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ABSENCE OF NEUTRINOS ON A LATTICE
(I). Proof by homotopy theory

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It is shown, by a homotopy theory argument, that for a general class of fermion theories on a Kogut–Susskind lattice an equal number of species (types) of left- and right-handed Weyl particles (neutrinos) necessarily appears in the continuum limit. We thus present a no-go theorem for putting theories of the weak interaction on a lattice. One of the most important consequences of our no-go theorem is that it is not possible, in strong interaction models, to solve the notorious species doubling problem of Dirac fermions on a lattice in a chirally invariant way.

1. Introduction

(a) It has been known for some time that incorporation of fermions on a lattice leads to further fermionic modes than those naively expected [1–3]. For example, a naive construction of a Weyl fermion (=neutrino) on a Kogut–Susskind lattice (i.e. discrete space and continuous time) yields eight Weyl fermions in the low energy regime. To be specific we consider the $3+1$ dimensional naive model of the Weyl equation on a lattice which is obtained by replacing $\boldsymbol{\partial}$ in the Weyl equation

$$i\frac{\partial}{\partial t}u(\boldsymbol{x})=\frac{1}{i}\boldsymbol{\sigma}\boldsymbol{\partial}u(\boldsymbol{x})$$

by differences

$$i\frac{\partial}{\partial t}u(\boldsymbol{x})=\sum_{i=1}^{3}\frac{1}{2i}\sigma_i\{u(\boldsymbol{x}+n_i)-u(\boldsymbol{x}-n_i)\}\,. \tag{1.1}$$

Here $u(\boldsymbol{x})$ is a two-component spinor. We take the lattice constant to be unity, and n_i denotes the unit vector in the x_i direction. In the momentum representation, eq. (1.1) takes the form

$$i\frac{\partial}{\partial t}\tilde{u}(\boldsymbol{p})=\sum_{i=1}^{3}\sigma_i\sin p_i\cdot\tilde{u}(\boldsymbol{p})\,.$$

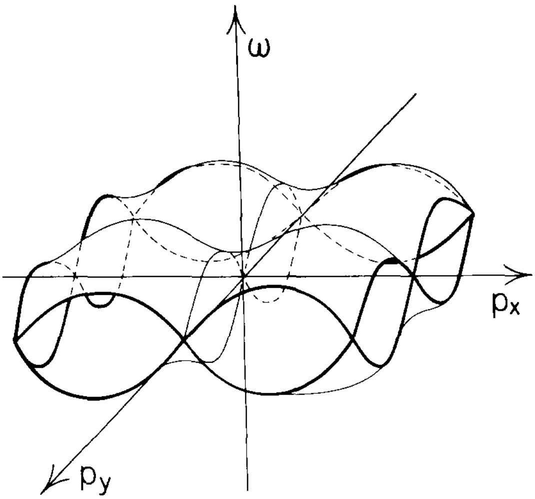

Fig. 1. The dispersion relation for the lattice fermion theory given by (1.1). The P_z dimension has been suppressed, and the drawing corresponds to $P_z = 0$. Just one Brillouin zone is drawn.

Since each sine function has two zeros in its dispersion relation for a period in a Brillouin zone in $\boldsymbol{p}$-space, there appear eight zeros of energy $\omega(\boldsymbol{p})$ at eight momentum values, as is depicted in fig. 1. These zeros represent Weyl particles in the dispersion relation in the long-wavelength limit.

It is the purpose of this article to formulate and prove a *no-go theorem*: the appearance of equally many right- and left-handed species (types) of Weyl particles with given quantum numbers is an unavoidable consequence of a lattice theory under some *mild assumptions*. Here species of neutrino means ν_e, ν_μ or ν_τ, etc. It should be stressed that our no-go theorem concerns *the number of species* of Weyl particles, but says *nothing about the actual number of Weyl particles* in the cosmos. The latter number is, of course, determined from how the states are filled or unfilled. The most important consequence of our no-go theorem is that *the weak interaction cannot be put on the lattice.*

Our "mild assumptions" include such important hypotheses as locality and exact conservation of discrete valued quantum number(s). Also, the charges are assumed to have a density defined from a finite region.

In a somewhat less general (Wilson model) and less rigorous form of this no-go theorem has very recently been put forward by Karsten and Smit [3a]. Firstly they argue for it from the Adler anomaly, but that would only give a theorem still allowing the standard $SU(3)\times SU(2)\times U(1)$ model with quarks and leptons to be put on a lattice. Since this will not be allowed by our theorem, the latter must be stronger. Secondly, they give, mainly for the Wilson model, a topological argument more similar to that applicable in $1+1$ dimensions, which we shall exhibit in our next article [12].

There are in the literature [2, 3, 4, 4a] various ingenious ways of getting rid of some of the extra and at first unwanted fermions in the low energy regime. Nobody seems, however, to have been able to get rid of all of them so that only one charged (with an exactly conserved quantum number) Weyl particle remains. If, however, no exactly conserved charge is required, a model with an odd number of Weyl particles can be built and we shall, in fact, present one in the successive article [12]. If one is interested in *strong interactions* and wants the Dirac particle rather than the Weyl particle, there should be no unsurmountable problem. A Dirac particle can be thought of as a composite of the components of two Weyl particles, a right-handed one and a left-handed one.

According to our no-go theorem it is *not*, however, *possible*[*] even in the strong interaction models, to keep chiral invariance conserved on the scale of the fundamental lattice. The important consequence of our work is to *discourage any attempt to construct chiral invariant lattice models for QCD.*

(b) We will consider the general class of lattice fermion theories for which the bilinear part of the action for the N-component complex fermion field $\psi(\boldsymbol{x})$ is of the form

$$S=-i\int \mathrm{d}t \sum_{\boldsymbol{x}} \dot{\psi}(\boldsymbol{x})\psi(\boldsymbol{x})-\int \mathrm{d}t \sum_{\boldsymbol{x},\boldsymbol{y}} \bar{\psi}(\boldsymbol{x})H(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y})\,, \tag{1.2}$$

with H a hamiltonian. Interactions of quartic or higher degree in ψ are neglected. These interactions do not change the dispersion relation that we are interested in. We in fact define the number of species of Weyl fermions so that it depends only on the bilinear part of the action and the dispersion relation and not on the interaction, which just causes scattering processes. The kinetic term in (1.2) is not the most general one, since we could have

$$\sum_{\boldsymbol{x},\boldsymbol{y}} \dot{\psi}(\boldsymbol{x})T(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y})\,. \tag{1.3}$$

With such a term we may risk the appearance of the singularities (poles) in the dispersion relation and, thus, particles with unbounded velocity. For our argument we shall need only the assumption that there are no such singularities. With eq. (1.2) we obtain the linear equation of motion

$$i\dot{\psi}(\boldsymbol{x})=\sum_{\boldsymbol{y}} H(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y})\,.$$

(If, however, we include a term like (1.3), the hamiltonian is effectively H/T.)

(c) We have assumed the following three conditions on the action (1.2):

(i) Locality of interaction, i.e. the hamiltonian satisfies $H(\boldsymbol{x}-\boldsymbol{y})\to 0$, when $|\boldsymbol{x}-\boldsymbol{y}|\to$ large, fast enough in the sense that the Fourier transform of $H(\boldsymbol{x})$ has continuous first derivative.

[*] See, as examples of non-chirally invariant ways [2, 3, 4a].

(ii) Translational invariance on the lattice (invariance under group translations by an integer number of lattice constants).

(iii) Hermiticity of the $N \times N$ matrix hamiltonian H (reality of S).

In momentum space the field takes the form

$$\tilde{\psi}(\boldsymbol{p}) = \frac{1}{V} \sum_{\boldsymbol{x}} e^{-i\boldsymbol{p}\cdot\boldsymbol{x}} \psi(\boldsymbol{x}) .$$

Since the momentum $\boldsymbol{p}$ is only unique modulo $2\pi\boldsymbol{n}$ ($\boldsymbol{n}$: set of integers) the independent field variables are only those inside the Brillouin zone with an interval, e.g.

$$-\pi \leq p_i \leq \pi , \qquad (i = 1, 2, 3) .$$

We should here mention the work of Drell, Weinstein and Yankielowicz [4]. Their method of discretizing the fermion theory is to replace ∇_μ by p_μ, i.e.

$$\partial_\mu \psi(\boldsymbol{x}) \to p_\mu \tilde{\psi}(\boldsymbol{p}) .$$

This may introduce a non-local interaction and violate our assumption (i).

The assumptions made for the charges Q (lepton number, say) of the theory are the following:

(i) Exact conservation of Q, even at scales where the lattice cutoff is relevant. Charge conservation means that the energy and momentum eigenstates are also charge eigenstates.

(ii) Q is locally defined, i.e.

$$Q = \sum_{\boldsymbol{x}} j^0(\boldsymbol{x}) .$$

The charge density $j^0(\boldsymbol{x})$ is a function of the field variables $\psi(y)$ related to y within a bounded distance from x.

(iii) Q is quantized. This is, for instance, the case if it generates an abelian closed subgroup of a compact group.

(iv) We also assume that Q is bilinear in the fermion field $\psi(x)$.

(d) It appears to be one of the important features of weak interactions and of the standard Weinberg–Salam model in particular, that right- and left-handed particles *do not* have the same hypercharge. In fact, in this way, parity and charge conjugation are broken in weak interaction processes. In the Weinberg–Salam model it is the different quantum number for right- and left-handed particles that prohibits masses for fermions modulo the Higgs mechanism. So if nature were indeed built on a lattice it should be possible to realize Weyl fermions with different quantum numbers for left- and right-handed ones. As we shall show, this is, however, just what cannot be done.

Thus we are faced with the following dilemma: we must give up either the idea that nature is based on a fundamental lattice cutoff, or some of our mild (helping) assumptions, or weak interaction phenomenology, i.e. the usual understanding of parity violation. The last possibility seems unacceptable indeed since the theory of weak interactions is well established on this point. As concerns the second possibility, we may be able to generalize our no-go theorem to an *amorphous lattice*. We hope to eliminate the assumption of locality of charge density and replace it by the assumption that the gauge field is coupled via the charge. Such generalizations will be presented in one of our forthcoming papers.

(e) In discussing the quantum numbers of the individual Weyl particles we mean a kind of chiral charge for the composite Dirac particles. The chiral charge* is, of course, defined as the difference in the number of right- and left-handed Weyl fermions present in the universe and has nothing to do with the number of species of Weyl fermions. Since a particle and its antiparticle have opposite quantum numbers, we should not consider the sector of opposite charge. Doing so would count the antiparticles of the particle once again. So we use the notation that we count only left-handed particles, letting the right-handed ones be represented by their antiparticles. Thus, our no-go theorem says that there are equal numbers of species of left-handed particles with one set of quantum numbers and with the opposite set.

If we have a chiral charge Q_{chiral}, we can consider the dispersion relation for those fermions with, say, $Q_{\text{chiral}} = -1$. This would correspond to left-handed particles and antiparticles of the right-handed particles of $Q_{\text{chiral}} = 1$. Since this dispersion relation contains only left-handed particles, this is against our no-go theorem. We can (somewhat imprecisely but still correctly) restate our no-go theorem: there are no *quantized conserved* chiral charges on a lattice. So we *cannot introduce chirally coupled (weak) gauge fields.*

(f) The ingredients of our proof are mainly topological** in character. In fact we have two topological arguments:

(i) Intuitive topological formulation. This proof will be relegated to the subsequent paper [12].

(ii) Algebraic topology. The argument considers mappings from closed surfaces homeopmorphic to the surface S_2 of a sphere imbedded in the Brillouin zone into the space CP^{N-1}. The space CP^{N-1} consists of rays of states superposed from the N fundamental fermion components. These mappings are classified into homotopy classes making up the group $\pi_2(CP^{N-1})$, which is isomorphic to the additive group of integers Z. We shall show that the classes corresponding to mappings from small S_2 spheres surrounding each degeneracy point in momentum space, which represents one species of Weyl particle, e.g. ν_e, ν_μ or ν_τ etc. are ± 1, depending on the handedness of the neutrinos in question. Then it is shown that the sum of the classes must be zero, i.e. the unit element of $\pi_2(CP^{N-1})$. Our no-go theorem follows from this sum rule.

* In refs. [4–6] chiral charge on a lattice is considered.

** A good textbook on topology is ref. [7] and for an excellent review of homotopy theory see ref. [8].

Both arguments (i) and (ii) are concerned with the topology in momentum space. It is crucial that *the space of momenta i.e. the Brillouin zone, makes up a torus* $S_1 \times S_1 \times S_1$.

This periodicity is the important way in which the lattice comes into the proof.

(g) Sect. 2 is devoted to defining the generic case of the hamiltonian. In sect. 3 we show how the Weyl particle comes out from the degeneracy point of the dispersion law in the continuum limit. Then, in sect. 4, we depict a strategy for the proof of the no-go theorem. Sects. 5 to 10 are devoted to giving a proof according to this strategy. Finally, in sect. 11, we draw the conclusions of this paper.

2. Generic case

Let us consider dispersion relations which look like e.g. fig. 1 [eq. (1.1)] given by the eigenvalue equation

$$H(\boldsymbol{p})\tilde{\psi}(\boldsymbol{p}) = \omega_i(\boldsymbol{p})\tilde{\psi}(\boldsymbol{p}) , \qquad (i = 1, \dots, N) . \tag{2.1}$$

We are interested in the types of dispersion relations that may arise from a generic set of hamiltonians. In fact, we do not need all such properties for our no-go theorem to work, but only assume the following:

(i) For almost all values of $\boldsymbol{p}$ there are N non-degenerate $\omega_i(\boldsymbol{p})$ which are ordered:

$$\omega_i(\boldsymbol{p}) > \omega_2(\boldsymbol{p}) > \cdots > \omega_N(\boldsymbol{p}) .$$

(ii) But at several separate points $\boldsymbol{p}_{\text{deg}}$, only a couple of levels are degenerate, e.g.

$$\omega_i(\boldsymbol{p}_{\text{deg}}) = \omega_{i+1}(\boldsymbol{p}_{\text{deg}}) .$$

Note that the two-level degeneracy is generic but the degeneracy of three or more is not. The reason is as follows. Consider the case of three-level degeneracy. The hamiltonian which is now a 3×3 matrix has the general form

$$H^{(3)}(\boldsymbol{p}) = \sum_{i=1}^{8} c_i^{(3)}(\boldsymbol{p})\lambda_i + d^{(3)}(\boldsymbol{p})1 ,$$

where λ_i are Gell–Mann's SU(3) matrices. For a three-level degeneracy the eight coefficients $c_i^{(3)}$ should satisfy the eight equations $c_i^{(3)}(p) = 0$ at $\boldsymbol{p} = \boldsymbol{p}_{\text{deg}}$, but, in reality, the parameters to be determined are just three $\boldsymbol{p}$. Thus these conditions are overdeterminant. On the other hand, in the case of two-level degeneracy we may have the 2×2 matrix hamiltonian

$$H^{(2)}(\boldsymbol{p}) = \sum_{i=1}^{3} \sigma^i c_i(\boldsymbol{p}) + d(\boldsymbol{p})1 .$$

The three parameters $c_i(\boldsymbol{p})$ should satisfy $c_i(\boldsymbol{p}) = 0$. These are just three equations for three unknowns $\boldsymbol{p}$, fixing the degenerate momentum $\boldsymbol{p}_{\text{deg}}$.

The above two properties (i) and (ii) can find phenomenological support, provided that the lattice theory is to be interpreted with a "renormalization" (or, rather, a redefinition) of the momenta, as will be explained in sect. 3.

Actually we do not fully need assumptions (i) and (ii); but to be able to formulate a theorem about Weyl fermions, we must assume that the low energy physics is described by particles with the same types of degeneracy as the Weyl equation has. That is, we need to assume that (by definition so to speak) each Weyl fermion represents two levels, being degenerate at one point but separate in a neighbourhood (in momentum space).

Then we can complete the proof given below by slightly modifying the hamiltonian into a generic one. It should not be too difficult to see that such a slight modification of the hamiltonian, and thus of the dispersion law, can be made without altering the number and handedness of the species of Weyl particles. So non-generic cases may be brought back to generic ones.

If we want a stable vacuum that is also in agreement with experiment, we should put $\omega_i(\boldsymbol{p}_{\rm deg})=0$. (However, there is a possibility of having an unstable vacuum and leaving $\omega_i(\boldsymbol{p}_{\rm deg})\neq 0$.) The following condition is required from phenomenology for getting a neutrino-like relativistically invariant dispersion relation in the continuum limit $p_0^2=\boldsymbol{p}^2$:

(iii) Zero energies in dispersion relations $\omega_i(\boldsymbol{p})=0$ are always achieved at degeneracy points $\boldsymbol{p}_{\rm deg}$. It should be mentioned that this is not a generic principle.

3. Weyl particle in the continuum limit

We may still have to consider N-component ψ's but for the behaviour near the degeneracy point it is the two linear combinations of field ψ_i and ψ_{i+1} that are of interest. We can therefore restrict ourselves to study a two-component case here. The eigenvalue equation near the degeneracy point $\boldsymbol{p}_{\rm deg}$ is given by

$$\begin{aligned} H^{(2)}(\boldsymbol{p})u^{(i)}(\boldsymbol{p}) &= \omega_i(\boldsymbol{p})u^{(i)}(\boldsymbol{p})\,, \\ H^{(2)}(\boldsymbol{p})u^{(i+1)}(\boldsymbol{p}) &= \omega_{i+1}(\boldsymbol{p})u^{(i+1)}(\boldsymbol{p})\,, \end{aligned} \tag{3.1}$$

with two component $u^{(i)}$ and $u^{(i+1)}$. The 2×2 matrix $H^{(2)}(\boldsymbol{p})$ can be expanded in a Taylor series around $\boldsymbol{p}_{\rm deg}$,

$$H^{(2)}(\boldsymbol{p})=\omega_{\rm deg}(\boldsymbol{p}_{\rm deg})+(\boldsymbol{p}-\boldsymbol{p}_{\rm deg})_k\sigma^\alpha V_\alpha^k+(\boldsymbol{p}-\boldsymbol{p}_{\rm deg})\boldsymbol{a}+\mathrm{O}((\boldsymbol{p}-\boldsymbol{p}_{\rm deg})^2)\,.$$

Here the constants $\boldsymbol{a}$ and V_α^k depend on the degeneracy point. Thus the eigenvalue equation (3.1) becomes in the lowest order of the expansion

$$(\boldsymbol{p}-\boldsymbol{p}_{\rm deg})_k\sigma^\alpha V_\alpha^k u(\boldsymbol{p})=\{\omega(\boldsymbol{p})-\omega_{\rm deg}(\boldsymbol{p}_{\rm deg})-(\boldsymbol{p}-\boldsymbol{p}_{\rm deg})\boldsymbol{a}\}u(\boldsymbol{p})\,. \tag{3.2}$$

We may define a new coordinate system the "practical momentum" ($\omega_{\rm pr}$, $p_{\rm pr}$) by

$$\begin{aligned} \omega_{\rm pr} &= \omega(\boldsymbol{p})-\omega_{\rm deg}(\boldsymbol{p}_{\rm deg})\,, \\ \boldsymbol{p}_{\rm pr} &= \boldsymbol{p}-\boldsymbol{p}_{\rm deg}\,. \end{aligned} \tag{3.3}$$

We may call the original $\boldsymbol{p}$ the "fundamental momentum". On the introduction of a new coordinate $(P_0, \boldsymbol{P})$ by

$$P_0 = \omega - \omega_{\rm pr} - \boldsymbol{p}_{\rm pr}\boldsymbol{a} ,\qquad P_\alpha = p_{{\rm pr}k} V^k_\alpha , \tag{3.4}$$

the equation of motion (3.1) becomes

$$\boldsymbol{\sigma P} u(\boldsymbol{p}) = P_0 u(\boldsymbol{p}) , \tag{3.5}$$

for small P_0 and $\boldsymbol{P}$. Then we get the usual relativistic neutrino-like dispersion relation $P_0^2 = \boldsymbol{P}^2$ in the long-wavelength limit. It should be stressed that each degeneracy point represents one species (type) of neutrino.

Now we have made a "renormalization" of momenta by eq. (3.3) to get eq. (3.5). This means that the momentum concept $\boldsymbol{p}_{\rm pr}$ used by physicists, who may not care for the lattice theory, deviates by an additive constant $\boldsymbol{p}_{\rm deg}$ from the one that is obtained by the Fourier transform of x on the lattice. In fact, we should subtract the momentum $\boldsymbol{p}_{\rm deg}$ in (3.3). If $\omega_i(\boldsymbol{p}_{\rm deg}) \neq 0$ (this is the generic situation) we should also "renormalize" the energy of a Weyl particle by the prescription in eq. (3.3).

Our philosophy corresponds to the assumption that the Dirac particles in nature are built up from Weyl components with different momentum. So the Higgs-like mechanism, to make mass terms, would have to break the conservation of the fundamental momentum p and conserve $\boldsymbol{p} - \boldsymbol{p}_{\rm deg}$.

The spin of the field $u(\boldsymbol{p})$ is to be determined. Eq. (3.5) for $u(\boldsymbol{p})$ is invariant under rotations generated by $\boldsymbol{J} = \boldsymbol{r} \times \boldsymbol{P} + \frac{1}{2}\boldsymbol{\sigma}$ with the definition $\boldsymbol{r} = i\partial/\partial \boldsymbol{P}$ since $[\boldsymbol{J}, \boldsymbol{\sigma P}] = 0$. The spin of u is $\frac{1}{2}\boldsymbol{\sigma}$. The state with $P_0 > 0$ (upper cone) has $\boldsymbol{\sigma p} > 0$ which means $+1$ helicity in the $(P_0, \boldsymbol{P})$ coordinate system.

To end this section we investigate the relations of the coordinate systems between $(P_0, \boldsymbol{P})$ and $(\omega_{\rm pr}, p_{\rm pr})$. Let us take a convention that coordinate $(\omega_{\rm pr}, \boldsymbol{p}_{\rm pr})$ is right-handed. In the p-coordinate system we take a basis vector $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ with $\boldsymbol{e}_1$ corresponding to $(P_x, P_y, P_z) = (1, 0, 0)$, $\boldsymbol{e}_2$ to $(P_x, P_y, P_z) = (0, 1, 0)$, and so on. Thus

$$\boldsymbol{e}_1 \to \boldsymbol{p}_{\rm pr} = V^{-1}\begin{pmatrix}1\\0\\0\end{pmatrix} ,$$

$$\boldsymbol{e}_2 \to \boldsymbol{p}_{\rm pr} = V^{-1}\begin{pmatrix}0\\1\\0\end{pmatrix} ,$$

and so on. These are basis vectors for the $\boldsymbol{p}_{\rm pr}$ coordinate system. Now we compute the handedness of the $\boldsymbol{p}$ coordinate system, that is, given by

$$\begin{aligned} \boldsymbol{e}_3 \cdot (\boldsymbol{e}_1 \times \boldsymbol{e}_2) &= \det\left((\boldsymbol{e}_1)_{p_{\rm pr}}, (\boldsymbol{e}_2)_{p_{\rm pr}}, (\boldsymbol{e}_3)_{p_{\rm pr}}\right) \\ &= \det(V^{-1}) . \end{aligned} \tag{3.7}$$

Here the external product $\times$ is defined in the usual way in $\boldsymbol{p}_{\mathrm{pr}}$ space and the subscript in $(\boldsymbol{e}_1)_{p_{\mathrm{pr}}}$ denotes the $\boldsymbol{e}_1$ vector in $\boldsymbol{p}_{\mathrm{pr}}$ space, (3.6). The sign of det V in (3.7) indicates the handedness of the $\boldsymbol{P}$ coordinate system: if det $V>0$ (<0) it is right (left) handed.

4. Strategy of the proof

The eigenvalue equation

$$H(\boldsymbol{p})|\omega_i(\boldsymbol{p})\rangle=\omega_i(\boldsymbol{p})|\omega_i(\boldsymbol{p})\rangle\,,\qquad (i=1,\ldots,N)\,,\tag{4.1}$$

determines a ray in an N-dimensional complex space for each value of the momentum $\boldsymbol{p}$ except for the cases $\omega_i(\boldsymbol{p})=\omega_{i+1}(\boldsymbol{p})$ or $\omega_i(\boldsymbol{p})=\omega_{i-1}(\boldsymbol{p})$. Thus $\boldsymbol{p}$ determines a point in the complex projective space CP^{N-1}. In the generic case the ith and $(i+1)$th levels are degenerate at points in the Brillouin zone space. The crucial property of $|\omega_i(\boldsymbol{p})\rangle$ for our proof is *periodicity*. The idea of the proof is the following:

(i) Draw an infinitesimal S_2 sphere around each degeneracy point (sect. 5) and consider maps determined by $|\omega_i(\boldsymbol{p})\rangle$ from the S_2 spheres into CP^{N-1} (sect. 6). Calculate $\pi_2(\mathrm{CP}^{N-1})$ and show that these maps correspond to the elements ±1 of Z which depend on the handedness of the particles corresponding to the degeneracy points in question (sect. 7). (We use the notation that S_n stands for a topological space homeomorphic to the sphere in an $n+1$ dimensional euclidean space.)

(ii) Consider the map determined by $|\omega_i(\boldsymbol{p})\rangle$ from the whole surface of the Brillouin zone box homeomorphic to an S_2 sphere into CP^{N-1} (sect. 8) and show that it belongs to the unit element in $\pi_2(\mathrm{CP}^{N-1})$ (sect. 9).

(iii) Prove that the class of map of (ii) is the sum of the classes of eigenray map from the small S_2 spheres mentioned under point (i).

(iv) The consistency of the two expressions for the element $\pi_2(\mathrm{CP}^{N-1})$ requires that

$$N_{\mathrm{r}}(i,i+1)-N_{\mathrm{r}}(i-1,i)=N_{\ell}(i,i+1)-N_{\ell}(i-1,i)\tag{4.2}$$

(sect. 10), where, e.g. $N_r(i,j)$ denotes the number of right-handed degeneracy points between levels i and j. If i denotes the lowest energy level above the Fermi (or Dirac) sea and $i+1$ the highest level in the sea, the number of right-handed species of Weyl particles thus equals $N_r(i,i+1)$.

For points (i) and (ii) we will compute explicitly the integers in Z isomorphic to $\pi_2(\mathrm{CP}^{N-1})$ by making use of the following explicit isomorphism of $\pi_2(\mathrm{CP}^{N-1})$ to Z [7]:

$$\pi_2(\mathrm{CP}^{N-1})\xleftarrow{j_*}\pi_2(\mathrm{S}_{2N-1},\mathrm{S}_1)\xrightarrow{\partial}\pi_1(\mathrm{S}_1)\xrightarrow{\Delta}\mathrm{Z}\,.\tag{4.3}$$

The homeomorphisms j_*, ∂ and Δ are naturally defined and we shall give the definitions below (sect. 5). In fact, they turn out to be isomorphisms.

5. $\pi_2(\mathrm{CP}^{N-1})$ for the degeneracy point

Consider the infinitesimal S_2 sphere around a degeneracy point in the Brillouin zone space. The mapping

$$f: S_2 \to \mathrm{CP}^{N-1} \tag{5.1}$$

determines the homotopy class $[f]$, an element in $\pi_2(\mathrm{CP}^{N-1})$. The function f is determined by the given H and the ith level. That is, for

$$\begin{gathered} p \in S_2 , \\ S_2 = \{p \,|\, |p| = \varepsilon \text{ (infinitesimal)}\} , \end{gathered} \tag{5.2}$$

f is given by

$$f(p) = \{Z|\omega_i(p)\rangle \,|\, Z \in \mathrm{C}\backslash\{0\}\} . \tag{5.3}$$

Now there is an isomorphism between $\pi_2(\mathrm{CP}^{N-1})$ and the relative homotopy group $\pi_2(S_{2N-1}, S_1)$ [7]. The elements of the latter are homotopy classes of functions of the type

$$g: E_2, S_1 \to S_{2N-1}, S_1 , \tag{5.4}$$

where E_2 is the disk

$$E_2 = \{x \in \mathrm{R}^2 | x^2 \leq 1\} ,$$

and the circle S_1 appearing on the left-hand side of the arrow in (5.4) is the circumference of E_2, i.e.

$$S_1 = \{x \in \mathrm{R}^2 | x^2 = 1\} .$$

S_{2N-1} is the space of complex N-tuples

$$S_{2N-1} = \{u \in \mathrm{C}^N \,|\, |u|^2 = 1\} ,$$

and S_1 appearing on the right-hand side of (5.4) is a subset of proportional vectors

$$S_1 = \{\mathrm{e}^{i\delta} u_0 \in \mathrm{C}^N \,|\, \delta \in \mathrm{R}\} ,$$

where u_0 is a fixed unit norm in the complex N-tuple, i.e.

$$u_0 \in S_{2N-1} .$$

The isomorphism between $\pi_2(\mathrm{CP}^{N-1})$ and $\pi_2(S_{2N-1}, S_1)$ is induced by the relation

$$f = \pi \circ g . \tag{5.5}$$

Here π is the projection

$$\pi : S_{2N-1} \to \mathrm{CP}^{N-1} ,$$

i.e. π assigns ray in CP^{N-1} to which that element belongs to an element in S_{2N-1}. The mapping (5.5) from g into f induces a mapping

$$j_*: \pi_2(S_{2N-1}, S_1) \to \pi_2(CP^{N-1}), \tag{5.6}$$

which is defined by

$$j_*([g]) = [f] = [\pi \circ g]. \tag{5.7}$$

In (5.5) it is understood that the infinitesimal S_2 sphere is identified with the manifold which is obtained from E_2 in such a way that all the points on the boundary $S_1 = \partial E_2$ are identified into one point, e.g. the south pole S. Thus we have a one to one correspondence between $S_2/\{S\}$ and the interior $\mathring{E}_2$ of E_2

$$\mathring{E}_2 = \{x \in R \,|\, |x|^2 < 1\}.$$

S is the point with

$$\frac{1}{\varepsilon} P_z = -1, \qquad P_x = P_y = 0. \tag{5.8}$$

The point on the S_2 sphere is expressed in spherical coordinates $(\varepsilon, \theta, \phi)$, with the restriction $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \theta \leqslant 2\pi$, by

$$\begin{aligned} P_x &= \varepsilon \sin\theta \cos\phi, \\ P_y &= \varepsilon \sin\theta \sin\phi, \\ P_z &= \varepsilon \cos\theta. \end{aligned} \tag{5.9}$$

This is unique except for the north pole N ($\theta = 0$) and the south pole S ($\theta = \pi$). In the coordinate system (θ, ϕ) the cartesian coordinate (x_1, x_2) or E_2 are expressed by

$$x_1 = \frac{\theta}{\pi} \cos\phi, \qquad x_2 = \frac{\theta}{\pi} \sin\phi. \tag{5.10}$$

This representation is unique except for the centre $\theta = 0$.

The above stated identification is

$$\text{N-pole of } S_2 \to \text{centre of } E_2,$$

$$\text{S-pole of } S_2 \to \text{whole boundary of } E_2.$$

Note that the north-pole and the centre of E_2 have the same non-uniqueness in coordinates. Thus, except for the correspondence of the south pole to the whole boundary $S_1 = \partial E_2$, there is a homeomorphism between S_2 and E_2.

6. $\pi_2(S_{2N-1}, S_1)$

We now construct an explicit form of g corresponding to f. Let us choose our basis in the space C^N so that $H(\boldsymbol{p})$ becomes diagonal at $\boldsymbol{p} = \boldsymbol{p}_{deg}$ and expand $H(\boldsymbol{p})$ around $\boldsymbol{p} = \boldsymbol{p}_{deg}$ in a Taylor series:

$$H(\boldsymbol{p}) = H(\boldsymbol{p}_{deg}) + (\boldsymbol{p} - \boldsymbol{p}_{deg})\boldsymbol{V} \begin{array}{cc} \begin{matrix} & & i & i+1 & & \end{matrix} & \\ \begin{pmatrix} 0 & & & & \\ & \ddots & & & 0 \\ & & 0 & & \\ & & & \boldsymbol{\sigma} \ddots & \\ & & & & \cdot 0 \\ & 0 & & & \\ & & & & 0 \end{pmatrix} & \begin{matrix} \\ \\ \\ i \\ i+1 \\ \\ \\ \end{matrix} \end{array}$$

$$+ (\boldsymbol{p} - \boldsymbol{p}_{deg}) \begin{pmatrix} b_i & & 0 \\ & \ddots & \\ 0 & & b_N \end{pmatrix} + (\boldsymbol{p} - \boldsymbol{p}_{deg}) \begin{array}{cc} \begin{matrix} & & i & i+1 & & \end{matrix} & \\ \begin{pmatrix} 0 & & & & & * \\ & \ddots & & & & \\ & & 0 & & & \\ & & & 0 & 0 & \\ & & & 0 & 0 & \\ & & & & & 0 \\ & * & & & & \ddots \\ & & & & & 0 \end{pmatrix} & \begin{matrix} \\ \\ \\ i \\ i+1 \\ \\ \\ \\ \end{matrix} \end{array}$$

$$+ O((\boldsymbol{p} - \boldsymbol{p}_{deg})^2) .$$

Note that the second term is important to get the eigenvector in lowest order in perturbation theory. We thus may have

$$|\omega_i(\boldsymbol{p})\rangle = \begin{array}{cc} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u \\ 0 \\ \vdots \\ 0 \end{pmatrix} & \begin{matrix} \\ \\ \\ i \\ i+1 \\ \\ \\ \end{matrix} \end{array} , \tag{6.2}$$

and

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} .$$

Here

$$\boldsymbol{\sigma P} u = |\boldsymbol{p}| u . \tag{6.3}$$

That is, we have the positive eigenvalue for the upper level i.

We define the function g by

$$g = |\omega_i(\boldsymbol{p})\rangle \tag{6.4}$$

where $\boldsymbol{p}$ is a point on an infinitesimal sphere S_2 around $\boldsymbol{P}_{\text{deg}}$. We choose to put the following restrictions on u:

(i) Normalization $|u| = 1$. This is necessary for $|\omega_i(\boldsymbol{p})\rangle$ to lie on the sphere S_{2N-1}.

(ii) Phase convention $\operatorname{Arg} u_1 = 0$, i.e.

$$\operatorname{Re} u_1 \geqslant 0\,, \qquad \operatorname{Im} u_1 = 0\,.$$

Under restrictions (i) and (ii), eq. (6.3) has a unique solution. A little calculation leads us to the solution

$$u = \begin{pmatrix} \dfrac{p_x - ip_y}{\sqrt{2|\boldsymbol{p}|(|\boldsymbol{p}| - p_z)}} \\ \sqrt{\tfrac{1}{2}(1 - p_z/|\boldsymbol{p}|)} \end{pmatrix},$$

or in spherical coordinates

$$u = \begin{pmatrix} \mathrm{e}^{-i\phi} \sin \frac{1}{2}\theta \\ \cos \frac{1}{2}\theta \end{pmatrix}. \tag{6.6}$$

It should be noticed that this u is not unique at $\theta = \pi$ (south pole). This phase ambiguity is due to our phase conventions which do not fix the phase for $u_1 = 0$. But since the S-pole corresponds to $S_1 = \partial E_2$ there is no problem. We can make g continuous and thus choose g on S_1 so that its value is the limiting one from an inside point of E_2.

We thus have constructed a representative g for the class $j_*([f])$ of $\pi_2(S_{2N-1}, S_1)$ in the form

$$g(\theta, \phi) = \begin{array}{c} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{e}^{-i\phi} \sin \frac{1}{2}\theta \\ \cos \frac{1}{2}\theta \\ 0 \\ \vdots \\ 0 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ \\ i \\ i+1 \\ \\ \\ \\ \end{array}. \tag{6.7}$$

7. Helicity for the degeneracy point

We are now equipped to find the image of $[g]$ via the boundary mapping ∂ in the explicit isomorphism (4.3) [7]. Since the map ∂ is induced by the restriction to the boundary $S_1 = \partial E_2$, i.e. $\theta = \pi$, the representative of $\partial([g]) \in \pi_1(S_1)$ is simply given by

$$g(\phi)|_{S_1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e^{-i\phi} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ i \\ i+1 \\ \\ \\ \end{matrix} \tag{7.1}$$

on putting $\theta = \pi$ in g of eq. (6.7). Here

$$g(\phi)|_{S_1} \in S_1 \subset S_{2N-1} .$$

The $S_1 \subset S_{2N-1}$ consists of all phase possibilities for $|\omega_i(\boldsymbol{p})\rangle$ at the south-pole. Obviously the map $g|_{S_1}$ is a homeomorphism from $S_1 \subset E_2$ on to $S_1 \subset S_{2N-1}$. The integer assigned to the class $[g|_{S_1}] \in \pi_1(S_1)$ by the isomorphism ∂ mentioned in eq. (4.3) is by definition nothing but the winding number. The form (7.1) indicates that the winding number is either $+1$ or -1, i.e.

$$\Delta \circ \partial \circ j_*^{-1}([f]) = \pm 1 .$$

When $\det V_i^\alpha > 0$ at a degeneracy point, the coordinate system $\boldsymbol{P}$ is right-handed. Then this integer is -1. This means that there are left-handed Weyl fermions described by the states in the ith (upper) level. If $\det V_i^\alpha < 0$ the coordinate system $\boldsymbol{P}$ is right-handed. Thus, one should correct the form of $g(\sigma, \phi)$ by changing P_x to $-P_x$ and we obtain

$$g(\phi)|_{S_1} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ e^{i\phi} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ i \\ i+1 \\ \\ \\ \end{matrix} .$$

The integer assigned to this is $+1$.

So far we have considered the case of the positive eigenvalue of the upper level i given by eq. (6.3). Replacing this by

$$\boldsymbol{\sigma P} u = -|\boldsymbol{p}| u ,$$

for the lower level $i+1$ has the same effect as switching $\boldsymbol{P}$ to $-\boldsymbol{P}$ in eq. (6.3). This is also equivalent to changing the coordinate system from right- to left-handed, and has the same effect as the sign change of $\det(V_i^\alpha)$. Thus, we conclude the following list of $\pi_2(CP^{N-1})$ integer elements. These correspond to the infinitesimal S_2 spheres

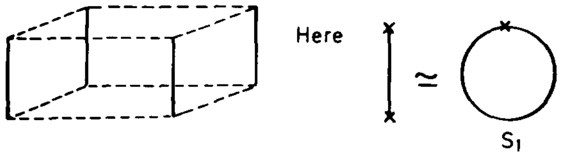

Fig. 2. The box on this figure illustrates the Brillouin zone. Four parallel edges are drawn heavily. They are mapped into a single curve in CP^{N-1} and the curve is indeed a closed one.

around the degeneracy points. These spheres are oriented by a right-handed coordinate system of $\boldsymbol{P}$ for the eigenray of the ith level.

$$\begin{array}{ll} \text{degeneracy of level } i \text{ with } i+1 & \\ \left.\begin{array}{l}\text{positive helicity of } i \\ \text{and negative one of } i+1\end{array}\right\} & -1 \text{ element} \\ \left.\begin{array}{l}\text{negative one of } i \\ \text{and positive one of } i+1\end{array}\right\} & +1\,. \end{array} \tag{7.2}$$

8. $\pi_2(CP^{N-1})$ for Brillouin zone surface

We shall show that the map $\hat{f}_{BS}$ from the surface of the Brillouin zone S_2 (box surface) (fig. 2) into CP^{N-1}, which is determined from the eigenrays $|\omega_i(\boldsymbol{p})\rangle$, belongs to the homotopy class corresponding to zero by the isomorphism $\Delta \circ \partial \circ j_*^{-1}$ of $\pi_2(CP^{N-1})$ with Z. The crucial property of this map belonging to a class corresponding to zero is the *periodicity* of $|\omega_i(\boldsymbol{p})\rangle$. The ith eigenrays on each of the six faces in fig. 2 are identical to those on the opposite faces. On the three sets of four parallel edges $|\omega_i(\boldsymbol{p})\rangle$ are the same. Also at the eight corners $|\omega_i(\boldsymbol{p})\rangle$ are the same.

The strategy of the proof is the following:

(i) A continuous deformation into $\hat{f}_d$ of the map $\hat{f}_{BS}$ allows us to replace it by one that maps all the 12 edges and the 8 corners into a single point.

(ii) Using this deformed map $\hat{f}_d$, it is shown to be a sum of six terms (elements) which split up into 3 pairs.

(iii) The sum of the terms in each pair is shown to be zero. Thus the sum of all 6 terms (elements) is zero.

In order to perform the deformation mentioned in (i), let us first consider how the Brillouin zone surface S_2 is imbedded in CP^{N-1} by the map f_{BS} determined from $|\omega_i(\boldsymbol{p})\rangle$. Periodicity implies that each of the six faces of the Brillouin zone surface is mapped by $\hat{f}_{BS}$ into the same piece of surface in CP^{N-1} as the opposite faces. Further, the three sets of four parallel edges of the Brillouin zone cube are mapped by $\hat{f}_{BS}$ into only one closed curve each, depicted in fig. 3. The resulting three curves in CP^{N-1} are closed since periodicity guarantees that all 8 corners of the cube are mapped by $\hat{f}_{BS}$

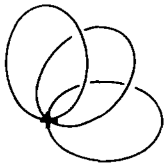

Fig. 3. The images of the three sets of four parallel edges.

into a single point in CP^{N-1}. It would be natural to use this point as a base point, i.e. we restrict our attention to maps mapping the south-pole of S_2 into this point.

Since CP^{N-1} is simply connected, we are guaranteed the existence of a way of deforming each of the three closed curves into the single point which is the image of the corners by the map $\hat{f}_{BS}$. The reader must convince himself that this deformation of $\hat{f}_{BS}$ restricted to the edges can be extended to a deformation of $\hat{f}_{BS}$ itself. By imagining the image of $\hat{f}_{BS}$ as a rubber sheet in CP^{N-1}, this is intuitively possible without spoiling the periodicity inherited by $\hat{f}_{BS}$. Thus, we have seen that $\hat{f}_{BS}$ is homotopic to a deformed mapping $\hat{f}_d$ which maps all the edges into a single point (the base point).

9. The unit element of $\pi_2(CP^{N-1})$

The next step of the proof, (ii), is trivial if we remember the definition of the *group composition of* π_2. Let us split the surface S_2 of the Brillouin zone cube into two parts as is shown in fig. 4a: the first one is just a single face and second one consists of all five other faces. Restriction of the previously constructed deformed map $\hat{f}_d$ to these two parts defines the two maps $\hat{f}_a$ and $\hat{f}_5$. Since $\pi_2(CP^{N-1})$ *is abelian* we denote the group composition law additively and thus we have

$$[\hat{f}_d] = [\hat{f}_a] + [\hat{f}_5]\,. \tag{9.1}$$

The first map $\hat{f}_a$, corresponding to the first part of the S_2 cube, is a mapping into CP^{N-1} from a bag made out of one face by identifying all pairs of edges as is illustrated in fig. 5, i.e.

$$\hat{f}_a : S_2 \text{ surface of the bag in fig. } 5 \to CP^{N-1}\,.$$

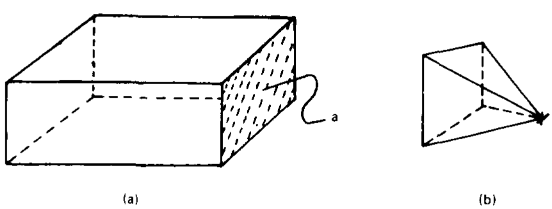

Fig. 4. The split of the surface of the Brillouin zone into a single face (a) and all five other faces illustrated in (b). Here the cross denotes the point into which surface a is contracted.

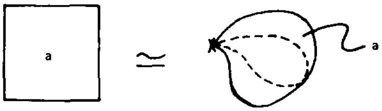

Fig. 5. The single face is homeomorphic to a bag. The cross in the bag denotes the point with which all pairs of edges are identified.

The second one is

$$\hat{f}_5 : S_2 \text{ surface in fig. 4b} \to CP^{N-1} .$$

Next we split $\hat{f}_5$ into $\hat{f}_b$ and $\hat{f}_4$,

$$[\hat{f}_5] = [\hat{f}_b] + [\hat{f}_4] ,$$

and so on. We here again split the second part of fig. 4b into third and fourth parts depicted in fig. 6. Finally, we have a decomposition

$$[\hat{f}_d] = [\hat{f}_a] + [\hat{f}_{\bar{a}}] + [\hat{f}_b] + [\hat{f}_{\bar{b}}] + [\hat{f}_c] + [\hat{f}_{\bar{c}}] ,$$

where each of the six terms corresponds to a face. We denote by $\hat{f}_a$ and $\hat{f}_{\bar{a}}$ the mappings corresponding to each face a and its antiface ā, as shown in fig. 7.

Step (iii). On the surfaces a and ā the orientations are opposite due to periodicity. Therefore $\hat{f}_a$ and $\hat{f}_{\bar{a}}$ are related to each other by a reflection, i.e.

$$\hat{f}_{\bar{a}} = \hat{f}_a \cdot \xi ,$$

where

$$\xi : S_2 \to S_2 .$$

S_2 is a face and ξ is a reflection in a big circle on S_2:

$$[\hat{f}_{\bar{a}}] = -[\hat{f}_a] . \tag{9.3}$$

Similar relations hold for b and $\bar{b}$, c and $\bar{c}$. So from eq. (9.2)

$$[\hat{f}_d] = 0 .$$

Since $[\hat{f}_d]$ is, of course, the same class as $[\hat{f}_{BS}]$, we conclude

$$[\hat{f}_{BS}] = 0 . \tag{9.4}$$

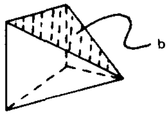

Fig. 6. The split of the surface in fig. 4b into a single face b and all four other faces.

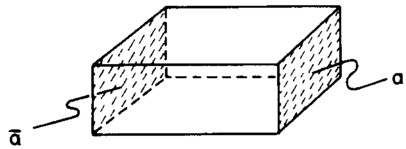

Fig. 7. Illustration of opposite faces.

10. Result

The surface of the Brillouin zone S_2 sphere considered in sect. 4 includes the infinitesimal S_2's around the degeneracy points. We deform all infinitesimal S_2 spheres to have a common base point. By the group composition law of π_2 we decompose

$$[\hat{f}_{BS}] = \sum_j [f_j] \tag{10.1}$$

where f_j denotes the mapping f_j: infinitesimal $S_2 \to CP^{N-1}$ and where j runs over all infinitesimal S_2 spheres. Since we have proved that the left-hand side of (10.1) is zero [eq. (9.4)], eq. (10.1) gives

$$\sum_j [f_j] = 0 .$$

That is to say, from eq. (7.2) the sum of the terms $+1$ or -1 corresponding to the degeneracy points for the ith and $(i+1)$th or $(i-1)$th levels are zero. Thus we obtain eq. (4.2):

$$N_r(i, i+1) - N_r(i-1, i) = N_\ell(i, i+1) - N_\ell(i-1, i) . \tag{4.2}$$

Here, e.g., $N_r(i, i+1)$ is the number of degeneracy points of the levels i and $i+1$ for which the upper ith has positive helicity.

We want, in fact, to show that

$$N_r(i-1, i) = N_\ell(i-1, i) \tag{10.2}$$

by induction using eq. (4.2). Now for the highest level $i = 1$, we have trivially

$$N_r(0, 1) = N_\ell(0, 1) = 0 ,$$

because there is simply no level number $i = 0$. Assuming that the eq. (10.2) is true for $i-1$, we find from eq. (4.2) that

$$N_r(i, i+1) = N_\ell(i, i+1) . \tag{10.3}$$

Especially, this equation is true for the number of i for which level i is unfilled while level $i+1$ is filled.

Since in this case each degeneracy point between ith and $(i+1)$th levels represents one species of Weyl particle (neutrinos), eq. (10.3) states our no-go theorem: there appear equal numbers of species of left- and right-handed Weyl particles.

11. Discussions

(a) It is easy to generalize our theorem to the case of many conserved charges Q (lepton number) associated with the fermion fields. Here we *assumed* that these Q's are represented by bilinear form in ψ

$$Q = \sum_{x} j_0(\boldsymbol{x}),$$

where

$$j_0(\boldsymbol{x}) = \sum_{\boldsymbol{y}} \bar{\psi}(\boldsymbol{x})\tilde{Q}(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y}).$$

For each combination of charges we have a separate class of fermion fields. There can be no transition from one such class to another, and to the approximation of a linear equation of motion there is no connection between these classes at all. Each class is therefore to be considered as a separate system of fermions. We can thus apply our no-go theorem to each class. This generalization is very worrisome. In fact, it threatens any hope of putting weak interactions on a lattice as we already discussed in sect. 1(d).

(b) In fact, were it not for the spontaneously broken gauge symmetries of the weak interaction, there would be no mass terms for any of the known fermions in the standard Weinberg–Salam model. It is precisely because of the different quantum numbers for different handed Weyl particles that mass terms are forbidden. That is, unless there are mass terms, no pair of handed Weyl particles can be combined into an ordinary Dirac four-component massive fermion. In discussing our no-go theorem we talked about the number of species of Weyl particles, i.e. neutrino-like particles. However, in nature many fermions—perhaps all, even neutrinos—may have masses. Massive fermions can be described as Weyl particles in the following two ways:

(i) one handed particle pairs up with the same handed one to become a Majorana fermion, or

(ii) one handed particle pairs up with the opposite handed one to become a Dirac particle.

These can precisely be done according to our no-go theorem.

In the case of (ii) there must be one right-handed and one left-handed Weyl particle and a mass term in the hamiltonian in order to cause a transition from one to the other. The alternative statement of our no-go theorem is that there are no quantized chiral charges due to the possible mass terms. For a low energy observer it is natural to redefine the momentum by the prescription (3.3). The subtraction constants ω_{deg} and $\boldsymbol{P}_{\text{deg}}$ in (3.3) may be different for different Weyl particles in one pair. The conservation of the "fundamental" momentum $\boldsymbol{p}$ (i.e. the momentum before subtraction of $\boldsymbol{P}_{\text{deg}}$) may exclude a mass term. We must say here that we do not know in detail how a necessary mass term comes about via the spontaneous breakdown of the "fundamental" momentum and this mass problem is an open question. Also, unless we know the mass term, the chiral charge is ill-defined. However, we do not know if the appropriate spontaneous breakdown of the "fundamental" momentum is unnatural, but it should at least be possible.

(c) It is of interest to note that our theorem *prevents any problem* with the Adler, Bell–Jackiw anomaly [10] in a *trivial manner*: there simply are no anomalies since they are cancelled between oppositely handed particles. But, of course, it should be remembered that in the correct theory of weak interactions anomaly-free conditions are *not* satisfied in such a trivial manner.

(d) There may be a possibility of evading our no-go theorem by abandoning the helping assumptions described in sect. 1(c) in the following ways:

(i) *Taking Q to be unquantized* may not be a proper way, since Q then does not go into a compact group generating a closed subgroup. One would thus have to claim that the guage group of the standard model is a low energy approximation only. It would be a serious complication contrary to the suggestion from phenomenology.

(ii) *Giving up locality of charge.* This will be the subject of our forthcoming paper [12] as was mentioned at the end of sect. 1(d). In that work we still need continuity of the fermion dispersion relation and thus cannot give up locality for the free fermion hamiltonian. If we did that we would allow the Drell, Weinstein and Yankielowicz model [4] as a counter example to our no-go theorem. Tom Banks has pointed out to Foerster how our theorem could be circumvented if the locality of fermion dispersion relation is not required. Then one can let the *light velocity of one or more neutrinos go to infinity* so that the light-cone steepens and effectively disappears.

(e) We have throughout this article discussed the Kogut–Susskind lattice. Can our results be taken over to the Wilson lattice [11]? Yes, *if* we understand it as a theory with a spatial lattice and discrete *imaginary* time. In fact, we can construct an operator e^{-H} which gives the development by one lattice unit along the imaginary time. Here H can be considered as the hamiltonian. In our dicussion of the Kogut–Susskind lattice we only used the existence of such a hamiltonian.

Had we, however, wanted a lattice theory with a discrete real time we would instead have constructed an operator e^{-iH}. Thus H is only unique modulo 2π, and so energy would be defined modulos 2π. The concept that one energy is lower than another then loses its meaning. This would make the question of how to fill a Dirac sea more delicate and may complicate the argument. However, this case of discrete real time seems not to be popular and we shall not go into it further.

(f) In concluding the paper we would like to mention the following point which will be investigated in our succeeding paper [12]:

In the present article we have considered the case where the fermion field $\psi(\boldsymbol{x})$ is *complex. A real field formulation for* $\psi(\boldsymbol{x})$ may open the possibility of describing a larger class of physical systems. In fact there exist lattice models with only one two-component fermion in the real field formulation, which cannot be described as a complex field. We shall illustrate this counter example to our no-go theorem. But, if we have a conserved non-zero charge, the charged fermions can be formulated in terms of complex fields.

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