Chapter 7

Green function

The Green function the fundamental response function of many body system. The single particle green function is defined as

$$G(t-t') = -\mathrm{i}\langle \phi \mid \mathcal{T}\psi(t)\psi^{\dagger}(t') \mid \phi\rangle \tag{7.1}$$

where ϕ is many body ground state . The operator $\psi(t)$ is defined on Heisenberg representation. We define the propagator as the Green function on the momentum space , namly

$$G(k,\omega) = -\mathrm{i}\int \frac{d^3k}{(2\pi)^2}\frac{d\omega}{2\pi}\langle \phi \mid \mathcal{T}\psi(x,t)\psi^{\dagger}(x',t') \mid \phi\rangle e^{\mathrm{i}k\cdot(x-x')-\omega(t-t')} \tag{7.2}$$

Figure 7.1: Single Fermion Propagator

7.1 Fermion Green function

Let's define th e ground state $\mid \phi\rangle$ as

$$\mid \phi\rangle = \prod_{|k|<k_f,\sigma} c^{\dagger}_{k\sigma} \mid 0\rangle \tag{7.3}$$

According to definition of Green function (7.1), the fermion green function could be calculated as

$$G(k,t-t') = -\mathrm{i}\langle \phi \mid \mathcal{T}c_{k\sigma}(t)c^{\dagger}_{k'\sigma'}(t') \mid \phi\rangle = -\mathrm{i}\theta(t-t')\delta_{kk'}(1-n_{k'\sigma'})e^{\mathrm{i}\omega_k(t'-t)} - \mathrm{i}\theta(t'-t)\delta_{kk'}n_{k\sigma}e^{\mathrm{i}\omega_k(t'-t)} \tag{7.4}$$

We consider the Fourier transformation to find the propagator

$$\begin{aligned} G(k,\omega) &= \int_{-\infty}^{+\infty} dt - \mathrm{i}\left(\theta(t)(1-n_{k\sigma}) - \theta(-t)n_{k\sigma}\right)e^{\mathrm{i}(\omega-\varepsilon_k)t} \\ &= -\mathrm{i}\lim_{\varepsilon\to 0}\int_0^{\infty}\theta_{k-k_F}e^{\mathrm{i}(\omega-\omega_k+\mathrm{i}\varepsilon)t} + \mathrm{i}\lim_{\varepsilon\to 0}\int_{-\infty}^{0}\theta_{k_F-k}e^{\mathrm{i}(\omega-\omega_k+\mathrm{i}\varepsilon)t} \\ &= \lim_{\varepsilon\to 0}\frac{\theta_{k-k_F}}{\omega-\omega_k+\mathrm{i}\varepsilon} + \frac{\theta_{k_F-k}}{\omega-\omega_k+\mathrm{i}\varepsilon} \\ &= \frac{1}{\omega-\omega_k+\mathrm{i}\varepsilon} \end{aligned} \tag{7.5}$$

The fermion green function consists two part. The first term stands for particle moving forwards in time. The second term stands for holes moving backwards in time. Hence, we define the free fermion Green function as

$$G(k,\omega) = \frac{1}{\omega - \omega_k + \mathrm{i}\varepsilon} = \tag{7.6}$$

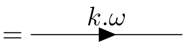

Figure 7.2: Single Fermion Propagator

7.2 Boson Green function

By the smae way , the bosonic green function can be calculated with definition. The non-interacting bosonic gas caould be described by hamiltonian

$$H = \sum_q \omega_q \left(b_q^\dagger b_q + \frac{1}{2} \right) \tag{7.7}$$

The ground state of hamiltonian (7.7) is just vacuum $\mid 0\rangle$. The physical field is defined as [1]

$$\phi_q = \sqrt{\frac{\hbar}{2m\omega_q}} \left(b_q + b_{-q}^\dagger \right) \tag{7.8}$$

The Green function for bosons could be defined by field ϕ_q .

$$\begin{aligned} G(q,t,t') = -\mathrm{i}\langle \phi \mid \mathcal{T}\phi_q(t)\phi_{q'}^\dagger(t') \mid \phi\rangle &= -\mathrm{i}\frac{\hbar}{2m\omega_q}\langle \phi \mid \mathcal{T}b_q(t)b_q^\dagger(t') \mid \phi\rangle - \mathrm{i}\frac{\hbar}{2m\omega_q}\langle \phi \mid \mathcal{T}b_{-q}^\dagger(t)b_{-q}(t') \mid \phi\rangle \\ &= -\mathrm{i}\frac{\hbar}{2m\omega_q}e^{\mathrm{i}\omega_q(t'-t)}\theta(t-t') - \mathrm{i}\frac{\hbar}{2m\omega_q}e^{-\mathrm{i}\omega_q(t'-t)}\theta(t'-t) \end{aligned} \tag{7.9}$$

We calculate the propagator from Eq(7.9).

$$\begin{aligned} G(q,\omega) &= -\mathrm{i}\frac{\hbar}{2m\omega_q}\int_{-\infty}^{+\infty} dt\theta(t)e^{\mathrm{i}(\omega-\omega_q)t} + \theta(-t)e^{\mathrm{ii}(\omega+\omega_q)t} \\ &= -\mathrm{i}\frac{\hbar}{2m\omega_q}\left(\int_0^{+\infty} dte^{\mathrm{i}(\omega-\omega_q+\mathrm{i}\varepsilon)t} + \int_{-\infty}^0 dte^{\mathrm{i}(\omega+\omega_q-\mathrm{i}\varepsilon)t} \right) \\ &= \frac{\hbar}{2m\omega_q}\left(\frac{1}{\omega-\omega_q+\mathrm{i}\varepsilon} - \frac{1}{\omega+\omega_q+\mathrm{i}\varepsilon} \right) \\ &= \frac{\hbar}{2m\omega_q}\frac{2\omega_q}{\omega^2-(\omega_q+\mathrm{i}\varepsilon)^2} \end{aligned} \tag{7.10}$$

Note:-

The bosonic green function contains two part. The first part involves forward boson emitting process. The second part involves boson absorbing process .

$$G(q,\omega) == \frac{\hbar}{2m\omega_q}\left(\frac{1}{\omega-(\omega_q+\mathrm{i}\varepsilon)} - \frac{1}{\omega+(\omega_q+\mathrm{i}\varepsilon)} \right) \tag{7.11}$$

At static limit $\omega \to 0$, then $G(q,\omega) = -\frac{\hbar}{2m\omega^2}$, it will induce effective atraction interaction.

[1] You can refer to scalar field quantization in field theory theory

7.3 Imaginary-time Green function

In this section , we discuss imaniary time Green function. The imaginary Green function is defined as

$$G_{\lambda\lambda'}(\tau-\tau') = -\langle \mathcal{T}\psi_\lambda(\tau)\psi^\dagger_{\lambda'}(\tau')\rangle = -\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta()}\psi_\lambda(\tau)\psi^\dagger_{\lambda'}(\tau')\right] \tag{7.12}$$

For a non-interacting system, the expectaton is

$$\langle\psi^\dagger_{\lambda'}\psi_\lambda\rangle = \delta_{\lambda'\lambda}\begin{cases} n(\varepsilon_\lambda) & \text{Bosons} \\ f(\varepsilon_\lambda) & \text{Fermions}\end{cases} \tag{7.13}$$

where $f(\varepsilon_\lambda), n(\varepsilon_\lambda)$ is the bosonic/fermionic distribution.

$$\begin{cases} f(\varepsilon_\lambda) = \dfrac{1}{e^{\beta\varepsilon_\lambda}-1} \\ n(\varepsilon_\lambda) = \dfrac{1}{e^{\beta\varepsilon_\lambda}+1}\end{cases} \tag{7.14}$$

The green function (7.12) could be written into

$$G_{\lambda\lambda'}(\tau-\tau') = -e^{\varepsilon_\lambda(\tau'-\tau)}\begin{cases} [\theta(\tau-\tau')(1+n(\varepsilon_\lambda))+\theta(\tau'-\tau)n(\varepsilon_\lambda)] & \text{Bosons} \\ [\theta(\tau-\tau')(1-f(\varepsilon_\lambda))-\theta(\tau'-\tau)f(\varepsilon_\lambda)] & \text{Fermions}\end{cases} \tag{7.15}$$

The most eminent property of imagnary-time Green function is the periodicty for bosons and anti-periodicity for fermions. We take $-\beta<\tau<0$, the Green function (7.12) expands into

$$\begin{aligned} G_{\lambda\lambda'}(\tau) &= -\langle\mathcal{T}c_\lambda(\tau)c^\dagger_{\lambda'}(0)\rangle = -\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}c^\dagger_{\lambda'}(0)c_\lambda(\tau)\right] \\ &= -\eta\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}c^\dagger_{\lambda'}(0)e^{H\tau}c_\lambda e^{-H\tau}\right] \quad \text{Trace identity} \\ &= -\eta\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}e^{H(\tau+\beta)}c_\lambda e^{-H(\tau+\beta)}c^\dagger_{\lambda'}(0)\right] \\ &= \eta G(\tau+\beta) \end{aligned} \tag{7.16}$$

Let's consider the identity for Green function

$$\begin{aligned} G(\tau) &= \frac{1}{\beta}\int_0^\beta G(\tau')\delta(\tau-\tau')d\tau' = \frac{1}{\beta}\int_0^\beta G(\tau')\beta\sum_{n=-\infty}^{\infty}e^{-\mathrm{i}\omega_n(\tau-\tau')} \\ &= \sum_{n=-\infty}^{+\infty}\left(\int_0^\beta G(\tau')e^{\mathrm{i}\omega_n\tau'}\right)e^{-\mathrm{i}\omega_n\tau} \\ &= \sum_{\omega_n}G(\mathrm{i}\omega_n)e^{\mathrm{i}\omega_n\tau} \end{aligned} \tag{7.17}$$

The Eq(7.17) tells us Matsubara representation

$$\begin{cases} G(\tau) = \displaystyle\sum_{\omega_n}G(\mathrm{i}\omega_n)e^{\mathrm{i}\omega_n\tau} \\ G(\mathrm{i}\omega_n) = \displaystyle\int_0^\beta e^{\mathrm{i}\omega_n\tau}G(\tau)d\tau \end{cases} \tag{7.18}$$

Eq(7.18) is just the Matsubara representation of imagninary-time Green function . The Eq(**??**) should match with properties(7.16) . We introduce Matsubara frequency for bosons and fermions respectively.

$$\omega_n = \begin{cases} \frac{2\pi n}{\beta} & \text{Bosons} \\ \frac{(2n+1)\pi}{\beta} & \text{Fermions} \end{cases} \tag{7.19}$$

Claim 7.1 .
The imaginary-time Green function adimits poperties below

$$\int_0^\beta G(\tau)e^{\mathrm{i}\omega_n\tau} = \int_{-\beta}^0 G(\tau)e^{\mathrm{i}\omega_n\tau} \tag{7.20}$$

Proof.

$$\int_{-\beta}^0 G(\tau)e^{\mathrm{i}\omega_n\tau} = \int_0^\beta G(\tau+\beta)e^{\mathrm{i}(\tau+\beta)}d\tau = \int_0^\beta G(\tau)e^{\mathrm{i}\omega_n\tau} \tag{7.21}$$

□

7.3.1 Lehmann representation

In many body physis, the Lehmann representation is obtained by assumption of time-independent hamiltonian. Now, we expand the Green function into eigenstates.

$$\begin{aligned} G(\tau) &= -\frac{1}{Z}\mathrm{Tr}\left(e^{-\beta H}e^{H\tau}\psi_\lambda(0)e^{-H\tau}\psi^\dagger_{\lambda'}(0)\right) \\ &= -\frac{1}{Z}\sum_{n,n'}\langle n \mid e^{-\beta H}e^{H\tau}\psi_\lambda(0)e^{-H\tau} \mid n'\rangle\langle n' \mid \psi^\dagger_{\lambda'}(0) \mid n\rangle \\ &= -\frac{1}{Z}\sum_{n,n'}\langle n \mid \psi_\lambda(0) \mid n'\rangle\langle n' \mid \psi^\dagger_\lambda(0) \mid n\rangle e^{-\beta E_n}e^{(E'_n-E_n)\tau} \end{aligned} \tag{7.22}$$

We transform the Green function (7.22) into frequencey space by formula (7.18).

$$\begin{aligned} G(\mathrm{i}\omega_n) &= \int_0^\beta e^{\mathrm{i}\omega_n\tau}G(\tau)d\tau = \sum_{n,n'}\langle n \mid \psi_\lambda(0) \mid n'\rangle\langle n' \mid \psi^\dagger_\lambda(0) \mid n\rangle e^{-\beta E_n}\int_0^\beta e^{(E_n-E'_n+\mathrm{i}\omega_n)\tau} \\ &= -\frac{1}{Z}\sum_{n,n'}\frac{\langle n \mid \psi_\lambda(0) \mid n'\rangle\langle n' \mid \psi^\dagger_\lambda(0) \mid n\rangle}{E'_n-E_n+\mathrm{i}\omega_n}e^{-\beta E_n}\left(e^{(E_n-E'_n+\mathrm{i}\omega_n)\beta}-1\right) \\ &= -\frac{1}{Z}\sum_{n,n'}\frac{\langle n \mid \psi_\lambda(0) \mid n'\rangle\langle n' \mid \psi^\dagger_\lambda(0) \mid n\rangle}{E'_n-E_n+\mathrm{i}\omega_n}\left(\eta e^{-\beta En'}-e^{-\beta E_n}\right) \\ &= \frac{1}{Z}\sum_{n,n'}\frac{\langle n \mid \psi_\lambda(0) \mid n'\rangle\langle n' \mid \psi^\dagger_\lambda(0) \mid n\rangle}{E'_n-E_n+\mathrm{i}\omega_n}\left(e^{-\beta En'}-\eta e^{-\beta E_n}\right) \end{aligned} \tag{7.23}$$

Question 2

If $-\beta < \tau < 0$. please derive Lehmann representation form of Green function.

7.3.2 Matsubara Green function for free fermion and free bosons

We obtain the fermionic green function for free fermion by using of (7.18)

$$G(\mathrm{i}\omega_n) = \int_0^\beta d\tau e^{\mathrm{i}\omega_n\tau}G(\tau) = -\int_0^\beta d\tau(1-f(\varepsilon_\lambda))e^{-(\varepsilon_\lambda - \mathrm{i}\omega_n)\tau} = \frac{1}{\varepsilon_\lambda - \mathrm{i}\omega_n}\frac{e^{-(\varepsilon_\lambda-\mathrm{i}\omega_n)\tau}-1}{1+e^{-\beta\varepsilon_\lambda}} = \frac{1}{\mathrm{i}\omega_n - \varepsilon_\lambda} \tag{7.24}$$

By the same way , we calculate the bosonic green function for free bosons.

$$G(\mathrm{i}\nu_n) = \int_0^\beta d\tau e^{\mathrm{i}\nu_n\tau}G(\tau) = \frac{1}{\mathrm{i}\nu_n - \varepsilon_\lambda} \tag{7.25}$$

Example 7.3.1 (.)
Calculate the finite temperature Green function for harmonics oscilators.

$$D(\tau) = -\langle \mathcal{T} x(\tau)x(0)\rangle \tag{7.26}$$

We expand the the Green function as

$$\begin{aligned} D(\tau) &= -\frac{\hbar}{2m\omega}\langle \mathcal{T}(b(\tau)+b^\dagger(\tau))(b(0)+b^\dagger(0))\rangle \\ &= -\frac{\hbar}{2m\omega}\left(\langle \mathcal{T}(b(\tau)b^\dagger(0))\rangle + \langle \mathcal{T}(b^\dagger(\tau)b(0))\rangle\right) \end{aligned}$$

The first term could be given by Eq(7.15) . And the second term has such form

$$\begin{aligned} \langle \mathcal{T}(b^\dagger(\tau)b(0))\rangle &= (\theta(\tau)n(\omega) + (n(\omega)+1)\theta(-\tau)) = n(\omega)+\theta(-\tau) = -n(-\omega)-1-\theta(-\tau) \\ &= -\left(n(-\omega)\theta(-\tau) + (1+n(-\omega))\theta(\tau)\right) \end{aligned}$$

Hence, the Green function can be founded by (7.24)

$$D(\tau) = \frac{\hbar}{2m\omega}\frac{2\omega}{(\mathrm{i}\nu_n)^2 - \omega^2} \tag{7.27}$$

7.4 Matsubara sum

We will often encounter summation for all Matsubara frequencies. In this section, we will develop complex contour integral method to deal with this problem. A very important example is the calculation of polarization bubble diagram

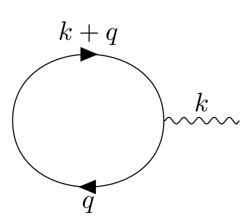

Figure 7.3: Polarization Bubble Feynman Diagram

The suscetibility is given by bubble diagram Figure(7.3)

$$
\begin{aligned}
\chi(q, \mathrm{i}\nu_n) &= -2\mu_B^2 T \sum_{k,n} G(k+q, \mathrm{i}\omega_n + \mathrm{i}\nu_n) G(k, \mathrm{i}\omega_n) \\
&= -2\mu_B^2 T \sum_{k,n} \frac{1}{\mathrm{i}\omega_n + \mathrm{i}\nu_n - \varepsilon_{k+q}} \frac{1}{\mathrm{i}\omega_n - \varepsilon_k}
\end{aligned} \tag{7.28}
$$

The minus sign originates from Fermionic loop . We consider such contour integral

$$
\int_{\mathcal{C}} f(z) \frac{1}{z + \mathrm{i}\nu_n - \varepsilon_{k+q}} \frac{1}{z - \varepsilon_k} = 2\pi\mathrm{i} \lim_{z \to \omega_n \cup \{\varepsilon_{k+1} - \mathrm{i}\nu_n, \varepsilon_k\}} \mathrm{Res}\left(f(z) \frac{1}{z + \mathrm{i}\nu_n - \varepsilon_{k+q}} \frac{1}{z - \varepsilon_k} \right) \tag{7.29}
$$

Hence, we can derive the susceptibility as

$$
\chi(q, \nu_n) = \sum_k \frac{f(\varepsilon_{k+q}) - f(\varepsilon_k)}{\mathrm{i}\nu_n - (\varepsilon_{k+q} - \varepsilon_k)} \tag{7.30}
$$

This result meets with Linhard response function.

Note:-

We notice that

$$
\begin{aligned}
\sum_{z \to z_n} \mathrm{Res}\left(f(z) \frac{1}{z + \mathrm{i}\nu_n - \varepsilon_{k+q}} \frac{1}{z - \varepsilon_k} \right) &= \sum_{z \to z_n} \left(\frac{z - \omega_n}{e^{\beta(z - \omega_n)} e^{\beta z_n} + 1} \frac{1}{z + \mathrm{i}\nu_n - \varepsilon_{k+q}} \frac{1}{z - \varepsilon_k} \right) \\
&= -\frac{1}{\beta} \sum_n \frac{1}{\mathrm{i}\omega_n + \mathrm{i}\nu_n - \varepsilon_{k+q}} \frac{1}{\mathrm{i}\omega_n - \varepsilon_k}
\end{aligned} \tag{7.31}
$$

We substitute Eq(7.31) into (7.29), the result is obvious.

7.5 Path integral

7.5.1 Coherent state

7.5.2 Bosonic path integral

In this section , we will derive bosonic path integral fromalism. Our start point is partition function . We consider write partition function into coherent state

$$
Z = \mathrm{Tr}\left(e^{-\beta H}\right) = \int \frac{dz d\bar{z}}{2\pi\mathrm{i}} e^{-z\bar{z}} \langle z \mid e^{-\beta H} \mid z \rangle \tag{7.32}
$$

We consider divide Bo;tzmann factor $e^{-\beta H}$ into time slices $e^{-\beta H} = \left(e^{-\Delta\tau H}\right)^N$. By using of completenes relation of coherent state, the partition function (7.33) could be written into

$$
Z = \mathrm{Tr}\left(e^{-\beta H}\right) = \int \prod_{i=0}^{N-1} \frac{dz_i d\bar{z}_i}{2\pi\mathrm{i}} e^{-z_i \bar{z}_i} \langle z_N \mid e^{-\Delta\tau H} \mid z_{N-1} \rangle \langle z_{N-1} \mid e^{-\Delta\tau H} \mid z_{N-2} \rangle \cdots \langle z_1 \mid e^{-\Delta\tau H} \mid z_0 \rangle \tag{7.33}
$$

The hamiltonian H is the polynomials about operator $b, b^\dagger$ with normal order. However, the Boltmann factor is not normal order . In other words, we cann't replace Boltmann factor by c number equivalents.

$$
\begin{aligned}
\langle z_k \mid e^{-\Delta\tau H} \mid z_{k+1} \rangle &= \langle z_k \mid 1 - \Delta\tau H[b^\dagger, b] \mid z_{k-1} \rangle = \langle z_k \mid 1 - \Delta\tau H[\bar{z}_k, z_{k-1}] \mid z_{k+1} \rangle = e^{\bar{z}_k z_{k-1}} \left(1 - \Delta\tau H[\bar{z}_k, z_{k-1}]\right) \\
&= e^{\bar{z}_k z_{k-1} - \Delta\tau H[\bar{z}_k, z_{k-1}]}
\end{aligned} \tag{7.34}
$$

We substitute (7.34) into (7.33) . The partition function becomes [2]

$$Z = \int \prod_{k=1}^{N-1} \frac{dz_k d\bar{z}_k}{2\pi \mathrm{i}} e^{(\bar{z}_{i+1}-\bar{z}_i)z_i - \Delta\tau H[\bar{z}_k, z_{k-1}]} \tag{7.35}$$

Furthermore, we take limit $N \to$ to continumm limits.

$$\sum_{k=0}^{N} (\bar{z}_{i+1}-\bar{z}_i)z_i - \Delta\tau H[\bar{z}_k, z_{k-1}] = \sum_{k=0}^{N} \Delta\tau \frac{(\bar{z}_{i+1}-\bar{z}_i)}{\Delta\tau} z_i - \Delta\tau H[\bar{z}_k, z_{k-1}] \to -\int_0^\beta d\tau\,(\bar{z}\partial z + H[\bar{z}, z]) \tag{7.36}$$

We define functional measurement as

$$\mathcal{D}[\bar{z}, z] = \int \prod_{k=1}^{N-1} \frac{dz_k d\bar{z}_k}{2\pi \mathrm{i}} \tag{7.37}$$

The partion function could be written into

$$Z = \int \mathcal{D}[\bar{z}, z] e^{-S} \qquad S = \int_0^\beta d\tau\,(\bar{z}\partial z + H[\bar{z}, z]) \tag{7.38}$$

7.5.3 Gaussian path integral

A most important path integral is Gaussian path integral . We start from action S_E

$$S_E = \int_0^\beta d\tau \bar{z}_\alpha(\tau)(\partial_\tau + h_{\alpha\beta}) z_\beta(\tau) \tag{7.39}$$

The action (7.39) could be written into Matsubara representation .

$$\begin{aligned} S_E &= \int_0^\beta d\tau \bar{z}_\alpha(\tau)(\partial_\tau + h_{\alpha\beta}) z_\beta(\tau')\delta(\tau-\tau') \\ &= \frac{1}{\beta}\sum_{\mathrm{i}\nu_n, \mathrm{i}\nu_n'} \bar{z}_\beta(\mathrm{i}\nu_n')\,(\mathrm{i}\nu_n + h_{\alpha\beta})\, z_\beta(\mathrm{i}\nu_n) \int_0^\beta d\tau e^{\mathrm{i}(\nu_n - \nu_n')\tau} \\ &= \sum_{\mathrm{i}\nu_n} \bar{z}_\beta(\mathrm{i}\nu_n)\,(\mathrm{i}\nu_n + h_{\alpha\beta})\, z_\beta(\mathrm{i}\nu_n) \end{aligned} \tag{7.40}$$

Claim 7.2 ,

We have such identity for coherent state integral

$$\int \prod_{k=1}^{n} \frac{dz_k d\bar{z}_k}{2\pi \mathrm{i}} e^{-\bar{z}_\alpha M_{\alpha\beta} z_\beta} = \frac{1}{\det(M)} \tag{7.41}$$

Proof. Let's consider the unitary transformation for quadratic form $\bar{z}_\alpha M_{\alpha\beta} z_\beta$.

$$\bar{z}_\alpha M_{\alpha\beta} z_\beta = z_\alpha (U^\dagger U M_\mu U^\dagger U)_{\alpha\beta} z_\beta = z_\alpha' (U M U^\dagger)_{\alpha\beta} z_\beta' = \sum_{n=1}^{\infty} \lambda_n \bar{z}_k' z_k' \tag{7.42}$$

[2] $Z_N = Z_0$

Hence, the integral (7.41) could be calculated as

$$\int \prod_{k=1}^{n} \frac{dz_k d\bar{z}_k}{2\pi \mathrm{i}} e^{-\bar{z}_\alpha M_{\alpha\beta} z_\beta} = \int \prod_{k=1}^{n} \frac{dz'_k d\bar{z}'_k}{2\pi \mathrm{i}} e^{-\sum_{k=1}^{n} \lambda_k \bar{z}'_k z_k} = \frac{1}{\det(M)} \tag{7.43}$$

□

Hence, we can write down the partion function as

$$Z = \frac{1}{\det(\partial_\tau + h)} \tag{7.44}$$

7.5.4 Fermonic path integral

We introduce the Grassmann coherent state to formulate fermioinc path integral. By the same way, we can write down the fermionic path integral .

We introduce the Grassmann variables to formulate fermionic path integral . The Grassmann varibales satisfies to anti-commutating propeties

$$\{\xi_i, \xi_j\} = 0 \quad \{\bar{\xi}_i, \bar{\xi}_j\} = 0 \quad \{\xi_i, \bar{\xi}_j\} = 0 \tag{7.45}$$

Furthermore, the Grassmann varibales also sati-commutate with fermioonic operators

$$\{\xi_i, c_j\} = \{\xi_i, c_j^\dagger\} = \{\bar{\xi}_i, c_j\} = \{\bar{\xi}_i, c_j^\dagger\} = 0 \tag{7.46}$$

Definition 7.5.1: .

Fermionic coherent state $\mid \xi\rangle$ is defined as

$$\mid \xi\rangle = e^{-\xi c^\dagger} \mid 0\rangle = (1 - \xi c^\dagger) \mid 0\rangle \tag{7.47}$$

The fermionic coherent state definition (7.5.4) could be generalized into many particles

$$\mid \xi_1, \xi_2, \cdots \xi_n\rangle = e^{-\sum_{i}^{n} \xi_i c_i^\dagger} \mid 0\rangle \tag{7.48}$$

If the function could be expanded into power series with Grassmann variables, then we call this function is Grassmann anlytical function.

$$\psi(\xi) = \psi_0 + \psi_1 \xi + \psi_2 \xi^2 + \cdots \tag{7.49}$$

We give the diffrentiation and integration rules for Grassmann variables .

$$\partial_\xi(\xi) = 1 \quad \partial_\xi(\bar{\xi}\xi) = -\bar{\xi} \tag{7.50}$$

The Eq(7.50) tells us anti-commutation relation between ∂_ξ and $\partial_{\bar{\xi}}$.

$$\{\partial_\xi, \partial_{\bar{\xi}}\} = \partial_\xi \partial_{\bar{\xi}} + \partial_{\bar{\xi}} \partial_\xi = 0 \tag{7.51}$$

Eq(7.50) suggests suh definition

$$\int d\xi 1 = \int d\xi \partial_\xi \xi = 0 \quad \int d\xi \xi = 1 \tag{7.52}$$

Thus, integration and differentiation is same for Grassmann varibales

$$\partial_\xi \equiv \int d\xi \tag{7.53}$$

Now we try to find resolution identity

$$I = \int d\xi d\bar{\xi} \mu(\xi) \mid \xi\rangle\langle \xi \mid \qquad \mu(\xi) = e^{-\bar{\xi}\xi} \tag{7.54}$$

The inner product could be defined as

$$\langle f \mid g\rangle = \int d\bar{\xi} d\xi e^{-\bar{\xi}\xi} \bar{f}(\xi) g(\xi) = \int d\bar{\xi} d\xi e^{-\bar{\xi}\xi} \mid (\bar{f}_0 + \bar{f}_1\bar{\xi})(\bar{g}_0 + \bar{g}_1\bar{\xi}) = \bar{f}_0 f_0 + \bar{g}_0 g_0 \tag{7.55}$$

Cases

We give some cases about resolution identity of fermionic coherent states

$$\mid \psi\rangle = \int \prod_{i=1}^{N} d\bar{\xi} d\xi e^{-\sum_{i=1}^{N} \bar{\xi}_i \xi_i} \psi(\bar{\xi}_1, \bar{\xi}_2, \cdot \bar{\xi}_n) \mid \xi_1, \xi_2, \cdots \xi_n\rangle \tag{7.56}$$

we consider annihilating fermion on coherent states

$$\begin{cases} c \mid \xi\rangle = c(\mid 0\rangle - \xi c^\dagger \mid 0\rangle) = \xi \mid 0\rangle = \xi_i \mid \xi\rangle \implies \langle \xi \mid c^\dagger = \bar{\xi}\langle \xi \mid \\ c^\dagger \mid \xi\rangle = c^\dagger(1 - \xi c^\dagger) \mid 0\rangle = c^\dagger \mid 0\rangle = -\partial_\xi \mid \xi\rangle \implies \langle \xi \mid c = \partial_\xi \langle \xi \mid = \end{cases} \tag{7.57}$$

We consider matrix element on the coherent states

$$\begin{cases} \langle \xi \mid c \mid \psi\rangle = \int d\bar{\xi} d\xi e^{-\bar{\xi}\xi} \langle \xi \mid c \mid \xi\rangle\langle \xi \mid \psi\rangle = \int d\bar{\xi} d\xi \xi \langle \xi \mid \psi\rangle = \int d\bar{\xi} \langle \xi \mid \psi\rangle = \partial_{\bar{\xi}} \psi(\bar{\xi}) \\ \langle \xi \mid c^\dagger \mid \psi\rangle = \int d\bar{\xi} d\xi e^{-\bar{\xi}\xi} \langle \xi \mid c^\dagger \mid \xi\rangle\langle \xi \mid \psi\rangle = \int d\bar{\xi} d\xi \xi \langle \xi \mid \psi\rangle = \int d\bar{\xi} d\bar{\xi}(1 - \bar{\xi}\xi)\bar{\xi}\langle \xi \mid \psi\rangle = \bar{\xi}\partial_{\bar{\xi}} \psi(\bar{\xi}) \end{cases}$$

The evaluation pf density operator N could be wriiten as

$$\begin{aligned} &\frac{\langle \xi_1, \xi_2, \cdots \xi_n \mid \hat{N} \mid \xi_1, \xi_2, \cdots \xi_n\rangle}{\langle \xi_1, \xi_2, \cdots \xi_n \mid \xi_1, \xi_2, \cdots \xi_n\rangle} = \sum_{i=1}^{N} \frac{\langle \xi_1, \xi_2, \cdots \xi_n \mid c_i^\dagger c_i \mid \xi_1, \xi_2, \cdots \xi_n\rangle}{\langle \xi_1, \xi_2, \cdots \xi_n \mid \xi_1, \xi_2, \cdots \xi_n\rangle} = \sum_{i=1}^{N} \bar{\xi}_i \frac{\langle \xi_1, \xi_2, \cdots \xi_n \mid c_i \mid \xi_1, \xi_2, \cdots \xi_n\rangle}{\langle \xi_1, \xi_2, \cdots \xi_n \mid \xi_1, \xi_2, \cdots \xi_n\rangle} \\ &= \sum_{i=1}^{N} \bar{\xi}_i \frac{\langle \xi_1, \xi_2, \cdots \xi_n \mid c_i \mid \xi_1, \xi_2, \cdots \xi_n\rangle}{\langle \xi_1, \xi_2, \cdots \xi_n \mid \xi_1, \xi_2, \cdots \xi_n\rangle} \\ &= \sum_i \bar{\xi}_i \xi \end{aligned} \tag{7.58}$$

Cases

We consider general form of hamiltonian

$$H = \sum_i \varepsilon_i c_i^\dagger c_i + V \sum_{i,j} c_i^\dagger c_k^\dagger c_k c_i \tag{7.59}$$

$$\langle H\rangle = \sum_i \varepsilon_k \bar{\xi}_i \xi_i + V \sum_{i,k} |\ \xi_i\ |^2 |\ \xi_k\ |^2 \tag{7.60}$$

In most cases , we need to evaluate the Grassmann Gaussian integal .

Theorem 7.1 .

$$\mathcal{Z}[\bar{\zeta}, \zeta] = \int \prod_{i=1}^{N} d\bar{\xi}_i d\xi_i e^{-\sum_{i,j} \bar{\xi}_i M_{ij} \xi + \bar{\xi}_i \zeta_i + \bar{\zeta}_i \xi_i} \tag{7.61}$$

Proof. Firstly , we consider more simple form , namly

$$\begin{aligned}
\int \prod_{i=1}^{N} d\bar{\xi}_i d\xi_i e^{-\sum_{i,j} \bar{\xi}_i M_{ij} \xi} &= \int \prod_{i=1}^{N} d\bar{\xi}_i d\xi_i (1 - M_{\sigma(1),1} \bar{\xi}_{\sigma(1)} \xi_1) \cdots (1 - M_{\sigma(n),n} \bar{\xi}_{\sigma(n)} \xi_n) \\
&= \int \prod_{i=1}^{N} d\bar{\xi}_i d\xi_i (-M_{\sigma(1),1} \bar{\xi}_{\sigma(1)} \xi_1)(-M_{\sigma(2),2} \bar{\xi}_{\sigma(2)} \xi_2) \cdots (-M_{\sigma(n),n} \bar{\xi}_{\sigma(n)} \xi_n) \\
&= \int \prod_{i=2}^{N} d\bar{\xi}_i d\xi_i d\bar{\xi}_1 M_{\sigma(1),1} \bar{\xi}_{\sigma(1)} (-M_{\sigma(2),2} \bar{\xi}_{\sigma(2)} \xi_2) \cdots (-M_{\sigma(n),n} \bar{\xi}_{\sigma(n)} \xi_n) \\
&\vdots \\
&= \int \int \prod_{i=2}^{N} d\bar{\xi}_i \bar{\xi}_{\sigma(1)} \bar{\xi}_{\sigma(2)} \cdots \bar{\xi}_{\sigma(n)} M_{\sigma(1),1} M_{\sigma(2),2} \cdots M_{\sigma(n),n} \\
&= \det(M)
\end{aligned} \tag{7.62}$$

Now we consider the shft the varibales ξ_i

$$\begin{aligned}
\sum_{ij} (\bar{\eta}_i + \bar{\alpha}_i) M_{ij} (\eta_j + \alpha_j) + \sum_i \bar{\zeta}_i (\eta_i + \alpha_i) + (\bar{\eta}_i + \bar{\alpha}_i)_i \zeta_i &= \sum_{ij} \bar{\eta}_i M_{ij} \eta_j + \sum_i \eta_i (\zeta_i + \sum_j M_{ij} \alpha_j) \\
\sum_i (\bar{\zeta}_i + \sum_j \bar{\alpha}_j M_{ji}) \eta_i + \sum_{ij} \bar{\alpha}_i M_{ij} \alpha_j + \sum_i \bar{\zeta}_i \alpha_i + \bar{\alpha}_i \zeta_i
\end{aligned} \tag{7.63}$$

We let $\zeta_i + M_{ij}\alpha_j = 0$, then (7.63) turns into

$$\sum_{ij} (\bar{\eta}_i + \bar{\alpha}_i) M_{ij} = \sum_{ij} \bar{\eta}_i M_{ij} \eta_j + \sum_{ij} \bar{\zeta}_i M_{ij}^{-1} \zeta_j - \sum_i \bar{\zeta}_i M_{ij}^{-1} \zeta_j - \sum_i \bar{\zeta}_i M_{ij}^{-1} \zeta_j \tag{7.64}$$

□

Hence, the Eq(7.61) could be derived with Eq(7.62)

$$Z = \int \mathcal{D}[\bar{\xi}, \xi] e^{-S} \qquad S = \int_0^\beta d\tau \left(\bar{\xi} \partial \xi + H[\bar{\xi}, \xi]\right) \tag{7.65}$$

By the same way , the fermionic path integral could be derived as

Question 3

Please review the process on section (7.5.2) to derive Eq(7.65)

7.6 Fluctuatio -dissipatio theorem

7.6.1 Kramers-Kronig theorem

The Kramers-Kronig theorem related the real part and imgnary part of susceptibility. The susceptibility is the frequency space has such form

$$\chi(\omega) = \mathcal{F}\left(-\mathcal{T}\langle[A(t), A(t')]\rangle\theta(t-t')\right) = \mathcal{F}(-\mathcal{T}\langle[A(t), A(t')]\rangle) * \mathcal{F}(\theta(t-t')) \tag{7.66}$$

Note:-

$$\mathcal{F}(t)) = \lim_{\varepsilon\to 0}\int_0^\infty dte^{t(-\varepsilon+\mathrm{i}\omega)}t = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon-\mathrm{i}\omega} = \mathrm{i}\left(\frac{1}{\omega} - \mathrm{i}\pi\delta(\omega)\right) \tag{7.67}$$

We substitute (7.67) into (7.66)

$$\chi(\omega) = \mathcal{F}(\chi''(t)) * F(\theta(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\mathrm{i}\left(\frac{1}{\omega-\omega'} - \mathrm{i}\delta(\omega-\omega')\right)\chi''(\omega') \tag{7.68}$$

Accrodin to (7.68), we can derive

$$\begin{aligned}\chi(\omega) = \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\frac{\chi''(\omega')}{\omega-\omega'} &= \frac{\mathrm{i}}{\pi}\int_0^\infty \frac{\chi''(\omega')}{\omega-\omega'} + \frac{\mathrm{i}}{\pi}\int_0^\infty\frac{\chi''(-\omega')}{\omega+\omega'}\\ &= \frac{\mathrm{i}}{\omega}\int_0^\infty\frac{(\omega+\omega')\chi''(\omega) + (\omega-\omega')\chi^{*\prime\prime}(\omega)}{\omega^2-\omega'^2}\\ &= \frac{\mathrm{i}}{\pi}\int_0^\infty\frac{2\omega\Re\chi''(\omega')}{\omega^2-\omega'^2}d\omega' - \frac{1}{\pi}\int_0^\infty\frac{2\omega'\Im\chi''(\omega')}{\omega^2-\omega'^2}d\omega'\end{aligned} \tag{7.69}$$

Hence, we could derive Kramers-Kronig relation

$$\begin{cases}\Re\chi(\omega) = -\dfrac{1}{\pi}\displaystyle\int_0^\infty\frac{2\omega'\Im\chi''(\omega')}{\omega^2-\omega'^2}d\omega'\\ \Im\chi(\omega) = \dfrac{\mathrm{i}}{\pi}\displaystyle\int_0^\infty\frac{2\omega\Re\chi''(\omega')}{\omega^2-\omega'^2}d\omega'\end{cases} \tag{7.70}$$