Chapter 13

Kosterilitz-Thouless transition

13.1 Algebraic order in the 2D XY model

The XY model is described by classical hamiltonian

$$H = -J \sum_{\langle i,j \rangle} S_i \cdot S_j \tag{13.1}$$

In the low temperature, the anglae difference is very small . We expand hamiltonian (13.1) into

$$\begin{aligned} H &= -J \sum_{\langle i,j \rangle} S_i \cdot S_j \\ &= -J \sum_{\langle i,j \rangle} \left(1 - \frac{1}{2}(\theta_i - \theta_j)^2 \right) \\ &= E_0 + \frac{J}{2} \int d^2 r (\nabla \phi(r))^2 \end{aligned} \tag{13.2}$$

13.1.1 Average magnetization

We calculate the average magnetization of hamiltonian (13.2)

$$\begin{aligned} \langle S_x \rangle = \langle \cos\theta(0) \rangle &= \frac{\mathrm{Tr}_{\theta_i}\left(e^{-\beta H} \cos\theta(r)\right)}{\mathrm{Tr}_{\theta_i}\left(e^{-\beta H}\right)} \\ &= \Re \left(\frac{1}{Z} \mathrm{Tr}_{\theta_i} \left(e^{-\beta H} e^{\mathrm{i}\theta(0)} \right) \right) \end{aligned} \tag{13.3}$$

We use the path integral to calculate (13.3)

$$\begin{aligned} \mathrm{Tr}_{\theta_i}\left(e^{-\beta H} e^{\mathrm{i}\cos\theta(r)}\right) &= \prod_k \int \mathcal{D}[\theta_k] \exp\left(-\beta \left(E_0 - \frac{Jk^2a^2}{2} \theta_k \theta_{-k} + \mathrm{i}\theta(k) \right) \right) \\ &= Z \prod_k \exp\left(-\frac{2}{Jk^2a^2\beta} \right) \end{aligned} \tag{13.4}$$

Combing with (13.3),13.4), the momentum has UV and IR cutoff , nmaly $k \in [\frac{\pi}{L}, \frac{\pi}{a}]$.

$$\langle S_x \rangle = \exp\left(-\frac{2k_BT}{J} \int \frac{k^2 dk}{(2\pi)^2} \frac{1}{k^2} \right) = \exp\left(-\frac{k_BT}{\pi J} \log\frac{L}{a} \right) = \left(\frac{a}{L} \right)^{\frac{k_BT}{\pi J}} \tag{13.5}$$

13.1.2 Correlation length

The correaltion function is defined as

$$G(r) = \langle e^{\mathrm{i}(\theta(r)-\theta(0))} \rangle = \frac{\mathrm{Tr}_{\theta_i}\left(e^{\mathrm{i}(\theta(r)-\theta(0))}e^{-\beta H}\right)}{\mathrm{Tr}_{\theta_i}\left(e^{-\beta H}\right)} \tag{13.6}$$

By the same way , we use path integral to calculate the correlation function

$$\mathrm{Tr}_{\theta_i}\left(e^{\mathrm{i}(\theta(r)-\theta(0))}e^{-\beta H}\right) = \prod_k \int \mathcal{D}[\theta_k]e^{-\beta E_0}\exp\left(-\frac{Jk^2a^2}{2k_BT}\theta_k\theta_{-k} + \theta_k(e^{\mathrm{i}kr}-1)\right) \tag{13.7}$$

In virtue of complex field θ_k, we split it into imaginary part and real part

$$\theta_k = \alpha_k + \mathrm{i}\beta_k \tag{13.8}$$

Hence, the Eq(13.7) turns into

$$\begin{aligned}\mathrm{Tr}_{\theta_i}\left(e^{\mathrm{i}(\theta(r)-\theta(0))}e^{-\beta H}\right) &= \prod_k e^{-\beta E_0}\int \mathcal{D}[\alpha_k]\exp\left(-\frac{Jk^2a^2}{2k_BT}\alpha_k^2 + \alpha_k(e^{\mathrm{i}kr}-1)\right)\int \mathcal{D}[\beta_k]\exp\left(-\frac{Jk^2a^2}{2k_BT} + \mathrm{i}\beta_k(e^{\mathrm{i}kr}-1)\right)\\ &= Z\exp\left(-\sum_k \frac{k_BT(e^{\mathrm{i}kr}-1)}{aJk^2a^2}\right)\end{aligned} \tag{13.9}$$

We consider summation on the bracket

$$\sum_k \frac{k_BT(e^{\mathrm{i}kr}-1)}{aJk^2a^2} \to \frac{k_BT}{4\pi^2 J}dk\int_0^{2\pi}\int_0^{\frac{\pi}{a}}\frac{1-\cos(kr\cos\theta)}{k}dk = \frac{k_BT}{2\pi J}\int_0^{\frac{\pi}{2}}\frac{1-J_0(kr)}{k}dk \tag{13.10}$$

On the ultraviolet region , the Bessel function $J_0(kr)$ tends to zero. The (13.10) could be approximated into [1]

$$\frac{k_BT}{2\pi J}\log\frac{r\pi}{a} \tag{13.11}$$

The behaviour of correlation function of (13.6) admits power law decay

$$G(r) \sim \left(\frac{r}{a}\right)^{-\eta(T)} \qquad \eta(T) = \frac{k_BT}{2\pi J} \tag{13.12}$$

In the low temperature , the correlation function admits long range behaviour. At the high temperature , the correlation length have exponential decay behaviour . The ferromagnetic order would be destroyed into disorder phase. We can make hypothesis that the system undergoes phase transition.

13.1.3 Vortices and entropy

Vortices are topological defects of filed $\theta(r)$ satisfy Laplace equation $\nabla^2\theta(r) = 0$. The nontrial solution of two dimensional Laplace equation is vortex solution.

$$\oint_{\mathcal{C}} \nabla\theta(r)\cdot d\ell = 2\pi n \tag{13.13}$$

[1] Firstly, you should make integral (13.10) into dimensionless integral

where n is the inding number . Can proliferation of vortex destroy ferromagnetic order? Now we will give the argument bases on free enenrgy . Lst's estimate the single voretex energy.

$$\varepsilon_0 = \frac{J}{2}\int d^3r\nabla\theta(r)\cdot dl = \frac{J}{2}\int_0^{2\pi} d\theta \int_a^L \frac{n^2}{r}dr = \pi J n^2 \log\frac{L}{a} \tag{13.14}$$

We can put single vortex into system with $\left(\frac{L}{a}\right)^2$ ways . The entropy could be derived with Boltzmann entropy

$$S = 2k_B \log\frac{L}{a} \tag{13.15}$$

Hence, the free energy to creation of single isolated vortex is

$$\Delta F = \Delta E - T\Delta S = \left(\pi J n^2 - 2k_B T\right)\log\frac{L}{a} \tag{13.16}$$

- $T < \frac{\pi J}{2}$. The creation of single vortex isn't favorable . The system tend to form vortex -anti-vortex bounded state to keep in netral.
- $T > \frac{\pi J}{2}$. The isolated vortex tend to proliferate.

13.2 Columb gas analogy

To proceed renoramlization group analysis, we should write down the partittion function . It's know to us that gradient field has no curl . We decompose the θ into regular part and singular part.

$$\nabla\theta = \vec{u}_{\text{reg}} + \vec{u}_s \tag{13.17}$$

The regular part is free from curl . However , the singular part will contribute votrtex integral . For example, we can take $\theta = \frac{y}{x}$, then this field correpsonds to vortex with winding number one. The singular field satisifes to

$$\oint_{\mathcal{C}} \nabla\theta(r)\cdot d\ell = \int d^2r\hat{z}\cdot\nabla\times(\vec{u}_s) = 2\pi n \tag{13.18}$$

Note:-

We can make ansat that

$$\nabla\times\vec{u}_s = 2\pi\sum_i n_i\delta(r-r_i)\hat{z} \tag{13.19}$$

We can set $\vec{u}_s = \nabla\times\psi\hat{z}$,

$$\nabla\times\vec{u}_s = \nabla\times(\hat{z}\psi) = \nabla\psi\times\hat{z} \tag{13.20}$$

We substitute it into (13.19)

$$\nabla^2\psi = -2\pi\sum_i\delta(r-r_i) \tag{13.21}$$

The solution ψ could be derived as

$$\psi = -\sum n_i \log \mid r - r_i \mid \tag{13.22}$$

In the Columb gas language , the physical meaning of ψ is just scalar potential genrated by charge density $\sum_i n_i \delta(r - r_i)$.

The continuum halmitoian could be written into regular field ϕ and singular field ψ

$$H = -\frac{J}{2} \int d^2r \left(\nabla\phi + \nabla \times (\psi \hat{z})\right)^2 = -\frac{J}{2} \int d^2r \left((\nabla\phi)^2 + 2\nabla\phi \cdot \nabla \times (\psi\hat{z}) + (\nabla \times (\psi\hat{z}))^2\right) \tag{13.23}$$

The first term on the (13.23) is just spin wave part, which could integrate out by gaussian integral. The second term could be written into total partial , which is vanishing on the boundary.

$$\begin{aligned}
\int d^2r \nabla\phi \cdot \nabla \times (\psi\hat{z}) &= \int d^2r \nabla\phi \cdot (\nabla\psi \times \hat{z}) = \int d^2r \hat{z} \cdot (\nabla\phi \times \nabla\psi) \\
&= \int d^2r \varepsilon_{ij} \partial_i \phi \partial_j \psi \\
&= \int d^2r \varepsilon_{ij} \partial_i (\phi \partial_j \psi) - \phi \varepsilon_{ij} \partial_i \partial_j \psi
\end{aligned} \tag{13.24}$$

Hence, the hamiltonian will simplified into

$$H = -\frac{J}{2} \int d^2r \left(\nabla^2\phi + \nabla^2\psi\right) \tag{13.25}$$

We make partial integral for the singular part

$$\begin{aligned}
\int d^r \nabla^2\psi = \int d^2r \nabla \cdot (\psi\nabla\psi) - \phi\nabla^2 a\psi &= -2\pi \sum_{i,j} n_i n_j \log \mid n_i - n_j \mid \\
= -H_{\text{core}} - 2\pi \sum_{i<j} n_i n_j \log \mid n_i - n_j \mid
\end{aligned} \tag{13.26}$$

The partition function could be obtained as

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}[\phi] e^{-\frac{J}{2}\int d^2r (\nabla\phi(r))^2} \sum_{N=0}^{\infty} \frac{1}{N!^2} \int\int \prod_{i=1}^{2N} \frac{dp_i^2 dx_i^2}{h^{2N}} e^{2\pi J \sum\limits_{i<j} n_i n_j \log|r_i - r_j|} \\
&= Z_{\text{spin}} \sum_{N=0}^{\infty} \frac{y_0^{2N}}{N!^2} \int \prod_{i=1}^{2N} \frac{dx_i^2}{a^2} \exp\left(2\pi J \sum_{i<j} n_i n_j \log \mid r_i - r_j \mid\right)
\end{aligned} \tag{13.27}$$

where y_0 is the dimensionless characteristic quantity $y_0 = \dfrac{\sqrt{2\pi m k_B T a^2}}{h}$

13.2.1 RG flow equation

We use the perturbative treatement to this sytem . We consider two chargeds at position s, s'. Our effective hamiltonian is the average of external charge

$$e^{H_{eff}(r-r')} = \langle e^{-2J\pi \log|r-r'|} \rangle \tag{13.28}$$

Let;s use the partition function (13.27) to write down the effective hamiltonian

$$
\begin{aligned}
\langle e^{-2K\pi \log|r-r'|}\rangle &= \frac{\sum\limits_{N=0}^{\infty} \frac{y_0^{2N}}{N!^2} \int \prod\limits_{i=1}^{2N} \frac{dx_i^2}{a^2} \exp\left(-2J\pi \log \mid r-r' \mid +2\pi J \sum\limits_{i<j} n_i n_j \log \mid r_i - r_j \mid\right)}{\sum\limits_{N=0}^{\infty} \frac{y_0^{2N}}{N!^2} \int \prod\limits_{i=1}^{2N} \frac{dx_i^2}{a^2} \exp\left(2\pi J \sum\limits_{i<j} n_i n_j \log \mid r_i - r_j \mid\right)} \\
&= \frac{\exp(-2J\pi \log \mid r-r' \mid)\left(1 + y_0^2 \int dsds' e^{-2J\pi \log|s-s'|+2J\pi D(r,r';s,s')}\right)}{\left(1 + y_0^2 \int dsds' e^{-2J\pi \log|s-s'|+2J\pi D(r,r';s,s')}\right)} \\
&= \exp(-2J\pi \log \mid r-r' \mid)\left(1 + y_0^2 \int dsds' e^{-2J\pi \log s-s'|}\left(e^{3J\pi D(r,r';s,s')} - 1\right)\right)
\end{aligned} \tag{13.29}
$$

where the $D(r, r'; s, s')$ is the interaction between external charge and the single dipole.

$$
D(r, r'; s, s') = \log \mid s - r \mid - \log \mid s - r' \mid + \log \mid s' - r' \mid - \log \mid s' - r \mid \tag{13.30}
$$

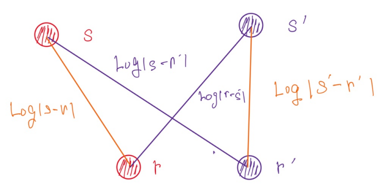

Figure 13.1: The interaction between the external charge and dipole

Note:-

n this short note, we will introduce central coordinate $X = \frac{s+s'}{2}$ and relative coordinate $xs - S6\prime$ to expand the interaction term $D(r, r'; s, s')$.

$$
\begin{cases}
\log \mid s - r \mid = \log \mid X + \frac{x}{2} - r \mid = \log \mid X - r \mid + \frac{x}{2} \cdot \nabla_X \log \mid X - r \mid + \frac{1}{2}\left(\frac{x}{2} \cdot \nabla_X\right)^2 \log \mid X - r \mid + \cdots \\
\log \mid s' - r \mid = \log \mid X - \frac{x}{2} - r \mid = \log \mid X - r \mid - \frac{x}{2} \cdot \nabla_X \log \mid X - r \mid + \frac{1}{2}\left(\frac{x}{2} \cdot \nabla_X\right)^2 \log \mid X - r \mid + \cdots \\
\log \mid s - r' \mid = \log \mid X + \frac{x}{2} - r' \mid = \log \mid X - r' \mid + \frac{x}{2} \cdot \nabla_X \log \mid X - r' \mid + \frac{1}{2}\left(\frac{x}{2} \cdot \nabla_X\right)^2 \log \mid X - r' \mid + \cdots \\
\log \mid s' - r' \mid = \log \mid X - \frac{x}{2} - r' \mid = \log \mid X - r' \mid - \frac{x}{2} \cdot \nabla_X \log \mid X - r' \mid + \frac{1}{2}\left(\frac{x}{2} \cdot \nabla_X\right)^2 \log \mid X - r' \mid + \cdots
\end{cases} \tag{13.31}
$$

We substitute the (13.31) into (13.30)

$$
D(r, r'; s, s') = x \cdot \nabla_X \left(\log \mid X - r \mid - \log \mid X' - r' \mid\right) \tag{13.32}
$$

We expand the $e^{2J\pi D(r,r';s,s')}$ to second order

$$
e^{2J\pi D(r,r';s,s')} = 1 + 2J\pi x \cdot \nabla_X \left(\log \mid X - r \mid - \log \mid X - r' \mid\right) + \frac{1}{2}\left(2J\pi x \cdot \nabla_X \left(\log \mid X - r \mid - \log \mid X - r' \mid\right)\right)^2 + \cdots \tag{13.33}
$$

The integral measurement could be expressed by varibales X, x as $\int d^2 s d^2 s' = \int d^2 x d^2 X$. We subtitute the (13.33) into (13.29)

$$\int dsds' e^{-2J\pi\log|s-s'|+2J\pi D(r,r';s,s')} = \int d^2xd^2X(x)^{-2J\pi}\left(1+2J\pi x\cdot\nabla_X\log\left|\frac{r-X}{r'-X}\right|+2J^2\pi^2\left(x\cdot\nabla_X\log\left|\frac{r-X}{r'-X}\right|\right)^2+c\dots\right. \tag{13.34}$$

Note:-

Let's analyze the trems on the (13.34)

$$\int d^2xd^2Xe^{-2J\pi\log x}x\cdot\nabla_X\log\left|\frac{r-X}{r'-x'}\right| = \int d^2xe^{-2J\pi\log x}x\cdot\int d^2X\nabla_X\log\left|\frac{r-X}{r'-x'}\right| = 0 \tag{13.35}$$

$$\begin{aligned}&\int d^2xd^2X\left(x\cdot\nabla_X\log\left|\frac{r-X}{r'-X}\right|\right)^2 = \int_0^\infty e^{-2J\pi\log x}x^3dx\int d^2X\\&\int_0^{2\pi}d\theta\left(\cos\theta\nabla_{X_1}\log\left|\frac{r-X}{r'-X}\right|+\sin\theta\nabla_{X_2}\log\left|\frac{r-X}{r'-X}\right|\right)^2\\&=\pi\int d^2xd^2X\left(x\cdot\nabla_X\log\left|\frac{r-X}{r'-X}\right|\right)^2=\pi\int_0^\infty e^{-2J\pi\log x}x^3dx\int d^2X\left(\nabla_X\log\left|\frac{r-X}{r'-X}\right|\right)^2\end{aligned} \tag{13.36}$$

We calculate the last part integral

$$\begin{aligned}\int d^2X\left(\nabla_X\log\left|\frac{r-X}{r'-X}\right|\right)^2 &= \left(\nabla_X\log\left|\frac{r-X}{r'-X}\right|\right)_{X\to\infty}-\int d^2X\log\left|\frac{r-X}{r'-X}\right|\nabla_X^2\log\left|\frac{r-X}{r'-X}\right|\\&=-\int d^2X\log\left|\frac{r-X}{r'-X}\right|(2\pi\delta(X-r)-2\pi\delta(X-r')\mid)\\&=-2\pi\int d^2X(\log\mid r-r\mid-\log\mid r'-r\mid-\log\mid r-r'\mid+\log\mid r'-r'\mid)\\&=4\pi\log\mid r-r'\mid\end{aligned} \tag{13.37}$$

Hence, the integral (13.36) turns into

$$\begin{aligned}&\exp(-2J\pi\log\mid r-r'\mid)\left(1+y_0^2\int dsds'e^{-2J\pi\log|s-s'|+2J\pi D(r,r';s,s')}\right)=exp(-2J\pi\log\mid r-r'\mid)\\&\left(1+8J^2\pi^4y_0^2\log\mid r-r'\mid\int_1^\infty x^{3-2J\pi}dx+\mathcal{O}(y_0^4)\right)\end{aligned} \tag{13.38}$$

We use the lattice cut-off to revise the divergent integral

We could write down the K_{eff} from the (13.38)

$$K_{\text{eff}} = K-4\pi^3K^2y_0^2\int_1^\infty dxx^{3-2\pi K} \tag{13.39}$$

To be convenient , we use the K^{-1}

$$K_{\text{eff}}^{-1}=\frac{1}{K}\frac{1}{1-4\pi^2Ky_0^2\int_1^\infty dxx^{3-2\pi K}}\simeq K^{-1}+4\pi^2y_0^2\int_1^\infty dxx^{3-2\pi K} \tag{13.40}$$

- Scale $x : 1 \mapsto b$

$$K_{\text{eff}}^{-1} = \left(K^{-1} + 4\pi^2 y_0^2 \int_1^b dx x^{3-2\pi K} \right) + 4\pi^2 y_0^2 \int_b^\infty dx x^{3-2\pi K} \tag{13.41}$$

- Rescale the $x : x \mapsto x/b$

$$\tilde{K}^{-1} = K^{-1} + 4\pi^3 \tilde{y}_0^2 \int_{1/b}^1 dx x^{3-2\pi K} \qquad \tilde{y}_0 = b^{2-\pi K} y_0 \tag{13.42}$$

We choose the infinitesimal renormalization parameter $b = e^l$

$$\tilde{y}_0 = y_0 \left(1 + (2 - \pi K)dl + \mathcal{O}(dl^2)\right) \implies \frac{dy_0}{dl} = (2 - \pi K) y_0 \tag{13.43}$$

$$\begin{aligned}
\tilde{K}^{-1} &= K^{-1} + 4\pi^3 \tilde{y}_0^2 \int_{1/b}^1 dx x^{3-2\pi K} = K^{-1} + 4\pi^3 \tilde{y}_0^2 \left. \frac{x^{4-\pi K}}{4 - \pi K} \right|_{\frac{1}{b}}^1 \\
&= K^{-1} + 4\pi^3 \tilde{y}_0^2 \frac{1 - e^{-(4-\pi K)l}}{4 - \pi K} \\
&= K^{-1} + 4\pi^3 l \tilde{y}_0^2 \implies \frac{d\tilde{K}^{-1}}{dl} = 4\pi^3 \tilde{y_0}^2
\end{aligned} \tag{13.44}$$

Now, we have derive the *RG* equation. To simplify problem, we focus on the behaviour of fixed point. The RG equation (13.43) tells us that $K = \frac{2}{\pi}$ is fixed point, which gives the critical temperature

$$T = \frac{\pi}{2} J/k_B \tag{13.45}$$

This result is meeted with vortice argument (13.16). We introduce new varibales $t = \frac{\pi}{2} - K^{-1}$ [2] to study the behviour near fixed point.

$$\begin{cases} \dfrac{dt}{dl} = 4\pi^3 y^2 \\ \dfrac{dy}{dl} = \dfrac{4}{\pi} t y \end{cases} \tag{13.46}$$

The Eq(**??**) tells us conserve quantity

$$\frac{d}{dl}\left(t^2 - \pi^4 y^2\right) = 0 \tag{13.47}$$

[2]The variable t is small quantity, $2 - \pi K = \frac{2t}{2+\pi/2} \simeq \frac{4}{\pi} t + \mathcal{O}(t^2)$