A NO-GO THEOREM FOR REGULARIZING CHIRAL FERMIONS

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We present a no-go theorem for regularizing chiral fermions in a general and abstract form, together with a review of our lattice no-go theorem for chiral fermions.

1. Lattice regularization [1,2] has been introduced to ensure the calculability of the long-distance behaviour of gauge theory and provides a laboratory [3] for the investigation of nonperturbative effects, e.g. confinement. To be able to investigate QCD we should be able to incorporate fermions on a lattice while maintaining symmetry, one of the crucial properties of the strong interaction.

It is, however, known that in the four-dimensional *naive* lattice Dirac fermion model there appear 15 unwanted species in the long-wavelength regime (species doubling) [4,5], which can be *avoided* in some way or other if chiral invariance is ignored. A similar problem arises when we consider chiral (Weyl) fermions in order to construct chirally invariant QCD or the Weinberg–Salam weak-interaction model on a lattice. That is, in four-dimenions, the naive introduction of, for example, a left-handed Weyl field ψ_L with a chiral charge, e.g. $\chi = -1$, on a lattice leads to the appearance of four right-handed species together with four left-handed ones in the long-wavelength regime. If one wants to avoid calamities such as nonlocality or non-discrete charges, one is forced to let all these fermions couple to a (attempted) chiral gauge field. We may refer to this disease as the chiral doubling problem. It is *unavoidable*. As a trivial consequence we do not obtain the correct Adler–Bell–Jackiw anomaly; instead, the axial U(1) current is conserved because of the cancellation of opposite-handed species.

Now the question is whether this disease – chiral doubling (and consequently species doubling) – can be evaded through a modification of the naive lattice action. Our answer is no, under some assumptions. We in fact prove the following no-go theorem for a general class of lattice chiral fermion theories [6,7]: it is a necessary consequence of the topological character of lattice theory that, when one handed Weyl fermion, e.g. ψ_L, is put on the lattice, an equal number of right- and left-handed Weyl fermions appears in the continuum limit, for a given charge combination of a compact group.

This disease is not peculiar to lattice theory, but some disease appears universally in all regularization schemes, for instance dimensional regularization. It is the purpose of the present article to abstract the following *general* no-go theorem for the regularization of chiral fermions without specifying particular schemes: there does not exist a regularized chiral fermion theory that has (1) invariance under global gauge group, (2) different number of right- and left-handed species for given charge combinations, (3) (correct) Adler–Bell–Jackiw anomaly, and (4) an action billinear in the Weyl field.

It should be noticed that in our lattice no-go theorem the assumption (3) of the general no-go theorem, appearance of the (proper) anomalies, is not needed.

So we see that our lattice and general no-go theorems are complementary to each other and none of them is contained completely in the other one.

The present paper is organized as follows. We shall start with a 1 + 1 dimensional naive chiral fermion model in order to illustrate the chiral doubling problem. We shall first review our lattice no-go theorem for the construction of chirally invariant QCD and Weinberg–Salam models on the lattice. It is instructive to review briefly our proofs of the theorem in order to give a perspective, although they have been given in detail in our previous papers [6,7]. We shall then next discuss some models, invented by Wilson [4], Susskind [5], Drell et al. [8] and ourselves [7], which violate some of our assumptions. Finally we shall formulate a quite general and abstract no-go theorem.

2. To indicate the point clearly, let us consider a simple example – the 1 + 1 dimensional [‡1] *naive* chiral (Weyl) fermion theory coupled to a chiral gauge field $\boldsymbol{A}$ with the action

$$S = \int \mathrm{d}t\, \mathrm{d}x \{-\mathrm{i}\dot{\bar{\psi}}_{\mathrm{L}}(x)\psi_{\mathrm{L}}(x) + \bar{\psi}_{\mathrm{L}}(x)[-\mathrm{i}\partial_x + A_1(x)]\,\psi_{\mathrm{L}}(x)\}\,, \tag{1}$$

and the equation of motion

$$\mathrm{i}\dot{\psi}_{\mathrm{L}}(x) = [\mathrm{i}\partial_x - A_1(x)]\,\psi_{\mathrm{L}}(x)\,, \tag{2}$$

where ψ_{L} stands for a left-handed Weyl field expressed in terms of the two-component Dirac field by $\psi_{\mathrm{L}} = \frac{1}{2}(1 - \gamma_5)$. For a parity-conserving theory one adds the term corresponding to ψ_{R} in (1) and repeats the same argument as for the ψ_{L} part to show chiral doubling and thus species doubling. We may assign a chirality $\chi = -1$ to ψ_{L} by means of the definiton of the chiral charge: $Q_5 = \int \mathrm{d}x\, \psi_{\mathrm{L}}^{\dagger}\psi_{\mathrm{L}}$.

Now go to the spatial lattice (and continuous time) by replacing ∂_x by a difference operation in eq. (2) to obtain

$$\mathrm{i}\dot{\psi}_{\mathrm{L}}(x) = \tfrac{1}{2}\mathrm{i}\{\exp[\mathrm{i}A_1(n)]\,\psi(n+1) - \exp[-\mathrm{i}A_1(n)]\,\psi(n-1)\}\,, \tag{3}$$

where n is integer and the lattice spacing is set equal to one. When we take the free part of (3) the dispersion relation is given by

$$\omega = -\sin p\,. \tag{4}$$

Since the momentum space forms a Brillouin zone we consider $-\pi \leqslant p \leqslant \pi$ and identify the points $p = -\pi$ and $p = \pi$. The dispersion relation (4) describes two excitations near $p = 0$ and $p = \pi$ in the long-wavelength regime, for which the chiral charges are both $\chi = -1$ by the definition of Q_5. This assignment of χ is, however, apparently in contraction with the actual "helicity" (which is, in fact velocity in 1 + 1 dimensions) of the particle near $p = \pi$, when we define a group velocity by

$$v = \partial\omega/\partial p\,. \tag{5}$$

The species near $p = \pi$ [‡2] has a velocity $v = 1$ and thus represents a right-moving particle, while the species near $p = 0$ represents a left-moving one. These correspond to the right- and left-handed species in 3 + 1 dimensions. Note that both handed species have $\chi = -1$, and then couple to one chiral gauge field A_1. Therefore Q_5 defined above cannot be the *conserved* chiral charge.

Another way of seeing the point is to compute the axial Ward identity. We know that there should be an Adler–Bell–Jackiw anomaly [9] for the axial U(1) current $J_\mu^5 = -\psi_{\mathrm{L}}(x)\gamma_\mu\psi_{\mathrm{L}}(x)$

$$\partial^\mu J_\mu^5(x) = -(4\pi)^{-1}\epsilon_{\mu\nu}F^{\mu\nu}(x)\,. \tag{6}$$

In the lattice theory described by the action (2) we have

$$\partial^\mu J_\mu^5(x) = 0\,, \tag{7}$$

because the cancellation of the anomaly terms takes place due to the chiral doubling of left- and right-moving particles in ψ_{L}. This again means that the particles described by ψ_{L} with $\chi = -1$ are left-moving near $p = 0$ and right-moving near $p = \pi$.

3. We next review [6,7] our lattice no-go theorem by considering a general class of lattice version of continuum four-dimension Weyl fermion theory, which contains only a left-handed field ψ_{L} coupled to the

‡1 The proof of the lattice no-go theorem for general class of lattice action in 1 + 1 dimensions is found in ref. [7].

‡2 Momentum p of the species near $p = \pi$ should be renormalized via $p_{\mathrm{practical}} = p - \pi$, in order to describe massless particles with $\omega^2 = p_{\mathrm{pr}}^2$. See refs. [6,7].

chiral gauge field $A_\mu(x)$, by the action

$$S = \int d^4x \{-i\dot{\bar{\psi}}_L(x)\psi_L(x) + \bar{\psi}_L(x)\boldsymbol{\sigma}[-i\partial - \boldsymbol{A}(x)]\,\psi_L(x)\}\,, \qquad (8)$$

with an invariance under the transformation of a compact group. The general class of lattice actions, which will describe a *would-be* left-handed Weyl fermion in the continuum limit given by the action (8) is of the form

$$S = -i\int dt \sum_{\boldsymbol{n}} \dot{\bar{\psi}}(\boldsymbol{n})\psi(\boldsymbol{n}) - \int dt \sum_{\boldsymbol{n},\boldsymbol{m}} \bar{\psi}(\boldsymbol{n})H(\boldsymbol{n}-\boldsymbol{m})I(\boldsymbol{n}-\boldsymbol{m})\psi(\boldsymbol{m})\,, \qquad (9)$$

with H a free hamiltonian [‡3]. The gauge field is introduced via

$$I(\boldsymbol{n}-\boldsymbol{m}) = \exp\left(ig \sum_{\boldsymbol{z}=\boldsymbol{n}+1/2}^{\boldsymbol{m}-1/2} A(\boldsymbol{z})\right). \qquad (10)$$

The ψ's denote N-component fermion fields and thus H is a $N \times N$ hermitian matrix. We have assumed (i) exact conservation of the (chiral) charges associated with the invariance of a compact group, e.g. chiral SU(3) × SU(3), chiral U(1) etc. in QCD, or chiral SU(2) × U(1) in the standard Weinberg–Salam model. The second assumption is (ii) locality of the interaction, $H(\boldsymbol{n}) \to 0$ (sufficiently fast) as $|\boldsymbol{n}| \to \infty$. It should be noted that assumption (i) indicates that a dispersion relation for the free hamiltonian H, $\omega = \omega(\boldsymbol{p})$, is assigned to a certain charge combination. The second assumption (ii) insists on the continuity of the dispersion relation and thus forbids any singularity.

Under these assumptions the topological character of the Brillouin zone manifold of momentum space – a torus $S_1 \times S_1 \times S_1$ – leads us to our no-go theorem in two ways as shown in some detail in our previous papers [6,7]. One is by an algebraic topology argument: a one-to-one correspondence is made between the helicity of the Weyl fermion and the element of the homotopy group $\Pi_2(CP^{N-1})$. On the other hand, the sum of the homotopy classes is zero because of the periodicity of the Brillouin zone. This means that there must be an equal number of right- and left-handed species of Weyl fermions in theories of the general class of actions (9). Notice that both these handed species couple to one chiral gauge field A_μ. This proof is found in ref. [6]. The second proof given in ref. [7] is more intuitive and understandable even if one does not have any knowledge of the mathematics of a topological manifold.

One of the consequences of our no-go theorem is that, if one wants to put the Weinberg–Salam model on the lattice – e.g. ν_L and e_L with $Y = -1$ coupled to $W^\pm_\mu$ via the interaction $\bar{\nu}_L\gamma^\mu e_L W^+_\mu + \bar{e}_L\gamma^\mu\nu_L W^-_\mu$ – one obtains additional ν_R and e_R with $Y = -1$ and they couple to $W^\pm_\mu$ via $\bar{\nu}_R\gamma^\mu e_R W^+_\mu + \bar{e}_R\gamma^\mu\nu_R W^-_\mu$. This means that the parity violating theory cannot be constructed, which contradicts weak-interaction phenomenology. It should be noted that the charge $Y = -1$ turns out to be assigned to both ν_L, e_L and ν_R, e_R and thus it cannot be the chiral charge.

As for QCD, we add the term for the right-handed Weyl field ψ_R in (8) given by

$$\int d^4x \{-i\dot{\bar{\psi}}_R(x)\psi_R(x) - \bar{\psi}_R(x)\boldsymbol{\sigma}[-i\partial - \boldsymbol{A}(x)]\,\psi_R(x)\} \qquad (11)$$

and the corresponding lattice version of (11) in (9). Repeating the same argument as that for the action (9) of $\psi_L(x)$ we show chiral doubling in ψ_R. Therefore in the lattice version of chirally invariant QCD described by the continuum action (9) + (11) species doubling occurs. All species, of course, couple to the gauge field A_μ. It should be noted that all species are distinct. We stress that a *conserved* chiral charge does not exist. We thus conclude that there is no chiral invariant resolution of the species doubling problem and hence that it is impossible to construct chirally invariant QCD on the lattice.

Let us now turn to models of lattice QCD which violate some of our assumptions. The Wilson [4] strategy is to put a generalized mass term in the action in order to prevent species doubling and thus breaks chiral

[‡3] One can take a kinetic term of the form

$$-i\int dt \sum_{\boldsymbol{n},\boldsymbol{m}} \dot{\bar{\psi}}(\boldsymbol{n})\,T(\boldsymbol{n}-\boldsymbol{m})\,\psi(\boldsymbol{m}).$$

This gives an effective hamiltonian H/T, which produces non-local singularities in the dispersion relation as was pointed out in ref. [6].

invariance explicitly. One can easily adjust mass parameters in order to obtain the correct anomaly as was shown explicitly by Karsten and Smit [10]. Another interesting model due to Susskind [5], is to put each component of the fermion field on its own series of sites. The discrete chiral symmetry given by $\psi \to \exp(ip_i)\gamma_5 \psi$ survives in the low-momentum regime $p_i \approx 0$, although the continuous chiral symmetry is absent simply because the particle and antiparticle are on different sites. A serious problem lies in the fact that the group of this single link translation is noncompact, so that the (chiral) charge is not quantized. Although the chirally invariant theory can be constructed by sacrificing locality of the interaction, i.e. H in (9), as has been done by Drell et al. [8], this introduces discontinuities in the dispersion relations and therefore some particles carry infinite velocity [‡4]. We constructed a model [7] that contains only one two-component field in the continuum limit by abandoning conservation of charge, which is conserved approximately toward the long-wavelength regime.

4. We now come to the main point of this article. It is worthwhile to formulate a quite general no-go theorem for regularizing chiral fermions without referring to specific regularization methods [‡5]. The no-go theorem states that it is not possible to construct a regularized theory of chiral fermions that satisfies the following four properties:

(1) There is invariance at least under the global part of the gauge group.

(2) The numbers of right- and left-handed species of Weyl fermions are different for a given combination of charges. Here by charges we mean generators of the global subgroup of the local gauge group.

(3) The theory has the (correct) Adler–Bell–Jackiw anomaly. We therefore exclude such models as the nonlocal ones of Drell et al. [8]. In such a model singularities due to nonlocal interactions give rise to a nonzero contribution to the axial Ward identity.

(4) The action is bilinear in the Weyl field.

[‡4] The renormalization procedure of the nonlocal singularities is argued by Rabin in ref. [11].

[‡5] A similar statement for the specific methods for specific models as our general no-go theorem can be found in the literature in many places, e.g. refs. [10] and [12]. A similar argument was given by Englert [13].

For simplicity think of the theory of a Weyl fermion coupled to the abelian gauge field (or abelian part of the non-abelian gauge theory). The proof of the no-go theorem is quite simple and goes as follows. Properties (1) and (4) ensure the existence of some conserved quantum number, e.g. number of Weyl fermions and the *conserved* Noether current given by

$$J_\mu^{(Q_i=a_i)}(x) = \sum_{\substack{\forall_{\psi^{(\alpha)}} \\ \text{with}(Q_i=a_i)}} \bar{\psi}^{(\alpha)}(x)\gamma_\mu \psi^\alpha(x) , \qquad (12)$$

and

$$\partial^\mu J_\mu^{(Q_i=a_i)}(x) = 0 . \qquad (13)$$

Here the $\psi^{(\alpha)}$ are the chiral fermions, $(Q_i = a_i)$ denotes the charge combination $(Q_1 = a_1, Q_2 = a_2, ..., Q_n = a_n)$ and the sum runs over all Weyl fields $\psi^{(\alpha)}$ with charges $Q_i = a_i$. On the other hand according to (2) the numbers of right- and left-handed species are different for at least one charge combination $(Q_i = a_i)$, $N_R(Q_i = a_i) \neq N_L(Q_i = a_i)$. One thus finds for the anomaly:

$$\partial^\mu J_\mu^{(Q_i=a_i)}(x) = [N_R(Q_i = a_i) - N_L(Q_i = a_i)]$$
$$\times \frac{1}{32\pi^2}\left(\sum_{i=1}^{n} g_i a_i F_{\mu\nu}^{(i)}\right)\left(\sum_{k=1}^{n} g_k a_k {}^*F^{(k)\mu\nu}\right), \qquad (14)$$

where the g_i denote the coupling constants associated with the charges a_i. The result (12) is apparently in contradiction with (11). Therefore the assumption of the existence of the regularization satisfying (1)–(4) cannot be maintained.

The lattice regularization of the chiral fermions considered in sections 1–3 described by the action (9) satisfies the properties (1), (2) and (4) of the general no-go theorem, but we did not assume (3), appearance of the proper anomalies. Instead, we assumed locality of interaction.

Finally let us sketch the familiar regularization schemes. The Pauli–Villars method breaks global invariance [property (1)] by introducing a regulator mass. In dimensional regularization there are two possibilities for construction of γ_5 [14]. One of the definitions $(\gamma_{[1}\gamma_2 \dots \gamma_{4+\epsilon]})$ gives a vanishing triangle diagram. The other one $(\gamma_{[1}\gamma_2\gamma_3\gamma_{4]})$ breaks global invariance.

Our conclusions may be summarized as follows: according to the general no-go theorem presented in this article, if we ***assume***, as our prejudice, the existence of a fundamental cutoff (a cutoff given by the gods) in a gauge invariant regularization, we are in crisis, since we are forced to have mirror particles [‡6]. Having mirror particles seems unnatural and one is led to ask: why should the right-handed component of the mirror particles not combine with the left-handed component of the ordinary particles, e.g. through the bare mass term? The non-appearance of mirror particles means that such a supposition is in contradiction with the series of seemingly healthy prejudices listed as assumptions for our general no-go theorem [‡7]. The way out of the crisis might be to have a fundamental cutoff in a gauge non-invariant form. This is, however, not attractive according to Veltman [‡8].

We may perhaps compare the problem of finding a regularizable theory of weak interactions with the apparent contradictions which Einstein had to resolve in postulating relativity: there the position was that there did not exist a theory, consistent with experiment, that satisfied both the principle of relativity and the concept of absolute simultaneity. Thus Einstein had to give up at least one presupposition, which seemed reasonable at that time.

Now it is up to the reader to play the role of Einstein and find out which of ***our*** prejudices has to be sacrificed: do we have to give up the compactness of the gauge group of the Weinberg–Salam model? Or do we have to introduce a supergravity theory in which the supersymmetries make regularization superfluous? Or do we need some completely new idea?

[‡6] By mirror particles we mean fermions coupled through their right-handed currents.

[‡7] Banks and Casher [15] even suspected that the theory for weak interactions is not renormalizable unless it contains mirror fermions.

[‡8] His argument [16] is that it is troublesome to have gauge non-invariance at high energies because it produces a huge photon–photon cross section, since the diagrams are highly divergent in high powers of the cutoff.

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