperature T is near critical temperature T_c

$$c_n = 2\beta\sum_k\left(-\frac{\partial f_k}{\partial \xi_k}\right)\xi_k^2 = 2k_B^2 T N(0)\int_{\frac{\hbar\omega_D}{k_BT}}^{\frac{\hbar\omega_D}{k_BT}} \frac{x^2e^x}{(1+e^x)^2}dx \approx 4k_B N(0) T\int_{-\infty}^{+\infty}\frac{x^2e^x}{(1+e^x)^2}dx \sim \frac{2\pi^2}{3}N(0)k_B^2 T \tag{5.36}$$

Note:-

We calculate the integral on the (5.36)

$$\int_0^{+\infty}\frac{x^2e^x}{(1+e^x)^2} = \int_0^\infty dx\frac{x^2e^{-x}}{(1+e^{-x})^2} = \sum_{n=1}^{\infty}\int_0^\infty x^2e^{-nx}dx == \Gamma(3)\eta(2) = \frac{\pi^2}{6} \tag{5.37}$$

At the critical temperature, the second term gives the specific heat between superconductor state and normal state.

$$c_n - c_s = -2\beta^2 \sum_k \xi_k - \frac{\partial f_k}{\partial(\beta\xi_k)} \frac{\partial \xi_k}{\partial \beta} = -\beta^2 N(0) \frac{d\Delta^2}{\partial \beta}\bigg|_{T=T_c} = k_B N(0) \frac{d\Delta^2}{dT}\bigg|_{T=T_c} \tag{5.38}$$

We can see that the specific heat is not continuous at T_c . Hence, this is second order phase transition. At the low temperature region $T \ll T_c$, We neglect the second term

$$c_{es} = 2k_B\beta \sum_k \left(-\frac{\partial f_k}{\partial \xi_k}\right) \xi_k^2 = 2\beta N(0) k_B \int_{-\infty}^{+\infty} \xi^2 e^{-\beta\xi} d\xi = 2\frac{\Delta^2(0)}{T} N(0) e^{-\frac{\Delta(0)}{k_B T}} \int_{-\infty}^{\infty} e^{-\frac{\varepsilon^2}{2k_B T \Delta(0)}}$$
$$= 2\frac{\Delta^2(0)}{T} N(0) \left(\frac{2\pi\Delta(0)}{k_B T}\right)^{0.5} e^{-\frac{\Delta(0)}{k_B T}} \tag{5.39}$$

5.2.3 Gap function dependdence on temperature

We start gap function (5.27) directly

$$1 = N(0)V \int_0^{\hbar\omega_D} \frac{\tanh \frac{1}{2}\beta\xi}{\xi} d\varepsilon = \frac{2N(0)V}{\beta} \int_0^{\infty} \sum_{n=-\infty}^{n+\infty} \frac{1}{\omega_n^2 + \xi^2} d\xi \approx \frac{2N(0)V}{\beta} \sum_{n=-\infty}^{n+\infty} \int_0^{\infty} \left(\frac{1}{\omega_n^2 + \varepsilon^2} - \frac{\Delta(T)^2}{(\omega_n^2 + \varepsilon^2)^2} + \cdots\right)$$
$$= N(0)V \left[\int_0^{\hbar\omega} \frac{\tanh \frac{1}{2}\beta h\varepsilon}{\varepsilon} d\varepsilon - \sum_{n=-\infty}^{n+\infty} \frac{2}{\beta} \int_0^{\hbar\omega} \frac{\Delta(T)^2}{(\omega_n^2 + \varepsilon^2)^2}\right]$$
$$= N(0)V \left[\int_0^{\hbar\omega} \frac{\tanh \frac{1}{2}\beta h\varepsilon}{\varepsilon} d\varepsilon - \sum_{n=-\infty}^{n+\infty} \frac{2}{\beta} \int_0^{+\infty} \frac{\Delta(T)^2}{(\omega_n^2 + \varepsilon^2)^2}\right]$$
$$= N(0)g \left[\log \frac{2e^{\gamma}\hbar\omega_D}{\pi k_B T} - \left(\frac{\Delta^2(T)}{\pi k_B^2 T^2}\right)^2 \sum_{n=0}^{n+\infty} \frac{1}{(n+1)^2}\right] \tag{5.40}$$

We use the integral () on the last step . We substitute (5.25) into (5.40)

$$\Delta(T) = \pi k_B T \sqrt{\frac{T_c - T}{\eta(3) T_c}} \tag{5.41}$$

Note:-

The Fermionic distribution function could be expressed into Poisson summation ,namly

$$f(\varepsilon) = \sum_{n=-\infty}^{+\infty} \frac{1}{\beta} \frac{1}{\mathrm{i}\omega_n - \varepsilon} \tag{5.42}$$

where the ω_n is he Matsubara frequency (7.19) .

$$\tanh \frac{\beta\varepsilon}{2} = \frac{e^{\beta\varepsilon} - 1}{e^{\beta\varepsilon} + 1} = f(-\varepsilon) - f(\varepsilon) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \frac{1}{\mathrm{i}\omega_n + \varepsilon} - \frac{1}{\mathrm{i}\omega_n - \varepsilon} = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \frac{2\varepsilon}{\omega_n^2 + \varepsilon^2} \tag{5.43}$$

$$\int_0^{+\infty} \frac{1}{(1+x^2)^2} dx = \frac{1}{2} \int_0^{+\infty} \frac{x^{-0.5}}{(1+x)^2} = \frac{1}{2} B(0.5, 1.5) = \frac{\pi}{4} \tag{5.44}$$

5.3 Susceptibility

5.4 Single particle tunneling

If we consider two system connected each other, the system could be described by

$$H = H_R + H_L + H_T \tag{5.45}$$

The tunneling hamiltonian could be described as

$$H_T = \sum_k T_{kq} c^\dagger_{kR} c_{qL} + h.c \tag{5.46}$$

The current is defined as [1]

$$I = -\frac{2e}{\hbar} \Im \left(\sum_{k,q} T_{kq} c^\dagger_{kR,\sigma} c_{qL,\sigma} \right) \tag{5.47}$$

Claim 5.2 .

The density operator on the tunneling process is defined as

$$I = -\frac{2e}{\hbar} \Im \left(\sum_{k,q} T_{kq} c^\dagger_{kR,\sigma} c_{qL,\sigma} \right)$$

We can use motion equation to write down the current operator

$$\begin{aligned} I = -e\frac{d\langle N_L \rangle}{dt} &= -2e\frac{1}{\mathrm{i}\hbar}[N_L, H] \\ &= -\frac{2e}{\hbar}\sum_{kq\sigma} T_{kq}[c^\dagger_{Lk\sigma} c_{Lk\sigma}, T_{qk} c^\dagger_{qR\sigma} c_{Lk\sigma} + T_{kq} c^\dagger_{Lk\sigma} c_{Rq\sigma}] \\ &= -\frac{2e}{\hbar} \Im \left(\sum_{k,q} T_{kq} c^\dagger_{kR,\sigma} c_{qL,\sigma} \right) \end{aligned} \tag{5.48}$$

According to fluctuation theorem , we need to evaluate the response function of current operator.

$$G(\tau) = -\langle \mathcal{T}_\tau A(\tau) A^\dagger(0) \rangle \tag{5.49}$$

Note:-

Accoring to Wick theorem

$$\begin{aligned} G(\tau) &= -\sum_{k,q;k',q'} T_{kq} T_{k'q'} \langle \mathcal{T} c^\dagger_{Rk}(\tau) c_{Lq}(\tau) c^\dagger_{Rk'} c_{Lq'} + c^\dagger_{Lk}(\tau) c_{Rq}(\tau) c^\dagger_{Lk'} c_{Rq'} \rangle \\ &= -\sum_{k,q} \mid T_{kq} \mid^2 [G_L(q,\tau) G_R(k,-\tau) + G_R(k,\tau) G_L(q,-\tau)] \end{aligned} \tag{5.50}$$

We transform it into Matsubra representation

[1] Tou can refer to Claim (5.4)

$$G(\mathrm{i}\omega_n) = \int_0^\beta e^{\mathrm{i}\omega_n\tau} G(\tau) d\tau = -\sum_{k,q} \mid T_{kq} \mid^2 \int_0^\beta d\tau \left[G_L(q,\tau)G_R(k,-\tau) + L \to R\right] e^{\mathrm{i}\omega_n\tau}$$
$$-\sum_{k,q} \mid T_{kq} \mid^2 \left[\sum_n G(q, p_n+\omega_n)G(k,p_n) + L \to R\right] \tag{5.51}$$

5.5 Cohenrence factor

5.6 Electromagnetic response

5.6.1 Linear response

In this section we discus the eletromagnetic response to superconducotr. We use linear response theory to study paramagnetic curent and diamagnetic current. The gauge potential is coupled into kinetic energy term

$$H = \int d^3x \psi^\dagger(x) \frac{1}{2m}\left(-\mathrm{i}\hbar\nabla + \frac{e}{c}\vec{A}\right)^2 \psi(x) + \int d^3x \psi(x)^\dagger(x)\psi^\dagger(x')V(x'-x)\psi(x')\psi(x) \tag{5.52}$$

The hamiltonian relying on vector potential A is just

$$H_1 = \frac{\mathrm{i}e\hbar}{mc}\int d^x\psi^\dagger(x)\left(\vec{A}\cdot\nabla + \nabla\cdot\vec{A}\right)\psi(x) = \mathrm{i}\mu_B \int d^x\psi^\dagger(3)\left(\vec{A}\cdot\nabla + \nabla\cdot\vec{A}\right)\psi(x) \tag{5.53}$$

On the momentum space, we derive the hamiltonian (5.53)

$$H_1 = \frac{1}{2}\mathrm{i}\mu_B\int d^3\psi^\dagger(x)\left(\vec{A}\cdot\nabla + \nabla\cdot\vec{A}\right)\psi(x) = \frac{1}{2}\mathrm{i}\mu_B\int d^3x \sum_{k_1,k,k_2} c^\dagger_{k_1} e^{-\mathrm{i}k_1x}\left(\vec{A}(q)e^{\mathrm{i}qx}\cdot\nabla c_{k_2}e^{\mathrm{i}k_2x} + \nabla\cdot(\vec{A}(q)e^{\mathrm{i}qx}c_{k_2}e^{\mathrm{i}k_2x})\right)$$
$$= -\mu_B\sum_{k,q}\vec{A}(q)\cdot(\vec{k}+\frac{q}{2})c^\dagger_{k+q}c_k(\vec{k}) \tag{5.54}$$

The hamiltonian (5.54) could be viewed as perturbation . If we use Feymann diagram to express the (5.53), then it turns into

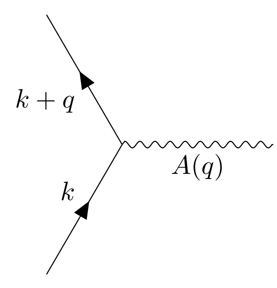

Figure 5.2: Electron souples with photon at vertex . Every vertex contributes to factor μ_B

Furthermore, we consider take Columb gauge $\nabla\cdot\vec{A} = 0$, then the perturbation hamiltonian (5.54) becomes into

$$H_I = -\frac{1}{2}\mu_B\sum_{k,q}(\vec{q}\cdot\vec{A}(q))\left(c^\dagger_{k+q\uparrow}c_{k\uparrow} - c^\dagger_{-k\downarrow}c_{-(k+q)\downarrow}\right) \tag{5.55}$$

In the superconductor region, we use Bogliubov particle formalism to discuss problem . Hence, we substitute the Hamltonian (5.55) with Bogliubov particle operator α_k to discuss problem .

$$
\begin{aligned}
H_I &= -\frac{1}{2}\mu_B \sum_{k,q} (\vec{k}\cdot\vec{A}(q)) \left(c^\dagger_{k+q\uparrow} c_{k\uparrow} - c^\dagger_{-k\downarrow} c_{-(k+q)\downarrow} \right) \\
&= -\frac{1}{2}\mu_B \sum_{k,q} (\vec{k}\cdot\vec{A}(q)) \left[(u_{k+q}\alpha^\dagger_{k+q\uparrow} - v_{k+q}\alpha_{-(k+q)\downarrow})(u_k\alpha_{k\uparrow} - v_k\alpha^\dagger_{-k\downarrow}) - (k+q \to -k, \uparrow\to\downarrow) \right] \\
&= -\frac{1}{2}\mu_B \sum_{k,q,\sigma} \sum_{k,q} (\vec{k}\cdot\vec{A}(q)) \left[(u_{k+q}u_k + v_{k+q}v_k)(\alpha^\dagger_{k+q\uparrow}\alpha_{k\uparrow} - \alpha^\dagger_{k+q\downarrow}\alpha_{k\downarrow}) + (-u_{k+q}v_k + u_k v_{k+q}) \left(\alpha^\dagger_{k+q\uparrow}\alpha^\dagger_{-k\downarrow} - \alpha_{k\uparrow}\alpha_{-(k+q)\downarrow} \right) \right]
\end{aligned} \tag{5.56}
$$

The current $\vec{j}(r)$ could be derived from (5.54).

$$
\vec{j}(r) = \frac{\delta H}{\delta \vec{A}(r)} = \frac{e\hbar}{2m\mathrm{i}} \left(\psi^\dagger \nabla\psi - (\nabla\psi)^\dagger \psi \right) - \frac{e^2}{mc^2}\psi^\dagger \vec{A}\psi = \vec{j}_1(r) + \vec{j}_2(r) \tag{5.57}
$$

The first term called paramagnetic current , which exists on noraml metal but vanishes on superconductor. The second term called diamagnetic curent. We could drive the paramagnetic current from (5.55),namly

$$
\begin{aligned}
j_1(r) &= -\frac{1}{2}\mu_B \sum_{k,q} (\vec{k}2 + \vec{q}) e^{\mathrm{i}\vec{q}\cdot\vec{r}} \left(c^\dagger_{k+q\uparrow} c_{k\uparrow} - c^\dagger_{-k\downarrow} c_{-(k+q)\downarrow} \right) \\
&= -\frac{1}{2}\mu_B \sum_{k,q,\sigma} \sum_{k,q} (\vec{k} + \frac{\vec{q}}{2}) \left[(u_{k+q}u_k + v_{k+q}v_k)(\alpha^\dagger_{k+q\uparrow}\alpha_{k\uparrow} - \alpha^\dagger_{k+q\downarrow}\alpha_{k\downarrow}) + (-u_{k+q}v_k + u_k v_{k+q}) \left(\alpha^\dagger_{k+q\uparrow}\alpha^\dagger_{-k\downarrow} - \alpha_{k\uparrow}\alpha_{-(k+q)\downarrow} \right) \right]
\end{aligned} \tag{5.58}
$$

We use the first perturbation to calculate the paramagnetic current. If we put the perturbation (5.55) into superconductor hamiltonian (5.21) . We use the first perturbation theory to calculate the superconductor ground state up to first order

$$
\mid \Omega\rangle = \mid \Omega\rangle_0 + \sum_l \mid l\rangle_0 \frac{{}_0\langle l \mid H_1 \mid \Omega\rangle_0}{E_l - E_0} \tag{5.59}
$$

where $\mid \Omega\rangle_0$ is the BCS ground state. The BCS is the vaccum of Bogliubov particles . Hence, the paramagnetic current only contributed by second term. The state $\mid l\rangle_0$ is defined as

$$
\mid l\rangle = \alpha^\dagger_{k+q\uparrow}\alpha^\dagger_{-k\downarrow} \mid \Omega\rangle_0 \tag{5.60}
$$

The BCS ground state $\mid \Omega\rangle$ doesn't contribute to paramagnetic current. It requires to immendiate state to $\mid l\rangle$ to carry current. We substitute current (5.58) into (5.59)

$$
\langle \vec{j}_1(r)\rangle = \sum_l \left[\frac{{}_0\langle\Omega \mid H_1 \mid l\rangle\langle l \mid H_1 \mid \Omega\rangle_0}{E_l - E_0} + \frac{\langle l \mid H_1 \mid \Omega\rangle_0 \langle l \mid j_1(r) \mid \Omega\rangle_0}{E_l - E_0} \right] \tag{5.61}
$$

Combing with (5.58,5.60) , the Eq(5.61) could be simplified into

$$
\langle \vec{j}_1(r)\rangle = \frac{1}{c}\mu_B^2 \sum_l \left[\frac{(-u_{k+q}v_k + u_k v_{k+q})^2}{\xi_{k+q} + \xi_k} (\vec{k} + \frac{\vec{q}}{2})(\vec{k}\cdot\vec{A}(q)) e^{\mathrm{i}\vec{q}\cdot\vec{r}} \right] \tag{5.62}
$$

If we consider the contributon brought by spin freedom , the current will become

$$\langle \vec{j}_1(r)\rangle = 2\frac{1}{c}\mu_B^2\sum_k\left[\frac{(-u_{k+\frac{q}{2}}v_{k-\frac{q}{2}}+u_{k+\frac{q}{2}}v_{k+\frac{q}{2}})^2}{\xi_{k+\frac{q}{2}}+\xi_{k-\frac{q}{2}}}\vec{k}(\vec{k}\cdot\vec{A}(q))e^{\mathrm{i}\vec{q}\cdot\vec{r}}\right] \tag{5.63}$$

Note:-

$$\begin{cases}\langle_{|}H_1 \mid \Omega\rangle_0 = -\frac{1}{2}\mu_B(\vec{k}\cdot\vec{A}(\vec{q}))(-u_{k+q}v_k+u_kv_{k+q})\\ \langle l \mid \vec{j}_1(r)\mid \Omega\rangle_0 = \frac{1}{2}\mu_B(\vec{k}\cdot\vec{A}(\vec{q}))(-u_{k+q}v_k+u_kv_{k+q})e^{\mathrm{i}\vec{q}\cdot\vec{r}}\end{cases} \tag{5.64}$$

Let's analysis the current direction . The term $(\vec{k}\cdot\vec{A}(q))\vec{k}$ could be viewed as two rank tensor

$$(\vec{k}\cdot\vec{A}(q))\vec{k}\sim k_x(k_x\hat{i}+k_y\hat{j}+k_z\hat{k}) \tag{5.65}$$

The current requires to preserve invariant under mirror reflection about xy, xz, yz plane, thereby the current propagate along x direction .

$$\vec{j}(r) = \frac{2e^2\hbar^2}{m^2c}\left(\frac{1}{4\pi}\int_0^{2\pi}\cos^2\phi d\phi\int_{-1}^{1}\sin^2\theta d\cos\theta\frac{N(0)}{2}\int_{-\infty}^{\infty}\frac{(-u_{k+\frac{q}{2}}v_{k-\frac{q}{2}}+u_{k+\frac{q}{2}}v_{k+\frac{q}{2}})^2}{\xi_{k+\frac{q}{2}}+\xi_{k-\frac{q}{2}}}\vec{A}(q)\right) \tag{5.66}$$

Note:-

We use (5.23) to calculate the Eq(5.66)

$$\begin{aligned}\left(-u_{k+\frac{q}{2}}v_{k-\frac{q}{2}}+u_{k+\frac{q}{2}}v_{k+\frac{q}{2}}\right)^2 &= \frac{1}{4}\left((1+\frac{\varepsilon_{k+\frac{q}{2}}}{\xi_{k+\frac{q}{2}}})(1-\frac{\varepsilon_{k-\frac{q}{2}}}{\xi_{k-\frac{q}{2}}})+(1-\frac{\varepsilon_{k-\frac{q}{2}}}{\xi_{k-\frac{q}{2}}})(1+\frac{\varepsilon_{k+\frac{q}{2}}}{\xi_{k+\frac{q}{2}}})\right)-\frac{1}{2}\frac{\Delta^2}{\xi_{k+\frac{q}{2}}\xi_{k-\frac{q}{2}}}\\ &= \frac{1}{2}\frac{\xi_{k+\frac{q}{2}}\xi_{k-\frac{q}{2}}-\varepsilon_{k+\frac{q}{2}}\varepsilon_{k-\frac{q}{2}}-\Delta^2}{\xi_{k+\frac{q}{2}}\xi_{k-\frac{q}{2}}}\end{aligned} \tag{5.67}$$

We discuss current for the normal metal cases and superconductor cases.

- Normal metal
- Superconductor

5.7 Electromagnetic asorbtion

The elctromagnetic asorbtion perturbative hamiltonian reads as

$$\Delta H = \sum_{k,q}(k+\frac{q}{2})A(q)c^\dagger_{k+q}c_k \tag{5.68}$$

The vector potential is time reversal odd . [2] Hence, this is case-II response . The initial state is BCS ground state. The final state is connected with Bogliubov particle creation operator $\alpha_k\alpha_{k'}$. We can writen down real part of optical conductivity with Fermin golden rule [3]

$$\Re\sigma(\omega) = \frac{2\pi}{\hbar}\sum_{k,k'}\mid N(k\sigma|k'\sigma')\mid^2((1-f(E_k))(1-f(E_{k'}))-f(E_k)f(E_{k'}))\,\delta(\hbar\omega-E_k-E_{k'}) \tag{5.69}$$

[2] $\vec{E} = -\frac{\partial\vec{A}}{\partial t}$.
[3] The two Bouliubov particle energy meets with frequency $\delta(\hbar\omega-E_k-E_{k'})$

At the zero temperatue, the Eq(5.69) could be written into

$$\Re\sigma(\omega) = -\frac{2\pi}{\hbar}N^2(0)\tilde{N}^2\int_{\Delta}^{\infty}dE\int_{\Delta}^{\infty}dE'\frac{EE'}{\sqrt{E^2-\Delta^2}\sqrt{E'^2-\Delta^2}}(uv'-u'v')^2\delta(\hbar\omega-E-E') \tag{5.70}$$

Note:-

The coherence factor could be calculated as [a]

$$\begin{aligned}(uv'-u'v')^2 = u^2v'^2+u'^2v^2-2uvu'v' &= \frac{1}{2}(1+\frac{\varepsilon_k}{\xi_k})(1-\frac{\varepsilon_k}{\xi_k})+\frac{1}{2}(1+\frac{\varepsilon_{k'}}{\xi_{k'}})(1-\frac{\varepsilon_{k'}}{\xi_{k'}})-\frac{1}{2}\frac{\Delta^2}{\xi_k\xi_{k'}}\\ &= \frac{1}{2}\left(1+\frac{\varepsilon_k\varepsilon_{k'}}{\xi_k\xi_{k'}}-\frac{\Delta^2}{\xi_k\xi_{k'}}\right)\\ &= \frac{1}{2}\left(1+\frac{\varepsilon_k\varepsilon_{k'}}{\xi_k\xi_{k'}}-\frac{\Delta^2}{\xi_k\xi_{k'}}\right)\end{aligned} \tag{5.71}$$

[a] $\varepsilon_k, \varepsilon_{k'}$ lies on Fermi surface.

Hence, we could written down the optical conductance for normal metal

$$\Re\sigma_n(\omega) = -\frac{2\pi}{\hbar}N^2(0)\tilde{N}^2\hbar\omega \tag{5.72}$$

We consider the relative radio at region $\omega \gg 2\Delta$

$$\begin{aligned}\Re\left(\frac{\sigma_s(\omega)}{\sigma_n(\omega)}\right) &= \frac{1}{\hbar\omega}\int_{\Delta}^{\hbar\omega-\Delta}\frac{E(E-\hbar\omega)-\Delta^2}{\sqrt{E^2-\Delta^2}\sqrt{(\hbar\omega-2\Delta)^2-\Delta^2}}dE\\ &= \frac{1}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}}\frac{y(x-y)-\frac{1}{4}}{\sqrt{x^2-\frac{1}{4}}\sqrt{(x-y)^2-\frac{1}{4}}}dy\\ &= \frac{1}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}}\frac{-(x-y)^2-\frac{1}{4}+x^2-xy}{\sqrt{x^2-\frac{1}{4}}\sqrt{(x-y)^2-\frac{1}{4}}}dy\\ &= \frac{1}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}}\sqrt{\frac{x^2-\frac{1}{4}}{(x-y)^2-\frac{1}{4}}}\end{aligned} \tag{5.73}$$

where $x=\frac{\hbar\omega}{2\Delta}, y=\frac{E}{2\Delta}$.

Note:-

We simplify the Eq(5.73) into elliplitic function

$$\begin{aligned}\frac{1}{x}\int_{\frac{1}{2}}^{x-\frac{1}{2}}\frac{y(x-y)-\frac{1}{4}}{\sqrt{x^2-\frac{1}{4}}\sqrt{(x-y)^2-\frac{1}{4}}}dy &= \frac{1}{x\sqrt{x^2-\frac{1}{4}}}\int_{\frac{1}{2}}^{x-\frac{1}{2}}\frac{y(x-y)-\frac{1}{4}}{\sqrt{(x-y)^2-\frac{1}{4}}}\\ &= \frac{1}{x\sqrt{x^2-\frac{1}{4}}}\int_{\frac{1}{2}}^{x-\frac{1}{2}}\frac{\frac{1}{2}x^2-\frac{1}{2}(x-y)^2-\frac{1}{4}}{\sqrt{(x-y)^2-\frac{1}{4}}}\end{aligned} \tag{5.74}$$

Let $y = x-\sqrt{\left(x-\frac{1}{2}\right)^2\cos^2\theta+\frac{1}{4}\sin^2\theta}$

$$\frac{1}{x\sqrt{x^2-\frac{1}{4}}}\int_0^{\frac{\pi}{2}}\frac{\left((x-\frac{1}{2})^2-\frac{1}{4}\right)\sin\theta\cos\theta}{\sqrt{\left(x-\frac{1}{2}\right)^2\cos^2\theta+\frac{1}{4}\sin^2\theta}}\left[\frac{\frac{1}{2}x^2-\frac{3}{8}}{\sqrt{\left((x-\frac{1}{2})^2-\frac{1}{4}\right)\cos^2\theta}}-\frac{1}{2}\sqrt{\left((x-\frac{1}{2})^2-\frac{1}{4}\right)\cos^2\theta}\right]$$
$$=\frac{1}{x\sqrt{x^2-\frac{1}{4}}}\sqrt{\left((x-\frac{1}{2})^2-\frac{1}{4}\right)}(\frac{1}{2}x^2-\frac{3}{8})\int_0^1\frac{dx}{\sqrt{\frac{1}{4}-\left((x-\frac{1}{2})^2-\frac{1}{4}\right)^2x^2}}$$

5.7.1 Green function method

To calculate conductance , we calculate the curent-curent correlation function $\Pi_{\mu\nu}$. The vertex function is $\frac{e}{\hbar}\frac{\partial\varepsilon_k}{\partial k_\mu}$

$$\Pi_{\mu\nu}(q,\mathrm{i}\nu_n)=\frac{e^2}{\hbar^2V}\sum_k\frac{\partial\varepsilon_{k+\frac{q}{2}}}{\partial k_\mu}\frac{\partial\varepsilon_{k-\frac{q}{2}}}{\partial k_\nu}\frac{1}{\beta}\sum_{\mathrm{i}\omega_n}\mathrm{Tr}\left(G(k+q,\mathrm{i}(\nu_n+\omega_n))G(k,\mathrm{i}\omega_n)\right) \tag{5.74}$$

where $G(k,\mathrm{i}\omega_n)$ is the Nambu-Gorkov green function. We use the Matsubara summation to calculate the trace part

$$\frac{1}{\beta}\sum_{\mathrm{i}\omega_n}\mathrm{Tr}\left(G(k+q,\mathrm{i}(\nu_n+\omega_n))G(k,\mathrm{i}\omega_n)\right)=\frac{1}{\beta}\sum_{\mathrm{i}\omega_n}\mathrm{Tr}\left(\frac{1}{\mathrm{i}(\omega_n+\nu_n)-\sigma_z\varepsilon_{k+q}-\Delta\sigma_1}\frac{1}{\mathrm{i}\omega_n-\sigma_z\varepsilon_k-\Delta\sigma_1}\right)$$
$$=\frac{1}{\beta}\sum_{\mathrm{i}\omega_n}\mathrm{Tr}\left(\frac{(\mathrm{i}(\omega_n+\nu_n)+\sigma_z\varepsilon_{k+q}+\Delta\sigma_1)(\mathrm{i}\omega_n+\varepsilon_k\sigma_3+\Delta\sigma_1)}{[(\mathrm{i}(\omega_n+\nu_n))^2-\varepsilon_{k+q}^2-\Delta^2][(\mathrm{i}\omega_n)^2-\varepsilon_k^2-\Delta^2]}\right)$$
$$=\frac{1}{\beta}\sum_{\mathrm{i}\omega_n}\left(\frac{(\mathrm{i}(\omega_n+\nu_n))(\mathrm{i}\omega_n)+\varepsilon_k\varepsilon_{k+q}+\Delta^2}{[(\mathrm{i}(\omega_n+\nu_n))^2-\varepsilon_{k+q}^2-\Delta^2][(\mathrm{i}\omega_n)^2-\varepsilon_k^2-\Delta^2]}\right)$$
$$=\frac{\xi_{k+q}(\xi_{k+q}-\mathrm{i}\nu_n)+\varepsilon_k\varepsilon_{k+q}+\Delta^2}{\xi_{k+q}\left[(\xi_{k+q}-\mathrm{i}\nu_n)^2-\xi_k^2\right]}f(\xi_{k+q})+\frac{\xi_k(\xi_k+\mathrm{i}\nu_n)+\varepsilon_k\varepsilon_{k+q}+\Delta^2}{\xi_k\left[(\xi_k+\mathrm{i}\nu_n)^2-\xi_{k+q}^2\right]}f(\xi_k)$$
$$-\frac{\xi_{k+q}(\xi_{k+q}+\mathrm{i}\nu_n)+\varepsilon_k\varepsilon_{k+q}+\Delta^2}{\xi_{k+q}\left[(\xi_{k+q}+\mathrm{i}\nu_n)^2-\xi_k^2\right]}(1-f(\xi_{k+q}))-\frac{\xi_k(\xi_k-\mathrm{i}\nu_n)+\varepsilon_k\varepsilon_{k+q}+\Delta^2}{\xi_k\left[(\xi_k-\mathrm{i}\nu_n)^2-\xi_{k+q}^2\right]}(1-f(\xi_k)) \tag{5.75}$$

We consider the zero temorature limit, the polarization (5.74) will become

$$\Pi_{\mu\nu}(q,\mathrm{i}\nu_n)=-\frac{e^2}{\hbar^2V}\sum_k\frac{\partial\varepsilon_{k+\frac{q}{2}}}{\partial k_\mu}\frac{\partial\varepsilon_{k-\frac{q}{2}}}{\partial k_\nu}\left[\frac{\xi_{k+q}(\xi_{k+q}+\mathrm{i}\nu_n)+\varepsilon_k\varepsilon_{k+q}+\Delta^2}{\xi_{k+q}\left[(\xi_{k+q}+\mathrm{i}\nu_n)^2-\xi_k^2\right]}+\frac{\xi_k(\xi_k-\mathrm{i}\nu_n)+\varepsilon_k\varepsilon_{k+q}+\Delta^2}{\xi_k\left[(\xi_k-\mathrm{i}\nu_n)^2-\xi_{k+q}^2\right]}\right] \tag{5.76}$$

Note:-

$$\Im\left[\frac{1}{(\xi_k-\nu-\mathrm{i}\varepsilon-\xi_{k+q})(\xi_k-\nu-\mathrm{i}\varepsilon+\xi_{k+q})}\right]=\Im\left[\frac{1}{2\xi_{k+q}}\left(\frac{1}{(\xi_k-\nu-\mathrm{i}\varepsilon-\xi_{k+q})}-\frac{1}{(\xi_k-\nu-\mathrm{i}\varepsilon+\xi_{k+q})}\right)\right]$$
$$=\frac{\pi}{2\xi_{k+q}}\left(\delta(\nu+\xi_{k+q}-\xi_k)-\delta(\nu-\xi_k-\xi_{k+q})\right) \tag{5.77}$$

$$\Im\left[\frac{1}{(\xi_k+\nu+\mathrm{i}\varepsilon-\xi_{k+q})(\xi_k+\nu+\mathrm{i}\varepsilon+\xi_{k+q})}\right]=\Im\left[\frac{1}{2\xi_{k+q}}\left(\frac{1}{(\xi_k+\nu+\mathrm{i}\varepsilon-\xi_{k+q})}-\frac{1}{(\xi_k+\nu+\mathrm{i}\varepsilon+\xi_{k+q})}\right)\right]$$
$$=\frac{\pi}{2\xi_{k+q}}\left(\delta(\nu+\xi_{k+q}+\xi_k)-\delta(\nu+\xi_k-\xi_{k+q})\right) \tag{5.78}$$

The imaginary part of polarization is

$$\Im\Pi_{\mu\mu}(q,\nu) = \frac{\pi e^2}{2V\hbar^2}\sum_k \frac{\partial\varepsilon_{k+\frac{q}{2}}}{\partial k_\mu}\frac{\partial\varepsilon_{k-\frac{q}{2}}}{\partial k_\nu}\left[\frac{\xi_k\xi_{k+q} - \varepsilon_k\varepsilon_{k+q} - \Delta^2}{\xi_k\xi_{k+q}}\right](\delta(\omega+\xi_k+\xi_{k+q}) + \delta(\omega-\xi_k-\xi_{k+q})) \quad (5.79)$$

We consider to calculate optical conductance for isotropic s-wave superconductors

$$\Re\sigma(\omega) = \frac{\pi e^2 v_F^2}{6}N^2(0)\int_{-\infty}^{\infty}dE\int_{-\infty}^{\infty}dE'\frac{E}{\sqrt{E^2-\Delta^2}}\frac{E'}{\sqrt{E'^2-\Delta^2}}\frac{EE'-\Delta^2}{EE'}\delta(\omega-E-E') \quad (5.80)$$

Then we consider to derive optical ratio of normal state metal and superconductor

$$\Re\left(\frac{\sigma_s(\omega)}{\sigma_n(\omega)}\right) = \frac{1}{\hbar\omega}\int_{-\infty}^{\infty}dE\frac{E(\omega-E)-\Delta^2}{\sqrt{E^2-\Delta^2}\sqrt{(\omega-E)^2-\Delta^2}} \quad (5.81)$$

Note:-

We notice the we can let $E = \dfrac{\omega+(\omega-2\Delta)x}{2}$ to simplify (5.81) into Elliptic integralc

$$\begin{aligned}\sqrt{E^2-\Delta^2}\sqrt{(\omega-E)^2-\Delta^2} &= [(E+\Delta)(E-\Delta)(\omega-E+\Delta)(\omega-E-\Delta)]^{0.5}\\ &= \frac{\omega-2\Delta}{2}(1+x)\frac{\omega+2\Delta+(\omega-2\Delta)x}{2}\frac{(\omega+2\Delta)-(\omega-2\Delta)x}{2}\cdot\frac{\omega-2\Delta}{2}(1-x)\\ &= \left((\frac{\omega}{2})^2-\Delta^2\right)^2(1-x^2)(1-\alpha^3x^2)\end{aligned} \quad (5.82)$$

where $\alpha = \dfrac{\omega-2\Delta}{\omega+2\Delta}$

$$E(\omega-E)-\Delta^2 = \frac{\omega+(\omega-2\Delta)x}{2}\frac{\omega-(\omega-2\Delta)x}{2} - \Delta^2 = \left((\frac{\omega}{2})-\Delta^2\right)^2(1-\alpha x^2) \quad (5.83)$$

Hence, the Eq(5.81) could be simplified into

$$\begin{aligned}\Re\left(\frac{\sigma_s(\omega)}{\sigma_n(\omega)}\right) = \frac{\omega-2\Delta}{\hbar\omega}\int_0^1\frac{1-\frac{1}{\alpha}+\frac{1}{\alpha}(1-\alpha^2x^2)}{\sqrt{(1-x^2)(1-\alpha^2x^2)}} &= (1-y)(1-\frac{y+1}{1-y})K\left(\frac{1-y}{y+1}\right)+(1+y)E(\frac{1-y}{1+y})\\ &= -2yK\left(\frac{1-y}{y+1}\right)+(1+y)E(\frac{1-y}{1+y})\end{aligned} \quad (5.84)$$

We consider two limits

$$\begin{cases}\lim_{y\to1}\Re\left(\frac{\sigma_s(\omega)}{\sigma_n(\omega)}\right) = 2E(0)-2K(0) = 0\\ \lim_{y\to0}\Re\left(\frac{\sigma_s(\omega)}{\sigma_n(\omega)}\right) = E(1) = 1\end{cases} \quad (5.85)$$

The behaviour of (5.81) is plotted in Figure(5.3).

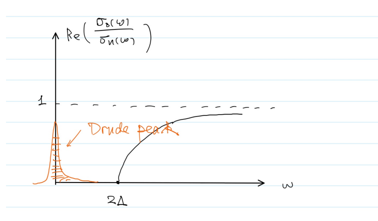

Figure 5.3: The conductance start response from 2Δ. If the frequency is infinite, optical conductance at superconductivity region is equal to normal state.