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ABSENCE OF NEUTRINOS ON A LATTICE (II). Intuitive topological proof

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An intuitive topological proof is given of the no-go theorem for putting Weyl fermions in weak interaction on a lattice, or for constructing chiral invariant lattice QCD, which was proved by a homotopy theory argument in our preceding paper (Absence I). This theorem hangs on the existence of the charge (e.g. fermion number), and thus on the complex-field formulation and on locality. If we relax the assumptions for the no-go theorem, for instance the existence of the charge, and thus use the real-field formulation, we can construct a model that has only one two-component field. We can assign this model an only approximately conserved charge.

1. Introduction

We have shown, in our recent paper (referred to as Absence I) [1], a no-go theorem for putting weak interactions on a lattice, which states that there appears an equal number of species of left- and right-handed Weyl particles for a general class of lattice fermion theories for each combination of quantum numbers (charges).

Essentially the same theorem, but less complete (only for simple models) and less rigorous has been known as a trouble by Wilson [2] and Susskind [3], and recently it has been reformulated by Karsten and Smit [4]* in the same context as them.

Our theorem says, for example, that if one wants to put on a lattice the left-handed electron neutrino ν_L and electron e_L with weak hypercharge $Y = \frac{1}{2}$ and the right-handed electron e_R with $Y = 1$ according to the Weinberg-Salam model, there must necessarily also exist a right-handed neutrino ν_R and an electron e_R with

* An even stronger theorem saying that the standard $SU(3) \times SU(2) \times U(1)$ model cannot be renormalized to non-perturbative accuracy unless there exist parity partners to the left coupling leptons and quarks is speculated by Casher (and Banks) [5].

$Y=\frac{1}{2}$ and a left-handed electron e_L with $Y=1$. This, of course, contradicts the phenomenology of weak interactions. We have two kinds of topological argument for proofs of our no-go theorem, one of which was given in Absence I. There we used algebraic topology and the homotopy group $\pi_2(CP^{N-1})$. In the present article we will give an alternative intuitive topological proof. The crucial ingredient of these proofs is that the momentum space of the lattice theory is periodic, i.e. forms the Brillouin zone

$$-\pi \leqslant p_i \leqslant \pi, \tag{1.1}$$

where the end points $p_i=-\pi$ and $p_i=\pi$ are identified. Topologically p_i runs on a circle S_1. Thus the momentum space makes up a hypertorus $S_1\times S_1\times S_1$. This can be seen from the fact that the Fourier transform of the field $\psi(n)$ with $\boldsymbol{n}$ integers,

$$\psi(\boldsymbol{p})=\sum_{\boldsymbol{n}} e^{-i\boldsymbol{p}\boldsymbol{n}}\psi(\boldsymbol{n})$$

is invariant under $\boldsymbol{p}\to\boldsymbol{p}+2\pi\times\boldsymbol{\kappa}$ for integers $\boldsymbol{\kappa}$.

The important consequence of our no-go theorem in lattice QCD is that it is not possible to keep chiral invariance on the scale of the fundamental lattice, e.g. a scale of the order of the Planck length (if there is a fundamental lattice with that lattice constant) when we eliminate unwanted lattice fermions [2, 3] in the low-energy regime.

It should be stressed that we make the important assumption of locality of the lattice theory. If we did not assume that, it is, according to Drell, Weinstein and Yankielowicz [6], possible.

Our general class of lattice fermion theories used in Absence I is described by the action of the form

$$S=-i\int dt\sum_{\boldsymbol{x}}\dot{\bar{\psi}}(\boldsymbol{x},t)\psi(\boldsymbol{x},t)-\int dt\sum_{\boldsymbol{x},\boldsymbol{y}}\bar{\psi}(\boldsymbol{x},t)H(\boldsymbol{x}-\boldsymbol{y})\psi(\boldsymbol{y},t) \tag{1.2}$$

for the N-component complex fermion field $\psi(x,t)$,

$$\psi=\begin{pmatrix}\psi_1\\ \vdots\\ \psi_N\end{pmatrix}. \tag{1.3}$$

Here N is neither a flavor nor color component number. We have assumed the following three mild conditions on the action (1.2):

(i) Locality of the interaction, i.e. $H(\boldsymbol{x})\to 0$ fast enough as $|\boldsymbol{x}|\to\infty$ that its Fourier-transformed $\tilde{H}(\boldsymbol{p})$ is a smooth function. Thus the eigenvalues $\omega_i(\boldsymbol{p})$ ($i=$

$1,\ldots,N$) are smooth except in the case that two eigenvalues $\omega_i(\boldsymbol{p})$ and $\omega_{i+1}(\boldsymbol{p})$ coincide.

(ii) Translational invariance on the lattice.

(iii) Hermiticity of the hamiltonian H.

We make the following assumptions for the charges Q (lepton numbers, weak hypercharges etc.):

(i) Q is exactly conserved even at the scale of the fundamental lattice.

(ii) Q is locally defined. That is, it is expressed as a sum of local charge densities

$$Q=\sum_{\boldsymbol{x}} j^0(\boldsymbol{x}).$$

(iii) Q is quantized.

As for Dirac particles in strong interactions, various ways of eliminating the unwanted fermions in the low-energy regime have been proposed. However, according to our no-go theorem, it is impossible to keep chiral invariance when one constructs lattice models for QCD avoiding species doubling or spectral multiplication. If one drops one or some of the assumptions for our no-go theorem, then one can construct models of lattice QCD. Indeed, Wilson [2], Susskind [3] and Banks and Casher [7] have been able to construct models free of spectral multiplication by breaking chiral invariance on the scale of the fundamental lattice, and Drell, Weinstein and Yankielowicz [6] have also been able to do so by breaking the locality of interaction.

In the present article we construct models with only one two-component fermion in the low-energy regime, dropping the assumption of the existence of a conserved charge. We thus take the fermion field $\psi(\boldsymbol{x})$ to be real. We can assign to the field components charges that are not conserved at the scale of the fundamental lattice, but approximately conserved in the low-energy regime only. Also, if one relaxes the assumption of a discrete spectrum for the charge, we can easily provide a counter example.

In sect. 2 we consider a general class of $1+1$ dimensional lattice fermion theories and give a proof of the no-go theorem in $1+1$ dimensions. Sect. 3 is devoted to presenting the intuitive topological proof in $3+1$ dimensions. In sect. 4 we give the real-field formulation and present an example which has only one two-component fermion. In sect. 5 we assign a non-conserved charge to this model. In sect. 6 we draw our conclusions.

2. The $1+1$ dimensional case

As an exercise we will consider the general class of $1+1$ dimensional fermion theories on the Kogut-Susskind lattice and state an analogous theorem to the no-go theorem in $3+1$ dimensions of Absence I. The analogy between these two cases is not precise: one of the reasons is that in $3+1$ dimensions the antiparticle of a Weyl

particle must be taken as a *CP*-transformed one and thus has opposite helicity. In 1 + 1 dimensions the antiparticle of a right mover (a right-moving particle) is also a right mover.

We consider a general class of lattice fermion theories in 1 + 1 dimensions whose action is the 1 + 1 dimensional version of the 3 + 1 dimensional case (1.2). We thus take ψ to be a complex N-component field in order to keep generality. We are interested in the 1 + 1 dimensional generic case analogous to that considered in sect. 2 of Absence I in 3 + 1 dimension.

For a generic (or one may equally well think of a random) hamiltonian in 1 + 1 dimensions there are *no degeneracies* of the energy levels at all. In fact, in order to have just two-level degeneracy [say $\omega_i(p)$ and $\omega_{i+1}(p)$], three parameters must be restricted in the hamiltonian $\tilde{H}(p)$. But since there is only one p, this can not be done for a generic hamiltonian.

It is generic, however, that the wave-packet velocity,

$$v_i = \left.\frac{\mathrm{d}\omega_i}{\mathrm{d}p}\right|_{p=p_f}, \tag{2.1}$$

is non-zero at the Fermi energy. Here a Fermi "surface" p_f, which is a point in 1 + 1 dimension, satisfies $\omega_i(p_f) = 0$ since it is natural to take the Fermi energy to be zero. (A generic dispersion relation is shown in fig. 1.) So in such a generic theory low-energy excitations of the vacuum are such that fermion states with p close to a p_f are excited. So for the particles relevant at low energy one finds

$$\omega_i(p) = (p - p_f)\left.\frac{\mathrm{d}\omega_i}{\mathrm{d}p}\right|_{p=p_f} + \mathrm{O}\big((p-p_f)^2\big). \tag{2.2}$$

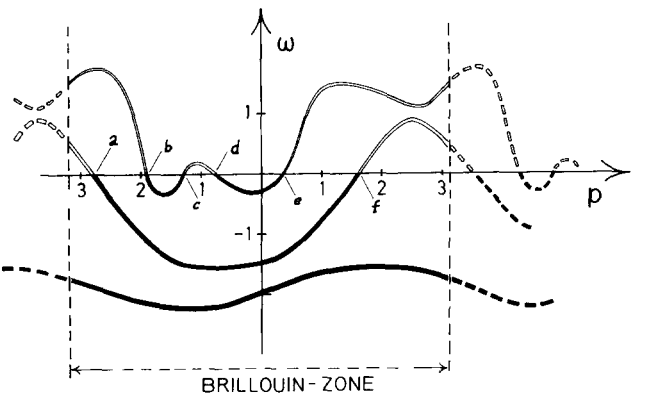

Fig. 1. Typical dispersion laws for 1 + 1 dimensional (complex) lattice Weyl fermion field theory. Each curve is closed since end points should be identified. Each of the crossing points a–f is a Fermi "surface" which represents one species of Weyl fermion.

When

$$\left.\frac{\mathrm{d}\omega_i}{\mathrm{d}p}\right|_{p=p_f} > 0\ (<0),$$

the particle at this crossing point with the $\omega_i = 0$ line is called a right (left) mover. It should be noted that the antiparticles have the same velocities as the corresponding particles. Assuming locality, $\omega_i(p)$ is analytic in p except for degeneracy points (which will be absent in the generic case) and a smooth curve $\omega_i(p)$ is defined in ω_i-p space. The crucial point is that, for a lattice theory, p is defined modulo 2π (in units of the inverse lattice constant) and thus runs on a circle S_1. So the curve is indeed closed and topologically it is a circle S_1. Obviously such curves must cross equally many times from $\omega_i(p) < 0$ to $\omega_i(p) > 0$ as they cross from $\omega_i(p) > 0$ to $\omega_i(p) < 0$. We can give an orientation to the curve such that it is along the increasing direction of p. Thus the curve going up through $\omega_i = 0$ means the existence of a right-mover and that going down is a left-mover. Topologically there must be the same number of upgoings and downgoings for the closed curve. Therefore there appear equally many right and left movers in the low-energy regime.

It should be noticed that we "renormalize" the momentum by defining a new practical momentum

$$p_{\mathrm{pr}} = p - p_{\mathrm{f}}, \tag{2.3}$$

(see sect. 3 in Absence I). The dispersion relation is then for small p_{pr}

$$\omega_i = \left.\frac{\mathrm{d}\omega_i}{\mathrm{d}p}\right|_{p=p_f} p_{\mathrm{pr}}. \tag{2.4}$$

This is considered to be a relativistically invariant dispersion relation for a massless mover along the right or left direction. This is true only when $\mathrm{d}\omega_i/\mathrm{d}p|_{p=p_f}$ is considered equal to the "velocity of light". It should be noticed that the extra particles with $p_{\mathrm{f}} \neq 0$ are not described by $\psi(n)$ itself, but they are described by

$$\psi'(n) = \mathrm{e}^{ip_{\mathrm{f}}n}\psi(n) \tag{2.5}$$

since in the Fourier transform

$$\psi(p) = \sum_n \mathrm{e}^{-ipn}\psi(n)$$

the transformation (2.3) should be done on the right-hand side.

3. Intuitive topological proof in 3 + 1 dimension

Let us consider the 3 + 1 dimensional case. We will use the periodicity of momentum space (1.1) in our proof of the no-go theorem. We start with a review of how to describe the Weyl fermion in the dispersion manifold which was discussed in detail in sect. 3 of Absence I.

3.1. TWO ENERGY LEVEL DEGENERACY DESCRIBES WEYL FERMION

As in Absence I we consider a general class of lattice fermion theories described by the action (1.2) for the N-component ψ (1.3) and thus the eigenvalue equation

$$H(p)\psi(p) = \omega_i(p)\psi(p), \qquad i = 1,\ldots,N, \tag{3.1}$$

with N ordered eigenvalues $\omega_1 > \omega_2 > \cdots > \omega_N$. When two energy levels, ith and $(i+1)$th coincide (are degenerate) at the momentum p_{deg},

$$\omega_i(p_{\mathrm{deg}}) = \omega_{i+1}(p_{\mathrm{deg}}) = 0,$$

as shown in fig. 2, one can expand the $N \times N$ matrix $H(p)$ near the degeneracy point p_{deg}. The relevant most general 2×2 hamiltonian $H^{(2)}(p)$ for $U(p)$, the two-component spinor describing the ith and $(i+1)$th levels, is of the form

$$H^{(2)}(p) = \omega_{\mathrm{deg}}(p_{\mathrm{deg}}) + (p - p_{\mathrm{deg}})b + (p - p_{\mathrm{deg}})_\kappa V_\alpha^\kappa \sigma^\alpha + \mathrm{O}\big((p-p)^2\big), \tag{3.2}$$

with constants b and $\mathbf{V}$. As in 1 + 1 dimensions, eq. (2.3), we renormalize the momentum by introducing a practical momentum through

$$p_{\mathrm{pr}} = p - p_{\mathrm{deg}}, \qquad \omega_{\mathrm{pr}} = \omega - \omega_{\mathrm{deg}}. \tag{3.3}$$

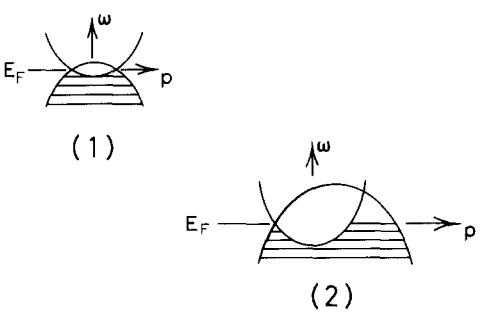

Fig. 2. The allowed Fermi surface (1), and not allowed (2). Here $\mathrm{E_F}$ denotes a Fermi surface.

Thus the $H^{(2)}(\boldsymbol{p})$ becomes

$$H^{(2)}(\boldsymbol{p}) = (\boldsymbol{p}_{\rm pr})_\kappa V_\alpha^\kappa \sigma^\alpha + (\boldsymbol{p} - \boldsymbol{p}_{\rm deg})\boldsymbol{b}. \tag{3.4}$$

We further define a new momentum by

$$P_0 = \omega_{\rm pr} - (\boldsymbol{p} - \boldsymbol{p}_{\rm deg})\boldsymbol{b}$$

$$P_\alpha = \pm (\boldsymbol{P}_{\rm pr})_\kappa V_\alpha^\kappa. \tag{3.5}$$

Here the sign $\pm$ depends on the sign of $\det V$ which determines the relative handedness of the $\boldsymbol{p}_{\rm pr}$ and $\boldsymbol{p}$ coordinates. We, however, take conventionally $\boldsymbol{p}_{\rm pr}$ to be the right-handed one. So the new $H^{(2)}(\boldsymbol{p})$ becomes

$$H^{(2)}(\boldsymbol{p}) = \boldsymbol{\sigma p} \tag{3.6}$$

and we obtain the right- or left-handed Weyl equation

$$\boldsymbol{P\sigma} U(\boldsymbol{p}) = \pm p_0 U(\boldsymbol{p}), \tag{3.7}$$

corresponding to the sign of $\det V$. Since the constant tensor $\mathbf{V}$ depends on the degeneracy point $p_{\rm deg}$, whether we get a right- or left-handed Weyl fermion also depends on the degeneracy point. We may call the degeneracy point a right- or left-handed one according to eq. (3.7). We have thus shown that each degeneracy point represents one species of Weyl fermion. If the Fermi energy surface lies at the degeneracy points of the ith and $(i+1)$th levels as depicted in fig. 2, each degeneracy point represents one species of Weyl particle, e.g. $\nu_{\rm L}, {\rm e}_{\rm L}, {\rm e}_{\rm R}$.

3.2. CLOSED CURVES

The no-go theorem was proved in Absence I by making use of the homotopy theory of $\pi_2({\rm CP}^{N-1})$. We shall present here a more intuitive topological proof.

We make use of curves in 4-dimensional ω-$\boldsymbol{p}$ space or on the 3-dimensional dispersion relation surface. They are defined by

$$\{(\boldsymbol{p}, \omega_i(\boldsymbol{p})) \,|\, \langle a|\omega_i(\boldsymbol{p})\rangle = 0\}.$$

Here

$$\langle a|\omega_i(\boldsymbol{p})\rangle = a_1\psi_1^{(i)} + a_2\psi_2^{(i)} + \cdots + a_N\psi_N^{(i)}. \tag{3.8}$$

The vector $|a\rangle = (a_1, a_2, \ldots, a_N)$ is an arbitrarily chosen constant in the complex N-dimensional vector space, which may be chosen to be the basis vector correspond-

ing to the field number one. The eq. (3.8) indeed specifies a 1-dimensional curve in the generic case. This is clear from the fact that eq. (3.8) fixes two variables since $\langle a|\omega_i(\boldsymbol{p})\rangle \in \mathrm{C}$ and that it is continuous and analytic complex-valued function of $\boldsymbol{p}$. The set of curves always passes through all the degeneracy points, as can be shown explicitly using the continuum 2-component Weyl equation near the degeneracy points, or by the following argument. In fact, let $|\omega_i(\boldsymbol{p}_{\mathrm{deg}})\rangle$ and $|\omega_{i+1}(\boldsymbol{p}_{\mathrm{deg}})\rangle$ be eigenstates at the degeneracy point $\boldsymbol{p}_{\mathrm{deg}}$. Then one can always construct a new state $|\bar{\omega}_i(\boldsymbol{p})\rangle$ by

$$|\bar{\omega}_i(\boldsymbol{p})\rangle = \alpha|\omega_i(\boldsymbol{p})\rangle + \beta|\omega_{i+1}(\boldsymbol{p})\rangle$$

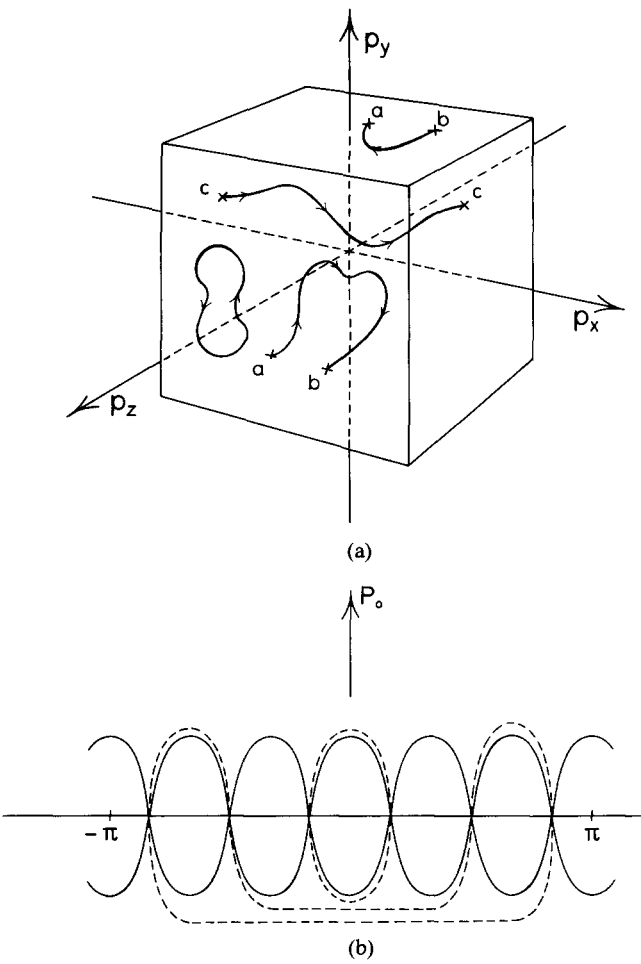

Fig. 3. Three kinds of closed curves in a Brillouin zone. Remember that opposite faces of the Brillouin zone are identified, so that especially the pairs of points denoted a, b and c are to be considered as only one point each. A typical closed curve (broken lines) in the dispersion ω-$\boldsymbol{p}$ space are drawn in (b), where p_x and p_y dimensions are suppressed.

near $\boldsymbol{P} = \boldsymbol{p}_{\text{deg}}$. This is approximately equal to an eigenstate for $\boldsymbol{p}$ close to $\boldsymbol{p}_{\text{deg}}$. We can always choose α and β such that

$$\langle a | \bar{\omega}_i(\boldsymbol{p}) \rangle = 0,$$

which is condition (3.8) for the curve.

Since the $\boldsymbol{p}$-space forms the Brillouin zone (1.1) and the curves continuously pass through all the degeneracy points between the energy levels ω_i and ω_{i+1}, such a curve must be closed or consists of a number of closed curves as illustrated in fig. 3.

If the curves have an orientation, such closed curves must pass equally many times upward and downward between the two energy levels ω_i and ω_{i+1} through all the degeneracy points.

3.3. ORIENTATION OF THE CURVE

We are now going to give an orientation assignment to the curve and show that, when the curve crosses the level $\omega = 0$ between levels ω_i and ω_{i+1} upward (or downward), the crossing point (= degeneracy point) is a right- (or left-) handed degeneracy point.

The eigenstate determined by the eq. (3.1) is unique modulo a phase factor if we impose the normalization condition $\langle \omega_i(\boldsymbol{p}) | \omega_i(\boldsymbol{p}) \rangle = 1$. We can choose the phase of $|\omega_i(\boldsymbol{p})\rangle$ to be analytic in a simply connected region which should not include the degeneracy points. An orientation of the curves is assigned by means of the phase rotation of $\langle a | \omega_i(\boldsymbol{p}) \rangle$ on a small circle S_1 around the curve (3.8). We choose the convention such that an increase of phase on S_1 should form a right-handed screw together with the oriented curve when we take a right-handed coordinate convention for $\boldsymbol{P}$.

Let us, as an example, consider the case that the curve is along $P_z > 0$ on the upper cone $P_0 > 0$ near the right-handed degeneracy point. Thus the two-component field U satisfies the eq.

$$\boldsymbol{p}\boldsymbol{\sigma} U(\boldsymbol{p}) = P_0 U(\boldsymbol{p}). \tag{3.9}$$

We draw a circle S_1 with radius R around the point on the curve at a distance d from the degeneracy point as depicted in fig. 4 and take $R \ll d$. The point Q on S_1 is given

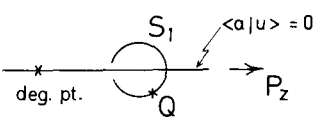

Fig. 4. The curve (3.8) along the p_z direction is described in the text.

by

$$p_x = R\cos\vartheta,$$
$$p_y = R\sin\vartheta,$$
$$p_z = d. \tag{3.10}$$

The normalized eigenvector U of (3.9) is solved in the limit $R/d \ll 1$:

$$U = \begin{pmatrix} 1 \\ \frac{R}{2d} e^{i\vartheta} \end{pmatrix}. \tag{3.11}$$

Since we obtain $U = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ on the curve $R = 0$, the constant vector $\langle a|$ must be $\langle a| = (0,1)$. Thus we have on S_1,

$$\langle a|\omega_i\rangle = \frac{R}{2p_z} e^{i\vartheta}. \tag{3.12}$$

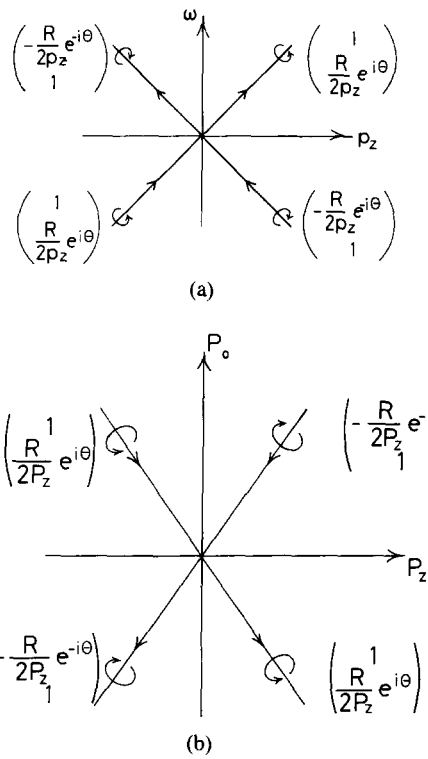

Fig. 5. This figure illustrates that around a "degeneracy point" all the curves of the type (3.8) are oriented away on one level and inward on the other one. The case illustrated in (a) is a right-handed particle (i.e. positive helicity) degeneracy point satisfying the right-handed Weyl equation, while the holes in the lower level would be left-handed. The case of a left-handed particle degeneracy satisfying left-handed Weyl equation is illustrated in (b).

According to the orientation convention the curve is oriented in the positive p_z direction because it forms a right-handed screw. Computing the case of $p_0 > 0$ and $p_z < 0$ cone, etc., we obtain fig. 5a for the right-handed degeneracy point. The case of the left-handed degeneracy point is depicted in fig. 5b. As we see in fig. 5 the curve continues through the degeneracy points conserving the orientation. It is noticed that the orientation determined in this way becomes the same all along the curve. The important fact is that the curve must be oriented away from the degeneracy point on the level sheet with right-handed particles (i.e. with positive helicity), while the curve must be inward oriented on the level with left-handed particles.

Finally the oriented curve in fig. 3 is closed and thus must go equally many times upward and downward through all the degeneracy points $\omega_i(\boldsymbol{p}_{\mathrm{deg}}) = \omega_{i+1}(\boldsymbol{p}_{\mathrm{deg}}) = 0$ between the two energy levels $\omega_i(\boldsymbol{p})$ and $\omega_{i+1}(\boldsymbol{p})$. When the Dirac sea ($\omega_{i+1}$ level) is filled and the Fermi surface lies at the degeneracy point $\omega = 0$ there appear equally many species of right- and left-handed Weyl fermions. This concludes the proof of our no-go theorem.

4. Real-field formulation

We have considered in sect. 3 and in Absence I, the case of the complex-field formulation assuming the existence of a conserved charge at all scale lengths. If we abandon this assumption we are led to use a real-field formulation. In this case there is double counting of the particles in the $\boldsymbol{p}$ representation in the Brillouin zone. Thus our no-go theorem may not hold. In fact we shall give an example of a lattice fermion theory with only one two-component fermion.

The reality condition of the fermion field is

$$\psi^*(x) = \psi(x), \tag{4.1}$$

where the asterisk denotes complex conjugation, if we consider the N-component fields $\psi(x)$ as Grassmann numbers, and hermitian conjugation, if we consider a second-quantized field $\psi(x)$. In momentum space eq. (4.1) becomes

$$\tilde{\psi}^*(-p) = \tilde{\psi}(p). \tag{4.2}$$

It should be stressed that $\tilde{\psi}(p)$ is an independent variable only over one half of the Brillouin zone $-\pi \leqslant p_i \leqslant \pi$ ($i = 1,2,3$). In order that the time development

$$i\dot{\psi}(x) = \sum_y H_{\mathrm{R}}(x-y)\psi(y) \tag{4.3}$$

does not violate (4.2), the $N \times N$ matrix $H_{\mathrm{R}}(x)$ must be purely imaginary:

$$H_{\mathrm{R}}^*(x) + H_{\mathrm{R}}(x) = 0. \tag{4.4}$$

Under a Fourier transformation (4.4) becomes

$$\tilde{H}_{\mathrm{R}}(p) = -\tilde{H}_{\mathrm{R}}^{*}(-p), \tag{4.5}$$

while the equation of motion (4.3) becomes

$$i\dot{\tilde{\psi}}(p) = \tilde{H}_{\mathrm{R}}(p)\tilde{\psi}(p).$$

It is easily seen that the real-field formulation is more general than the complex one. Any theory with a complex field ψ can, in fact, be reformulated in terms of real fields $\mathrm{Re}\psi$ and $\mathrm{Im}\psi$. When the N-component field ψ *is complex*, we can thus introduce a $2N$-component column vector

$$\begin{pmatrix} \mathrm{Re}\psi(x) \\ \mathrm{Im}\psi(x) \end{pmatrix} \tag{4.6}$$

which obeys the equation of motion

$$i\frac{\partial}{\partial t}\begin{pmatrix} \mathrm{Re}\psi(p) \\ \mathrm{Im}\psi(p) \end{pmatrix} = \tilde{H}_{\mathrm{C}}(p)\begin{pmatrix} \mathrm{Re}\psi(p) \\ \mathrm{Im}\psi(p) \end{pmatrix}.$$

Here

$$\tilde{H}_{\mathrm{C}}(p) = \begin{pmatrix} \frac{1}{2}\big(\tilde{H}(p) - \tilde{H}^{*}(-p)\big) & \frac{1}{2}i\big(\tilde{H}(p) + \tilde{H}^{*}(-p)\big) \\ \frac{1}{2i}\big(\tilde{H}(p) + \tilde{H}^{*}(-p)\big) & \frac{1}{2}\big(\tilde{H}(p) - \tilde{H}^{*}(-p)\big) \end{pmatrix}$$

and $\tilde{H}(p)$ is a Fourier transform of the $N \times N$ hamiltonian contained in the action (1.1). One may remark that $\tilde{H}_{\mathrm{C}}(p)$ is of the special form

$$\tilde{H}_{\mathrm{C}}(p) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}, \tag{4.7}$$

where A and B are $N \times N$ matrices. This form (4.7) is not a consequence of the real-field condition (4.5). Thus *there is indeed a restriction* (4.7) *for the complex-field formulation* (4.6).

The most general theory is obtained using the real-field formulation. In fact, if we impose fermion number conservation on a real-field formulation by requiring invariance under the $\mathrm{O}(2) \cong \mathrm{U}(1)$ symmetry

$$\begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_{2N}(x) \end{pmatrix} \to \begin{pmatrix} I\cos\theta & I\sin\theta \\ -I\sin\theta & I\cos\theta \end{pmatrix} \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_{2N}(x) \end{pmatrix},$$

where I is an $N \times N$ unit matrix, we are led to a complex-field hamiltonian. Thus, we in general lose fermion number conservation in the real-field formulation.

The example which we shall construct has *two real fermion fields* $\psi(x)$:

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}.$$

In order to have one two-component fermion we must have two degeneracy points which are connected by the symmetry

$$(\omega, \boldsymbol{p}) \to (-\omega, -\boldsymbol{p}). \tag{4.8}$$

We assume $\omega_{\text{deg}} = 0$ (see sect. 2 of Absence I). We may take the hamiltonian of the form

$$\tilde{H}_{\text{R}}(\boldsymbol{p}) = A(\boldsymbol{p})\sigma_x + B(\boldsymbol{p})\sigma_y + C(\boldsymbol{p})\sigma_z, \tag{4.9}$$

for simplicity. Hermiticity requires A, B and C to be real. The condition for the real-field formalism

$$\tilde{H}_{\text{R}}(\boldsymbol{p}) = -\tilde{H}^*_{\text{R}}(-\boldsymbol{p})$$

requires then that

$$A(-\boldsymbol{p}) = -A(\boldsymbol{p}), \qquad B(-\boldsymbol{p}) = B(\boldsymbol{p}), \qquad C(-\boldsymbol{p}) = -C(\boldsymbol{p}). \tag{4.10}$$

Note that $\tilde{H}_{\text{R}}(\boldsymbol{p})$ is periodic under

$$p_i \to p_i + 2\pi, \qquad (i = 1,2,3).$$

Thus A, B and C may be thought of as Fourier series in $\boldsymbol{p}$. We should bear in mind that a degeneracy point $\boldsymbol{p} = \boldsymbol{p}_{\text{deg}}$ occurs when

$$A(\boldsymbol{p}_{\text{deg}}) = B(\boldsymbol{p}_{\text{deg}}) = C(\boldsymbol{p}_{\text{deg}}) = 0. \tag{4.11}$$

Since we are looking for an example with a single neutrino and, since we have to count only half the Brillouin zone, we must have two degeneracy points $\boldsymbol{p}_{\text{deg}_1}$ and $\boldsymbol{p}_{\text{deg}_2}$ where

$$\boldsymbol{p}_{\text{deg}_1} = -\boldsymbol{p}_{\text{deg}_2}. \tag{4.12}$$

Let us choose a simple example. We may take

$$A(p) = a \sin p_x, \qquad C(p) = c \sin p_z. \tag{4.13}$$

There are four lines for which A and C in (4.13) are zero when thought of as imbedded in a Brillouin zone. See fig. 6. They are given by

$$(1) \qquad p_x = p_z = 0,$$

$$(2) \qquad p_x = \pi, \quad p_z = 0,$$

$$(3) \qquad p_x = 0, \quad p_z = \pi,$$

$$(4) \qquad p_x = p_z = \pi. \tag{4.14}$$

Line (4) is an identification of four parallel edges of the Brillouin zone. Next we choose the zero surface for $B(\boldsymbol{p})$ as an approximate sphere with center $\boldsymbol{p} = 0$ which intersects with only (1). We choose

$$B(\boldsymbol{p}) = b(\cos p_x + \cos p_y + \cos p_z) - \kappa. \tag{4.15}$$

The parameters b and κ are adjusted to satisfy $B(\boldsymbol{p} = 0) > 0$ and $B < 0$ when any of the components of $\boldsymbol{p}$ equals $\pm\pi$. We can arrange this by taking $b > 0$ and $3b > \kappa > b$. In this way we get just two degeneracy points at

$$\boldsymbol{p}_{\text{deg}} = (0, \pm|p_y|_{\text{deg}}, 0). \tag{4.16}$$

Here $|p_y|_{\text{deg}}$ is determined from

$$b\cos|p_y|_{\text{deg}} + 2b - \kappa = 0,$$

i.e.

$$|p_y|_{\text{deg}} = \arccos\frac{\kappa - 2b}{b}. \tag{4.17}$$

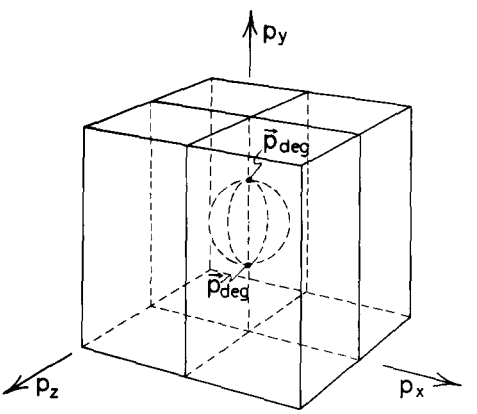

Fig. 6. The curves in the Brillouin zone along which two of the three coefficient functions $A(\boldsymbol{p})$, $B(\boldsymbol{p})$ and $C(\boldsymbol{p})$ are zero. At the degeneracy points they are all three zero. They are here denoted by $\boldsymbol{p}_{\text{deg}}$.

The hamiltonian (4.9) can now be written down:

$$\tilde{H}_{\mathrm{R}}(\boldsymbol{p}) = a\sigma_x \sin p_x + b\sigma_y(\cos p_x + \cos p_y + \cos p_z) + c\sigma_z \sin p_z - \kappa\sigma_y. \quad (4.18)$$

The dispersion relation derived from (4.18) is depicted in fig. 7.

The x-representation is given by

$$H_{\mathrm{R}}(\boldsymbol{x}) = \int_{-\pi}^{\pi} \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{e}^{i\boldsymbol{p}\boldsymbol{x}} \tilde{H}_{\mathrm{R}}(\boldsymbol{p}).$$

The non-zero components of this $H(\boldsymbol{x})$ for special values of $\boldsymbol{x}$ are:

$$H_{\mathrm{R}}(1,0,0) = \frac{-1}{2i} a\sigma_x + \tfrac{1}{2} b\sigma_y,$$

$$H_{\mathrm{R}}(-1,0,0) = \frac{1}{2i} a\sigma_x + \tfrac{1}{2} b\sigma_y,$$

$$H_{\mathrm{R}}(0,1,0) = \tfrac{1}{2} b\sigma_y,$$

$$H_{\mathrm{R}}(0,-1,0) = \tfrac{1}{2} b\sigma_y,$$

$$H_{\mathrm{R}}(0,0,1) = \frac{-1}{2i} c\sigma_z + \tfrac{1}{2} b\sigma_y,$$

$$H_{\mathrm{R}}(0,0,-1) = \frac{1}{2i} c\sigma_z + \tfrac{1}{2} b\sigma_y,$$

$$H_{\mathrm{R}}(0,0,0) = -\kappa\sigma_y. \quad (4.19)$$

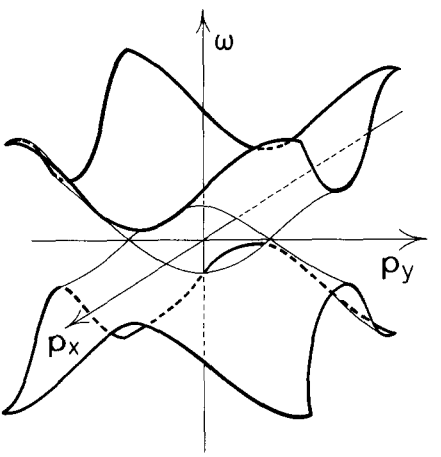

Fig. 7. The dispersion law of the hamiltonian (4.18) given in the text.

We can trivially construct the lagrangian corresponding to (4.19). The Schrödinger equation is given by (4.3),

$$\begin{aligned}\frac{\partial}{\partial t}\psi(x_1,x_2,x_3) = &\left(\tfrac{1}{2}a\sigma_x - \tfrac{1}{2}ib\sigma_y\right)\psi(x_1-1,x_2,x_3)\\ &+\left(-\tfrac{1}{2}a\sigma_x - \tfrac{1}{2}ib\sigma_y\right)\psi(x_1+1,x_2,x_3)\\ &-\tfrac{1}{2}ib\{\psi(x_1,x_2+1,x_3)+\psi(x_1,x_2-1,x_3)\}\\ &+\left(\tfrac{1}{2}c\sigma_z - \tfrac{1}{2}ib\sigma_y\right)\psi(x_1,x_2,x_3-1)\\ &+\left(-\tfrac{1}{2}c\sigma_z - \tfrac{1}{2}ib\sigma_y\right)\psi(x_1,x_2,x_3+1)\\ &+i\kappa\sigma_y\psi(x_1,x_2,x_3).\end{aligned}$$

This leads to the lagrangian

$$L=\sum_x i\,{}^t\psi(x)\dot{\psi}(x)-\sum_y {}^t\psi(x)H_R(x-y)\psi(x-y), \tag{4.20}$$

where ${}^t\psi(x)$ denotes the transpose, $H_R(y)$ is given by (4.19), and the remaining values of $H_R(y)$ are zero.

An especially nice model (with real fields) showing precisely one Weyl-type particle was presented to us by the referee. It is given by the action

$$S=\tfrac{1}{2}\sum_x \bar{\psi}_x\sigma^\mu\left(\psi_{x+\hat{\mu}}-\psi_{x-\hat{\mu}}\right)-i\eta\sum_{x,\hat{\mu}}\varepsilon^{\alpha\beta}\left(\psi^\alpha_{x+\hat{\mu}}+\psi^\alpha_{x-\hat{\mu}}-2\psi^\alpha_x\right)\psi^\beta_x+\text{h.c.}$$

Here $\hat{\mu}$ is a unit vector in the direction μ. The spinor indices α,β runs through 1,2. The first term provides the usual type of species multiplication. It is the most naive model, but the terms with coefficient η gives a Majorana particle type mass to the species doublers. One may imagine both a Kogut-Susskind lattice and a Wilson type lattice version of this model.

An important achievement of this model is that it contrary to the first presented real-field model having a single Weyl particle this last one has retained symmetry under discrete 90° rotations.

5. Assignment of non-conserved charges

We have constructed in sect. 4 a lattice model which describes only one two-component fermion in the low-energy regime. The hamiltonian (4.18) of this model does not, of course, satisfy condition (4.7) for the complex-field formulation. Thus, it

is impossible to assign conserved charges to this model. We can, however, assign locally defined but only approximately conserved charges $\tilde{Q}(\boldsymbol{p})$.

It should be stressed that if we do not require that the charge be locally defined, it is extremely easy to define a conserved charge since we could assign $\tilde{Q}(\boldsymbol{p})$ the value $+1$ for some values of the momentum $\boldsymbol{p}$ and -1 for the opposite $\boldsymbol{p}$ value if we have real fields. In this way one could easily assign $\tilde{Q}(\boldsymbol{p}) = +1$ to the states in the neighborhood of the degeneracy point at $\boldsymbol{p}_{\mathrm{deg}_2}$ say. One might then either leave the charge undefined outside such regions in $\boldsymbol{p}$ space or let $\tilde{Q}(\boldsymbol{p})$ have discontinuities as a function of $\boldsymbol{p}$ meaning the charge not being locally defined.

If we, however, want the charge to be locally defined and at the same time quantized, there must for all values of the momentum $\boldsymbol{p}$ in the Brillouin zone exist equally many eigenstates with a given charge eigenvalue. To a model with only one Weyl particle we can then only assign an approximately conserved charge. That is, however, possible: in fact, we shall show that in the case where we modify the model of sect. 4 with two real components (ψ_1, ψ_2) into a four-component model $(\psi_1, \psi_2, \psi_3, \psi_4)$, the modified model looks like that of the complex-field formulation (4.6) with non-conserved charges, but the hamiltonian does not satisfy the condition for the complex-field formulation (4.7) with conserved charges (i.e., is not complex linear). In fact, we want here to construct a model in which *charges are not conserved but are approximately conserved in the low-energy regime only.*

There should be only two degeneracy points, given by (4.16), which are connected by the symmetry (4.8) in order to describe one Weyl fermion in the low-energy region. Thus the dispersion relation for $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ should be the same as in fig. 2 and that for $\begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$ should have no degeneracy points. Therefore, the field components ψ_3 and ψ_4 may be chosen to have flat dispersion relations as depicted in fig. 8. The hamiltonian for

$$\tilde{\psi}(\boldsymbol{p}) = \begin{pmatrix} \tilde{\psi}_1(\boldsymbol{p}) \\ \vdots \\ \tilde{\psi}_4(\boldsymbol{p}) \end{pmatrix}$$

may be written, e.g.,

$$\tilde{H}(\boldsymbol{p}) = \begin{pmatrix} \tilde{H}_{\mathrm{R}}(\boldsymbol{p}) & 0 \\ 0 & \sigma_y \end{pmatrix}, \tag{5.1}$$

where $\tilde{H}_{\mathrm{R}}(\boldsymbol{p})$ is the 2×2 matrix hamiltonian for ψ_1 and ψ_2 given by (4.18). The eq. (5.1) satisfies the reality condition for the field $\tilde{\psi}(\boldsymbol{p})$. The charges $\tilde{Q}(\boldsymbol{p})$ that we will construct have the following properties: The charges $\tilde{Q}(\boldsymbol{p})$ are not conserved, i.e.

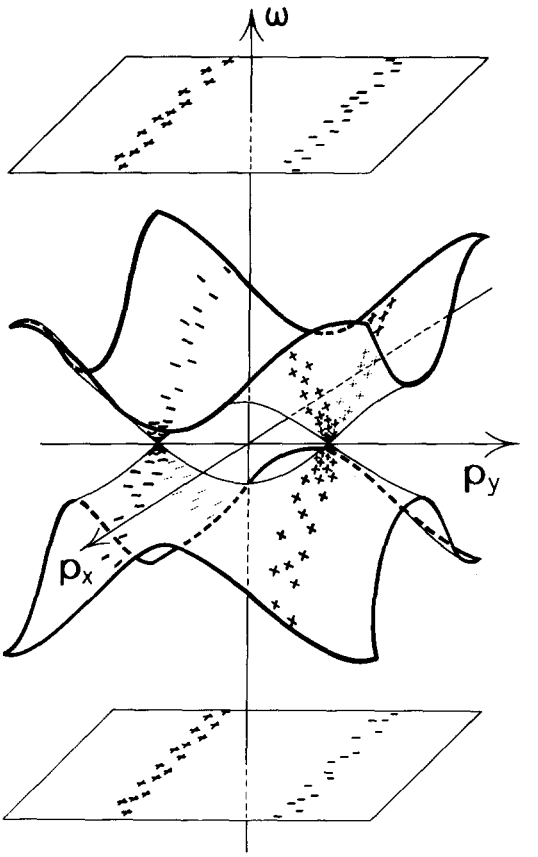

Fig. 8. The dispersion law for our model with four-component field with the assignment of non-conserved charges denoted by +'s and −'s. Over most of the dispersion relation we assign no definite charge, but a superposition.

$[\tilde{H}(p), \tilde{Q}(p)] \neq 0$ except for the low-energy regime. Thus the eigenstates of $\tilde{H}(p)$ are mixed states of the eigenstates of $\tilde{Q}(p)$. In the low-energy regime $\tilde{Q}(p)$ is conserved and we can assign definite charges to each component of $\psi(p)$.

In lattice fermion theories we should renormalize or shift the momentum by

$$p_{\mathrm{pr}} = p - p_{\mathrm{deg}},$$

where p is the original fundamental momentum and p_{pr} denotes the practical momentum as discussed in sect. 3 of Absence I. Thus, low energy means p_{pr} small, i.e. $p \approx p_{\mathrm{deg}}$. We assign charges to each field components of ψ in fig. 5 near the degeneracy points $p \approx p_{\mathrm{deg}}$ (i.e. for $p - p_{\mathrm{deg}}$ small on the scale of the lattice). Taking

the basis $(\psi_1\psi_3\psi_2\psi_4)^{\mathrm{T}}$, the charge operator $\tilde{Q}(\boldsymbol{p})$ should satisfy

$$\tilde{Q}(\boldsymbol{p})\begin{pmatrix}\tilde{\psi}_1(\boldsymbol{p})\\ \tilde{\psi}_3(\boldsymbol{p})\\ \tilde{\psi}_2(\boldsymbol{p})\\ \tilde{\psi}_4(\boldsymbol{p})\end{pmatrix} = \begin{pmatrix}+1 & & & 0\\ & -1 & & \\ & & +1 & \\ 0 & & & -1\end{pmatrix}\begin{pmatrix}\tilde{\psi}_1(\boldsymbol{p})\\ \tilde{\psi}_3(\boldsymbol{p})\\ \tilde{\psi}_2(\boldsymbol{p})\\ \tilde{\psi}_4(\boldsymbol{p})\end{pmatrix}, \qquad \text{near } \boldsymbol{p}=\boldsymbol{p}_{\mathrm{deg}}. \tag{5.2}$$

Notice that here we formulated $\tilde{Q}(\boldsymbol{p})$ with the order of the fields $\tilde{\psi}_i(\boldsymbol{p})$ in the columns permuted a bit. Eq. (5.2) should be true near the degeneracy point with

$$\boldsymbol{p}_{\mathrm{deg}} = (0, +|p_y|_{\mathrm{deg}}, 0),$$

say. For the reflected degeneracy point with

$$\boldsymbol{p}_{\mathrm{deg}} = (0, -|p_y|_{\mathrm{deg}}, 0),$$

we should instead have (5.2) with an extra minus sign on the right-hand side. This may be seen from eq. (5.4) or by noting that the degeneracy point reflected by (4.8) describes the *CP* antiparticles which have opposite charge.

The charge operator is now searched for in the form

$$\tilde{Q}(\boldsymbol{p})\begin{pmatrix}\tilde{\psi}_1(\boldsymbol{p})\\ \tilde{\psi}_3(\boldsymbol{p})\\ \tilde{\psi}_2(\boldsymbol{p})\\ \tilde{\psi}_4(\boldsymbol{p})\end{pmatrix} = \begin{pmatrix}n(\boldsymbol{p})\sigma & 0\\ 0 & n(\boldsymbol{p})\sigma\end{pmatrix}\begin{pmatrix}\tilde{\psi}_1(\boldsymbol{p})\\ \tilde{\psi}_3(\boldsymbol{p})\\ \tilde{\psi}_2(\boldsymbol{p})\\ \tilde{\psi}_4(\boldsymbol{p})\end{pmatrix}. \tag{5.3}$$

The reality condition of the field reads for the charge matrix

$$\tilde{Q}(\boldsymbol{p}) = -\tilde{Q}^*(-\boldsymbol{p}) \tag{5.4}$$

and the eigenvalues of $\tilde{Q}(\boldsymbol{p})$ are ± 1, i.e.

$$|\boldsymbol{n}(\boldsymbol{p})| = 1. \tag{5.5}$$

We must take $\boldsymbol{n}(\boldsymbol{p})$ real. Since the reality condition (5.4) is to be satisfied at $\boldsymbol{p}=0$,

$$(\boldsymbol{n}(\boldsymbol{p}=0)\sigma)^* = -(\boldsymbol{n}(\boldsymbol{p}=0)\sigma).$$

Here $\boldsymbol{n}(\boldsymbol{p}=0)$ should be of the form

$$\boldsymbol{n}(\boldsymbol{p}=0) = (0, n_y(\boldsymbol{p}=0), 0). \tag{5.6}$$

Near the degeneracy point $p = p_{\text{deg}}$ the charge $\tilde{Q}(p)$ is approximately conserved, so that

$$n(p)\sigma = n_z\sigma_z \qquad \text{at } p = p_{\text{deg}}. \tag{5.7}$$

As a solution to these two conditions (5.6) and (5.7) we may take the form

$$\begin{aligned} n(p) &= \left(0, \cos\left(\frac{\pi}{2|p_y|_{\text{deg}}}\right), \sin\left(\frac{\pi}{2|p_y|_{\text{deg}}}\right)\right) \\ &= (0, \cos p_y, \sin p_y), \end{aligned} \tag{5.8}$$

provided we choose b and κ such that $|p_y|_{\text{deg}} = \frac{1}{2}\pi$ from (4.17). So we should choose $\kappa = 2b$.

The charge operator (5.3) then becomes

$$\tilde{Q}(p) \begin{pmatrix} \psi_1(p) \\ \psi_3(p) \\ \psi_2(p) \\ \psi_4(p) \end{pmatrix} = \begin{pmatrix} \tilde{q}(p) & 0 \\ 0 & \tilde{q}(p) \end{pmatrix} \begin{pmatrix} \psi_1(p) \\ \psi_3(p) \\ \psi_2(p) \\ \psi_4(p) \end{pmatrix}, \tag{5.9}$$

where

$$\begin{aligned} \tilde{q}(p) &= \sigma_y \cos\left(\frac{\pi}{2|p_y|_{\text{deg}}} p_y\right) + \sigma_z \sin\left(\frac{\pi}{2|p_y|_{\text{deg}}} p_y\right) \\ &= \sigma_y \cos p_y + \sigma_z \sin p_y. \end{aligned}$$

6. Conclusion

We have given intuitive topological proofs of the no-go theorem in 1 + 1 (sect. 2) and 3 + 1 (sect. 3) dimensions. The no-go theorem for putting *weak interactions* on a lattice is stated as follows: in the general class of lattice fermion theories the appearance of an equal number of species of right- and left-handed Weyl particles is an unavoidable consequence of a lattice structure. The consequent no-go theorem for lattice QCD of *strong interactions* is that chiral invariance cannot be preserved. This no-go theorem holds under the following five main assumptions. The action of the lattice fermion theory should satisfy (1) locality of the interaction and (2) translational invariance. For the charge Q (lepton number, weak hypercharge etc.) we make the assumptions: (3) conservation of Q on the scale of the fundamental cutoff, (4) quantized Q and (5) locality of Q, i.e. a sum of the local charge density.

If one relaxes one or some of the assumptions one can construct a lattice fermion theory with only one two-component field. Indeed, in sects. 4 and 5 we explicitly constructed an example with one two-component charged fermion in the low-energy regime. The charge of this model is not conserved, but is approximately conserved in the continuum limit.

The strong interaction models for lattice QCD invented by Wilson [2], Susskind [3] and Banks and Casher [7] break chiral invariance in order to avoid spectral multiplication. Drell, Weinstein and Yankielowicz [6] were able to construct a chirally invariant lattice QCD model, but it breaks locality of the interaction.

We also owe the reader an example showing that the assumption of a quantized charge (i.e. a charge with a discrete set of eigenvalues) is essential. This is, however, rather easily provided since we can, for example, take the two real fermion field model built up in sect. 4 and assign a charge to it described in momentum representation by

$$\tilde{Q}(\boldsymbol{p}) = 1 \sin p_y .$$

Here 1 is the 2×2 unit matrix and $\tilde{Q}(\boldsymbol{p})$, of course, is defined by the total charge operator:

$$Q = \iiint_{-\pi}^{\pi} \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} \psi^+(\boldsymbol{p}) \tilde{Q}(\boldsymbol{p}) \tilde{\psi}(\boldsymbol{p}),$$

where $\tilde{\psi}(\boldsymbol{p})$ is the second quantized, here two component, momentum representation field. This charge matrix $\tilde{Q}(\boldsymbol{p})$ obeys the condition (5.4) for consistency with the reality of $\tilde{\psi}(\boldsymbol{x})$. Further, it is local because, as a polynomial in e^{ip_y}, it corresponds to a finite number of hopping terms. In fact, it corresponds in $\boldsymbol{x}$ representation to the matrix $Q(\boldsymbol{x})$ which is non-zero only for two values of $\boldsymbol{x}$, and for those we have

$$Q(0,1,0) = -\frac{1}{2i} 1,$$

$$Q(0,-1,0) = \frac{1}{2i} 1.$$

The abelian charge defined this way obeys all the requirements except that it has a continuous spectrum. However, we have achieved a lattice model with only one charged Weyl fermion. So our theorem is not true unless under the assumption of a quantized charge spectrum. In the low-energy regime this "counter example" has an approximate quantization of the charge since it there only takes the values

$$\pm \sin |p_{y_{\mathrm{deg}}}| = \pm \sin \arccos \frac{\kappa - 2b}{b} = \pm \sqrt{1 - \left(\frac{\kappa - 2b}{b} \right)^2} .$$

Our no-go theorem is a threat to a previous work on dynamical stability of local gauge symmetry by ourselves and Foerster [8]. We argued there that gauge invariance could be obtained in a natural, dynamically stable way. Actually we put it in by definition, at the same time introducing a new variable Ω. In this way gauge invariance becomes exact. Our no-go theorem, however, makes it difficult to see how fermions in a weak interaction theory can be imbedded.

Finally we would like to express the expectations of Banks and Casher that results similar to ours can be obtained even without the assumption of a lattice. It may therefore also be hoped that one can examine the possibilities of allowing for less and less lattice-like regularizations. For instance, one might as first steps think of generalizations to

(i) an amorphous lattice (i.e. no translational invariance);

(ii) a Presnajder-type of cutoff (a regularization in which the lattice torus Brillouin zone is replaced by some other momentum space manifold).

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