Supplemental material

LANDAU LEVEL PROJECTION

We consider electrons confined to the two-dimensional x-y plane and in a transverse magnetic field:

$$\hat{H}_0 = \frac{1}{2m}\left(\hat{\boldsymbol{P}} - e\boldsymbol{A}(\boldsymbol{r})\right)^2. \quad (1)$$

In the Landau gauge, $\boldsymbol{A}(\boldsymbol{r}) = B(0, x, 0)$ $(\boldsymbol{r} = (x, y, z))$, translations along the y direction leave the Hamiltonian invariant such that the momentum in this axis, p_y, is a good quantum number. On a torus of size $L_x \times L_y$ the wave function of the first Landau level reads:

$$\phi_{p_y}(\boldsymbol{r}) = \frac{1}{\sqrt{L_y}}\frac{1}{\sqrt{l_B\sqrt{\pi}}}e^{-(x/l_B - \mathrm{sign}(B)p_y l_B)^2/2}e^{ip_y y}. \quad (2)$$

Here the magnetic length scale is defined as $l_B^2 = \frac{\phi_0}{2\pi|B|}$, with $\phi_0 = \frac{h}{e}$, and the number of magnetic fluxes piercing the system, $N_\phi = \frac{|B|V}{\phi_0}$, is an integer so as to guarantee uniqueness of the wave function. Finally the momentum in the y-direction is given $p_y = \frac{2\pi n}{L_y}$ with $n \in 1, \cdots, N_\phi$. From here onwards we will consider the case of $B > 0$.

The orbital wave function of the first Landau level of free electrons in a magnetic field and that of the zero energy Landau level (ZLL) in graphene are identical. In graphene, however, there is an SU(4) symmetry such the electron carries an additional flavor index, $a \in 1, \cdots, 4$. Let $\hat{c}_{p_y,a}$ destroy an electron in the ZLL with flavor index a and momentum p_y. These operators satisfy canonical fermion commutation rules:

$$\left\{\hat{c}^\dagger_{p_y,a}, \hat{c}_{p'_y,a'}\right\} = \delta_{a,a'}\delta_{p_y,p'_y}, \ \left\{\hat{c}_{p_y,a}, \hat{c}_{p'_y,a'}\right\} = 0. \quad (3)$$

Our Hamiltonian is defined in terms of projected field operators.

$$\hat{\psi}_a(\boldsymbol{r}) = \sum_{p_y=1}^{N_\phi}\phi_{p_y}(\boldsymbol{r})\hat{c}_{a,p_y}. \quad (4)$$

Since the ZLL does not span the Hilbert space the projected field operators do not satisfy the fermion canonical commutation rules, and before formulating the AFQMC we have to express everything in terms of the canonical operators $\hat{c}_{a,p_y}$. Defining the Fourier transform of the four component spinor:

$$\hat{\boldsymbol{\psi}}^\dagger_{\boldsymbol{p}} = \frac{1}{\sqrt{V}}\int_V d^2\boldsymbol{r}e^{i\boldsymbol{p}\cdot\boldsymbol{r}}\hat{\boldsymbol{\psi}}^\dagger(\boldsymbol{r}) \quad (5)$$

we obtain

$$\begin{aligned}\hat{H} &= \sum_{i=0}^{5}\int_V d^2\boldsymbol{r}\frac{U_i}{2}[\hat{\psi}^\dagger_a(\boldsymbol{r})O^i_{ab}\hat{\psi}_b(\boldsymbol{r}) - C(\boldsymbol{r})\delta_{i,0}]^2 \\ &= \sum_{i=0}^{5}\frac{U_i}{2}\sum_{\boldsymbol{q}}\sum_{\boldsymbol{q'}}\int_V d^2\boldsymbol{r}(\frac{1}{V}e^{i\boldsymbol{q}\cdot\boldsymbol{r}}\hat{N}^i(\boldsymbol{q}))(\frac{1}{V}e^{i\boldsymbol{q'}\cdot\boldsymbol{r}}\hat{N}^i(\boldsymbol{q'})) \\ &= \frac{1}{2V}\sum_{i=0}^{5}\sum_{\boldsymbol{q}}\hat{N}^i(\boldsymbol{q})U_i\hat{N}^i(-\boldsymbol{q})\end{aligned} \quad (6)$$

Here, O^0, is the unit matrix and $\hat{\boldsymbol{\psi}}^\dagger(\boldsymbol{r})O^i\hat{\boldsymbol{\psi}}(\boldsymbol{r}) = \frac{1}{V}\sum_{\boldsymbol{q}}e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}\hat{N}^i(\boldsymbol{q})$ for $i = 1\cdots5$ and $\hat{\boldsymbol{\psi}}^\dagger(\boldsymbol{r})\hat{\boldsymbol{\psi}}(\boldsymbol{r}) - C(\boldsymbol{r}) = \frac{1}{V}\sum_{\boldsymbol{q}}e^{-i\boldsymbol{q}\cdot\boldsymbol{r}}\hat{N}^0(\boldsymbol{q})$.

Neglecting the constant background term at $i = 0$, the *density* operators $\hat{N}^i(\boldsymbol{q})$, can be expressed in terms of the canonical operators $\hat{\boldsymbol{c}}^\dagger_{p_y}$:

$$\begin{aligned}\hat{N}^i(\boldsymbol{q}) &= \sum_{\boldsymbol{p}}\hat{\boldsymbol{\psi}}^\dagger_{\boldsymbol{p}}\boldsymbol{O}^i\hat{\boldsymbol{\psi}}_{\boldsymbol{p}-\boldsymbol{q}} \\ &= \frac{1}{V}\sum_{\boldsymbol{p}}\int_V\int_{V'}d^2\boldsymbol{r}d^2\boldsymbol{r'}e^{i\boldsymbol{p}\cdot\boldsymbol{r}}e^{-i(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{r'}} \\ &\quad \sum_{k_1}\hat{\boldsymbol{c}}^\dagger_{k_1}\phi^*_{k_1}(\boldsymbol{r})\boldsymbol{O}^i\sum_{k_2}\hat{\boldsymbol{c}}_{k_2}\phi_{k_2}(\boldsymbol{r'}) \\ &= \frac{1}{V}\sum_{\boldsymbol{p}}\sum_{k_1}\sum_{k_2}\hat{\boldsymbol{c}}^\dagger_{k_1}\boldsymbol{O}^i\hat{\boldsymbol{c}}_{k_2}\int_V\int_{V'}d^2\boldsymbol{r}d^2\boldsymbol{r'} \\ &\quad \left(\frac{1}{\sqrt{L_y}}\frac{\pi^{-\frac{1}{4}}}{\sqrt{l_B}}e^{-ik_1y}e^{-\frac{1}{2}(\frac{x}{l_B}-k_1l_B)^2}e^{i\boldsymbol{p}\cdot\boldsymbol{r}}\right) \\ &\quad \cdot\left(\frac{1}{\sqrt{L_y}}\frac{\pi^{-\frac{1}{4}}}{\sqrt{l_B}}e^{+ik_2y'}e^{-\frac{1}{2}(\frac{x'}{l_B}-k_2l_B)^2}e^{i(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{r'}}\right) \\ &= \sum_{\boldsymbol{p}}\frac{2l_B\pi^{1/2}}{L_x}e^{ip_xp_yl_B^2}e^{-l_B^2p_x^2/2} \\ &\quad e^{i(p_x-q_x)(p_y-q_y)l_B^2}e^{-l_B^2(p_x-q_x)^2/2}\hat{\boldsymbol{c}}^\dagger_{p_y}\boldsymbol{O}^i\hat{\boldsymbol{c}}_{p_y-q_y} \\ &= \frac{1}{2\sqrt{\pi}}e^{-l_B^2\boldsymbol{q}^2/4}\sum_{p_y}e^{iq_xp_yl_B^2}\hat{\boldsymbol{c}}^\dagger_{p_y+\frac{q_y}{2}}\boldsymbol{O}^i\hat{\boldsymbol{c}}_{p_y-\frac{q_y}{2}}\end{aligned} \quad (7)$$

In the last step, the sum over p_x is carried out by changing sums to integrals and taking the limit $L_x \to \infty$. With the substitution $k = p_y + \frac{q_y}{2}$ and

$$\begin{aligned}\hat{n}^i(\boldsymbol{q}) = &\sum_{k=1}^{N_\phi}\sum_{a,b=1}^{4}F(\boldsymbol{q})e^{\frac{i}{2}(2k-q_y)l_B^2q_x}(\hat{c}^\dagger_{a,k}O^i_{a,b}\hat{c}_{b,k-q_y} \\ &- 2\delta_{q_y,0}\delta_{i,0})\end{aligned} \quad (8)$$

the Hamiltonian reads:

$$\hat{H} = \frac{1}{8\pi V}\sum_{i=0}^{5}\sum_{\boldsymbol{q}}\hat{n}^i(\boldsymbol{q})U_i\hat{n}^i(-\boldsymbol{q}). \quad (9)$$

In the above, $a(\mathrm{b}) = 1, 2, 3$ and 4 is the flavor index, and $F(\mathbf{q}) \equiv e^{-\frac{1}{4}(q_x^2+q_y^2)l_B^2}$. The background term $2\delta_{q_y,0}\delta_{i,0}$ can easily be verified by Fourier transform the real space background $C(\boldsymbol{r})$ (see main text).

As we will shown in the next subsection, this exponential decaying factor is essential for the QMC simulation since it provides a natural cutoff for the momenta $\boldsymbol{q}$. Finally, setting the magnetic unit length to unity such that $\frac{2\pi}{V} = \frac{1}{N_\phi}$ we obtain:

$$\hat{H} = \frac{1}{16\pi^2 N_\phi} \sum_{i=0}^{5} \sum_{\boldsymbol{q}} \hat{n}^i(\boldsymbol{q}) U_i \hat{n}^i(-\boldsymbol{q}) \tag{10}$$

FIERZ IDENTITY AND ABSENCE OF THE NEGATIVE SIGN PROBLEM

To avoid the negative sign problem in the QMC simulations we use the Fierz identity to rewrite Eq. (6) as:

$$H = \frac{1}{2N_\phi} \sum_{i=0}^{3} \sum_{\boldsymbol{q}} \hat{n}^i_{-\mathbf{q}} g^i \hat{n}^i_{\mathbf{q}} \tag{11}$$

Instead of the original density operators in Eq. (27), the $n^i (i = 0, 1, 2, 3)$ operators are based on 4 matrix:

$$\mathbb{I}_4, \tau_x \otimes \mathbb{I}_2, \tau_y \otimes \mathbb{I}_2, \tau_z \otimes \mathbb{I}_2. \tag{12}$$

Eq. (11) is identical to Eq. (6) when $\frac{g^0}{8\pi^2} = U_0 + U$, $\frac{g^1}{8\pi^2} = \frac{g^2}{8\pi^2} = -2U$, and $\frac{g^3}{8\pi^2} = 2U$. The form of Eq. (11) is also used in the discussion of quantum hall ferromagnetism.[?] Here we consider the SO(5) symmetric point and set $U_i = -U$ for $i \in 1, \cdots, 4$. The absence of sign problem holds for the region of $U_0 \geq -U$, follows from the work of Ref.[?] and is discussed in detail in reference.[?] The above matrix structure also gives an explicit $SU(2)$ symmetry which holds for each Hubbard-Stratonovich field configuration.

TROTTER ERRORS

Since $n(\boldsymbol{q})^\dagger = n(-\boldsymbol{q})$, the exponential of operators at each time slice is given by:

$$\begin{aligned} &e^{-\frac{\Delta\tau}{2N_\phi}(\hat{n}^i_{\boldsymbol{q}} g^i \hat{n}^i_{-\boldsymbol{q}} + \hat{n}^i_{-\boldsymbol{q}} g^i \hat{n}^i_{\boldsymbol{q}})} \\ =&e^{-\frac{\Delta\tau}{4N_\phi}[g^i(\hat{n}^i_{\boldsymbol{q}} + \hat{n}^i_{-\boldsymbol{q}})^2 - g^i(\hat{n}^i_{\boldsymbol{q}} - \hat{n}^i_{-\boldsymbol{q}})^2]} \end{aligned} \tag{13}$$

To ensure hemiticity, we use a symmetric Trotter decomposition:

$$Z = \mathrm{Tr}\left[\prod_{m=1}^{N} e^{-\frac{\Delta\tau}{2}\hat{H}_m} \prod_{n=N}^{1} e^{-\frac{\Delta\tau}{2}\hat{H}_n}\right]^{L_\tau} \tag{14}$$

where $\hat{H}_m$ corresponds to the $N = 2 \times 4 \times N_q$ operators $\pm\frac{g^i}{4N_\phi}(\hat{n}^i_{\boldsymbol{q}} \pm \hat{n}^i_{-\boldsymbol{q}})^2$. N_q is the number of momentum points used for the simulation. As we will see below N_q scales as N_ϕ.

For two operators $\hat{H}_1$ and $\hat{H}_2$ the leading order error produced in the symmetric Trotter decomposition reads:

$$\begin{aligned} &e^{-\frac{\Delta\tau}{2}\hat{H}_1} e^{-\Delta\tau\hat{H}_2} e^{-\frac{\Delta\tau}{2}\hat{H}_1} \\ =&e^{-\Delta\tau(\hat{H}_1+\hat{H}_2) + \frac{\Delta\tau^3}{12}[2\hat{H}_1+\hat{H}_2,[\hat{H}_1,\hat{H}_2]]} + \mathcal{O}(\Delta\tau^4) \end{aligned} \tag{15}$$

Iterating the above formula gives:

$$\prod_{m=1}^{N} e^{-\frac{\Delta\tau}{2}\hat{H}_m} \prod_{n=N}^{1} e^{-\frac{\Delta\tau}{2}\hat{H}_n} = e^{-\Delta\tau((\sum_{m=1}^{N}\hat{H}_m)+\hat{\lambda})} + \mathcal{O}(\Delta\tau^4) \tag{16}$$

where

$$\begin{aligned} \hat{\lambda} \equiv &-\frac{\Delta\tau^2}{12}(\sum_{m=1}^{N-1}\sum_{m'=m+1}^{N}[2\hat{H}_m + \hat{H}_{m'}, [\hat{H}_m, \hat{H}_{m'}]] \\ &+ \sum_{m=1}^{N-1}\sum_{m'=m+1}^{N}\sum_{m''=m+1}^{N}[\hat{H}_{m'}, [\hat{H}_m, \hat{H}_{m''}]](1-\delta_{m',m''})) \end{aligned} \tag{17}$$

and $\delta_{m',m''}$ the Kronecker delta. Using time dependent perturbation theory, we then obtain:

$$\begin{aligned} &\left(\prod_{m=1}^{N} e^{-\frac{\Delta\tau}{2}\hat{H}_m} \prod_{n=N}^{1} e^{-\frac{\Delta\tau}{2}\hat{H}_n}\right)^{L_\tau} \\ =&e^{-\beta\hat{H}} - e^{-\beta\hat{H}}\int_0^\beta d\tau e^{\tau\hat{H}}\hat{\lambda}e^{-\tau\hat{H}} + \mathcal{O}(\Delta\tau^3) \end{aligned} \tag{18}$$

with $L_\tau = \frac{\beta}{\Delta\tau}$ the number of time slices. $\hat{\lambda}$ is measure of the leading order error on the free energy density:

$$\begin{aligned} f_{QMC} \equiv &-\frac{1}{\beta V}\ln\mathrm{Tr}\,(\prod_{m=1}^{N} e^{-\frac{\Delta\tau}{2}\hat{H}_m} \prod_{n=N}^{1} e^{-\frac{\Delta\tau}{2}\hat{H}_n})^{L_\tau} \\ =&f + \frac{1}{\beta V}\int_0^\beta d\tau\langle e^{\tau\hat{H}}\hat{\lambda}e^{-\tau\hat{H}}\rangle + \mathcal{O}(\Delta\tau^3) \\ =&f + \frac{1}{V}\langle\hat{\lambda}\rangle + \mathcal{O}(\Delta\tau^3) \\ =&f + \frac{1}{2\pi N_\phi}\langle\hat{\lambda}\rangle + \mathcal{O}(\Delta\tau^3) \end{aligned} \tag{19}$$

In the above we have set $l_B = 1$ so as to replace V by N_ϕ, and $f = -\frac{1}{\beta V}\ln\mathrm{Tr}\,e^{-\beta\hat{H}}$. Since the interacting operators for different masses i do not commute with each other, the Trotter decomposition breaks the SO(5) symmetry of Hamiltonian (a SU(2) symmetry is left due to the Fierz identity in Eq. (11)).

To evaluate the expectation value of $\hat{\lambda}$, we first evaluate the commutator of two density operators:

$$\begin{aligned}
&[\hat{n}^i(\boldsymbol{q_1}), \hat{n}^j(\boldsymbol{q_2})] \\
=&F(\boldsymbol{q_1})F(\boldsymbol{q_2})\sum_k \hat{\boldsymbol{c}}_k^\dagger \{ e^{\frac{i}{2}(2k-(q_{1y}+q_{2y}))l_B^2(q_{1x}+q_{2x})} \\
&(2\cos(\theta_{\boldsymbol{q_1},\boldsymbol{q_2}})[O^i,O^j] + 2i\sin(\theta_{\boldsymbol{q_1},\boldsymbol{q_2}})\{O^i,O^j\})\} \\
&\hat{\boldsymbol{c}}_{k-(q_{1y}+q_{2y})} \\
=&\frac{F(\boldsymbol{q_1})F(\boldsymbol{q_2})}{F(\boldsymbol{q_1}+\boldsymbol{q_2})}\{\hat{n}^{[O^i,O^j]}(\boldsymbol{q_1}+\boldsymbol{q_2})2\cos(\theta_{\boldsymbol{q_1},\boldsymbol{q_2}}) \\
+&\hat{n}^{\{O^i,O^j\}}(\boldsymbol{q_1}+\boldsymbol{q_2})2i\sin(\theta_{\boldsymbol{q_1},\boldsymbol{q_2}})\}
\end{aligned} \tag{20}$$

where $\theta_{\boldsymbol{q_1},\boldsymbol{q_2}} = \frac{l_B^2}{2}(q_{1y}q_{2x} - q_{1x}q_{2y})$, and

$$\begin{aligned}
\hat{n}^{[O^i,O^j]}(\boldsymbol{q}) &\equiv \sum_{k=1}^{N_\phi} F(\boldsymbol{q}) e^{\frac{i}{2}(2k-q_y)l_B^2 q_x}(\hat{c}_k^\dagger [O^i,O^j]\hat{c}_{k-q_y}) \\
\hat{n}^{\{O^i,O^j\}}(\boldsymbol{q}) &\equiv \sum_{k=1}^{N_\phi} F(\boldsymbol{q}) e^{\frac{i}{2}(2k-q_y)l_B^2 q_x}(\hat{c}_k^\dagger \{O^i,O^j\}\hat{c}_{k-q_y})
\end{aligned} \tag{21}$$

Since the density operators do not commute we can estimate the magnitude of the Trotter error as follows. Let $||\hat{A}|| \equiv \max_{|\Psi\rangle,|||\Psi\rangle||=1}||\hat{A}|\Psi\rangle||$. Since the Hamiltonian $\sum_m H_m$ is an extensive quantity, $||\sum_m H_m|| \propto N_\Phi$. Here m runs over a set of order N_Φ momenta, hence implies that typically, $H_m \propto N_\Phi^0$. Using this to estimate the systematic error, yields the result:

$$f_{QMC} = f + \mathcal{O}\left(\Delta\tau^2 N_\Phi^2\right). \tag{22}$$

Hence, to keep the Trotter error under control we have to scale $\Delta\tau$ as $1/N_\Phi$.

The Trotter error in our model has a different scaling behavior, than for models with only local interaction such as the Hubbard model. For local interactions $||\lambda||$ scales as N_Φ, such that the systematic error on the free energy density is size independent.

An improved estimator is introduced, based on the SO(5) invariant structure factor:

$$S(\boldsymbol{q}) = \frac{1}{N_\phi}\sum_{i=1}^{5}\langle \hat{n}^i_{\boldsymbol{q}}\hat{n}^i_{-\boldsymbol{q}}\rangle. \tag{23}$$

The magnetization and correlation ratio used for the scaling analysis in the main part of the paper is based on the above structure factor.

Fig. 1 shows a numerical comparison of the correlation ratio for multiple system sizes. In Fig. 1 (a) we consider a constant $\Delta\tau$ while in Fig. 1 (b) we scale $\Delta\tau$ with the volume: $\Delta\tau = 25.6\pi^2/N_\phi$. As mentioned previously, our Trotter decomposition breaks the SO(5) symmetry such that a convenient measure of the finite time step systematic error is the discrepancy between the Néel and VBS order parameters. At constant $\Delta\tau = 3.2\pi^2$ the correlation ratio defined from the Néel, VBS and SO(5)

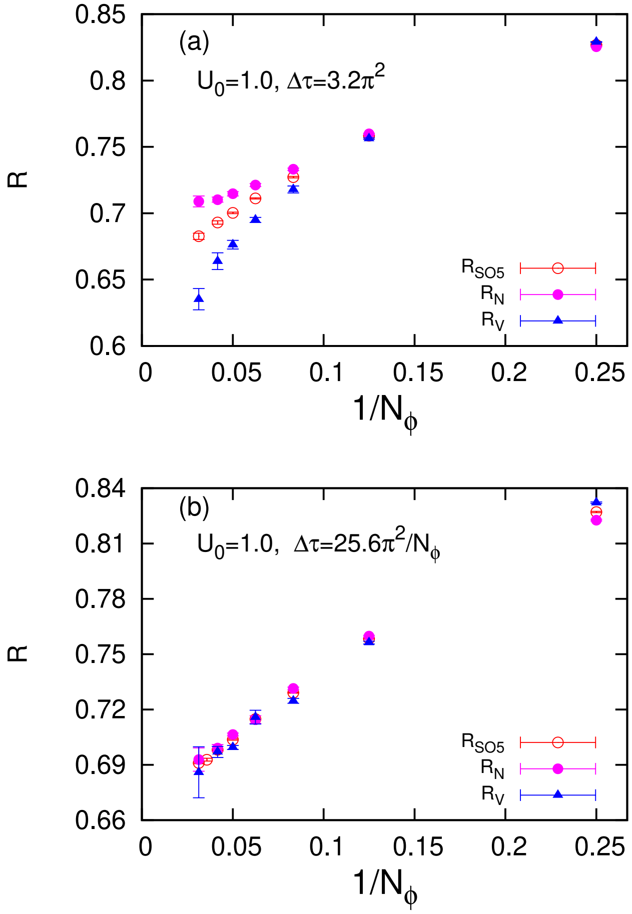

FIG. 1. Correlation ratio R of Neel, VBS order and the improved estimator for a fix $\Delta\tau$ as $3.2\pi^2$ (a), and for a linear scaling of $\Delta\tau = 25.6\pi^2/N_\phi$ (b). The simulation is based on $U_0 = 1.0$, of system sizes $N_\phi = 4, 8, 12, 16, ...32$, with $\beta = 1$.

order parameters progressively differ as a function of system size. On the other hand, for simulations where we keep $\Delta\tau N_\phi$ constant, see Fig. 1 (b), no $SO(5)$ symmetry breaking up to $N_\phi = 32$ is apparent. In all our simulations we have kept $\Delta\tau N_\phi$ constant.

CUTOFF

The effective interacting strength in Eq. (11) is controlled by a momentum dependent function $F(\boldsymbol{q})$ in Eq. (27):

$$F(\boldsymbol{q}) = e^{-\frac{1}{4}(q_x^2+q_y^2)l_B^2} \tag{24}$$

The exponential decay of the interacting strength gives a natural cutoff in the momentum space. In particular, we we can consider momenta satisfying $F(\boldsymbol{q}) > F_{min}$. As shown in Fig. 2, for $N_\phi = 4, 8$ and 12 at $U_0 = U = 1$, the cutoff dependence of the correlation ratio is negligible

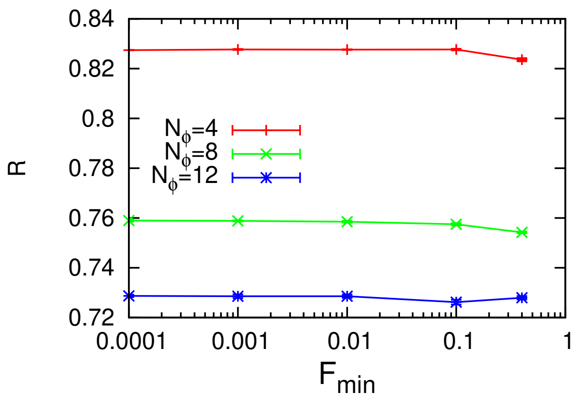

FIG. 2. Correlation ratio as a function of F_{min} for $N_\phi = 4, 8$ and 12, at $U_0 = U = 1, \beta = 160\pi^2, \Delta\tau = 3.2\pi^2$.

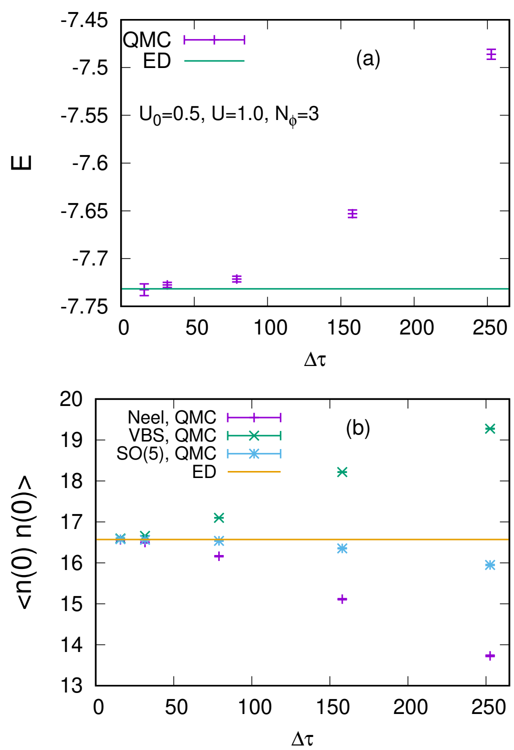

FIG. 3. Ground state energy (a) and magnetic order parameter (b) based on AFQMC as a function of $\Delta\tau$, as well as ED. The calculation is performed at $U_0 = 0.5, U = 1.0$ for $N_\phi = 3$.

up to $F_{min} = 0.01$. In our calculations, we have chosen $F_{min} = 0.01$. Setting $l_B = 1$ implies that the number of $\boldsymbol{q}$-vectors we consider for a given cutoff scales as N_ϕ.

COMPARISON TO EXACT DIAGONALISATION

A benchmark calculation of QMC with the exact diagonalization(ED) is performed, based on comparing exact ground state of a half filled system, to a finite temperature AFQMC simulation at low enough temperature ($\beta = 320\pi^2$). As an example we consider $U_0 = 0.5, U = 1.0$ at $N_\phi = 3$. In Fig. 3, we see that the two methods show consistent results for the ground state energy and the SO(5) invariant correlation function at zero momentum in the limit of small $\Delta\tau$. Both the Neel(VBS) correlation function based on the average value of density operators of only the $i = 1, 2, 3$ $(4, 5)$ term in Eq. (23) are equally shown in Fig. 3(b).

U_0	$\langle m \rangle_0$	η	χ^2/DOF
-1.0	0.03(1)	0.33(2)	1.51
-0.5	0.01(1)	0.28(2)	1.32
0.0	0.03(1)	0.29(2)	1.78
0.25	0.028(7)	0.27(1)	0.92
0.5	0.04(1)	0.28(2)	0.25
1.0	0.05(1)	0.28(2)	1.17
2.0	0.064(7)	0.26(2)	0.98
4.0	0.11(1)	0.26(2)	1.75
8.0	0.27(1)	0.38(2)	1.11

TABLE I. Collective fit using Eq. (25).

FITTING OF THE MAGNETIZATION: PROXIMITY TO FIX-POINT COLLISION

At a critical point, we expect the order parameter to scale as $m \propto L^{-\frac{\eta+z}{2}}$ where L is the linear length of the system. On the other hand, in the ordered phase $m = m_0 + b/L + c/L^2 + \cdots$ is an analytical function. The fitting form we heuristically suggest, reflects the notion of an ordered phase in the proximity of a critical point. On length scales smaller than the correlation length, we expect a power-law that gives way to an analytical form once we can resolve the correlation length. We have found that for vanishing b and c,

$$m = m_0 + aN_\phi^{-\frac{\eta+z}{4}}, \tag{25}$$

we obtain a good fit (see Table I) when all the system sizes are included. Note that $L \propto \sqrt{N_\Phi}$. The η exponent is robust as a function of U_0, except for the point of $U_0 = 8$. On the other hand, the extrapolated magnetization becomes nonzero when $U_0 \geq 0.25$.

U_0 DEPENDENCE OF $1/g$

In this section, we derive the dependence of the coupling strength parameter $1/g$ in the non-linear sigma model on U_0/U in our Hamiltonian. We perform a gradient expansion calculation to integrate out fermions, in order to get an effective bosonic action which contains only the order parameters.

First we write our Hamiltonian using a Fierz identity, such that the $SU(4)$ invariant interaction (U_0 term) is transformed to a sum of interacting terms over all traceless 4×4 matrices:

$$\begin{aligned}\hat{H} =&\frac{1}{2N_\phi}\sum_{\boldsymbol{q}}\left(U_0 \hat{n}^0(\boldsymbol{q})\hat{n}^0(-\boldsymbol{q}) - U\sum_{i=1}^{5}\hat{n}^i(\boldsymbol{q})\hat{n}^i(-\boldsymbol{q})\right)\\ =&\frac{1}{2N_\phi}\left[-\widetilde{U}\sum_{i=1}^{5}\sum_{\boldsymbol{q}}\hat{n}^i(\boldsymbol{q})\hat{n}^i(-\boldsymbol{q})\right.\\ &\left.-\widetilde{V}\sum_{i<j}\sum_{\boldsymbol{q}}\hat{S}^{ij}(\boldsymbol{q})\hat{S}^{ij}(-\boldsymbol{q})\right]\end{aligned} \tag{26}$$

where

$$\begin{aligned}\hat{n}^i(\boldsymbol{q}) =&\sum_{k=1}^{N_\phi}\sum_{a,b=1}^{4} e^{-\frac{\boldsymbol{q}^2 l_B^2}{4}} e^{\frac{i}{2}(2k-q_y)l_B^2 q_x}(\hat{c}^\dagger_{a,k}O^i_{a,b}\hat{c}_{b,k-q_y}\\ &-2\delta_{q_y,0}\delta_{i,0}),\\ \hat{S}^{ij}(\boldsymbol{q}) =&\sum_{k=1}^{N_\phi}\sum_{a,b=1}^{4} e^{-\frac{\boldsymbol{q}^2 l_B^2}{4}} e^{\frac{i}{2}(2k-q_y)l_B^2 q_x}(\hat{c}^\dagger_{a,k}\Gamma^{ij}_{a,b}\hat{c}_{b,k-q_y}),\end{aligned} \tag{27}$$

and the 10 generators Γ^{ij} of the SO(5) group are defined for $i<j$:

$$\Gamma^{ij} \equiv -\frac{i}{2}[O^i, O^j]. \tag{28}$$

The identity in Eq. 26 follows from the Fierz identity, with

$$\widetilde{U} = U + \frac{1}{5}U_0 \qquad \widetilde{V} = \frac{1}{5}U_0. \tag{29}$$

From now on we adopt Einstein summation notation for the 4 fermion flavor index, and denote by N_ϕ the number of flux quanta piercing the lattice.

Using Hubbard-Stratonovich transformations, the partition function can be recast as:

$$\begin{aligned}Z =& \operatorname{Tr} e^{-\beta H}\\ =&\int\prod_{\boldsymbol{q},\tau}\prod_{l}\prod_{i<j} d\pi'_l(\boldsymbol{q},\tau)d\Pi_{ij}(\boldsymbol{q},\tau)\mathrm{Tr_F}\mathcal{T}\\ &\exp\left\{-\int d\tau\sum_{\boldsymbol{q}}\left[\sum_{i=1}^{5}\frac{\widetilde{U}N_\phi}{2}\pi'_i(\boldsymbol{q},\tau)\pi'_i(-\boldsymbol{q},\tau)\right.\right.\\ &+\sum_{i<j}\frac{\widetilde{V}N_\phi}{2}\Pi_{ij}(\boldsymbol{q},\tau)\Pi_{ij}(-\boldsymbol{q},\tau)\\ &+\sum_{i=1}^{5}\widetilde{U}\pi'_i(-\boldsymbol{q},\tau)\hat{n}_i(\boldsymbol{q},\tau)\\ &\left.\left.+\sum_{i<j}\widetilde{V}\Pi_{ij}(-\boldsymbol{q},\tau)\hat{S}^{ij}(\boldsymbol{q},\tau)\right]\right\}.\end{aligned} \tag{30}$$

The constraint of ${\pi'}^\dagger_i(\boldsymbol{q},\tau) = \pi'_i(-\boldsymbol{q},\tau)$ and $\Pi^\dagger_{ij}(\boldsymbol{q},\tau) = \Pi_{ij}(-\boldsymbol{q},\tau)$ holds and $\mathcal{T}$ denotes time ordering. Next we introduce Grassmann variables as eigenvalues of the canonical fermion annihilation operator to obtain:

$$\begin{aligned}Z = \int\prod_{\boldsymbol{q},\tau}\prod_{l}\prod_{i<j} d\pi'_l(\boldsymbol{q},\tau)d\Pi_{ij}(\boldsymbol{q},\tau)\prod_{k,\tau}\\ \int dc^\dagger_k(\tau)dc_k(\tau)e^{-S(\pi',\Pi,c^\dagger,c)}\end{aligned} \tag{31}$$

where

$$\begin{aligned}S =& \int d\tau\sum_{\boldsymbol{q}}\sum_{k}\left[c^\dagger_k(\tau)\frac{\partial}{\partial\tau}c_k(\tau)\right.\\ &-\sum_{i=1}^{5}\left(\widetilde{U}\pi'_i(-\boldsymbol{q},\tau)e^{-\frac{1}{4}\boldsymbol{q}^2 l_B^2}e^{ikl_B^2 q_x}c^\dagger_{k-q_y/2}(\tau)O_i c_{k+q_y/2}(\tau)\right.\\ &\left.+\frac{\widetilde{U}N_\phi}{2}\pi'_i(\boldsymbol{q},\tau)\pi'_i(-\boldsymbol{q},\tau)\right)\\ &-\sum_{i<j}\left(\widetilde{V}\Pi_{ij}(-\boldsymbol{q},\tau)e^{-\frac{1}{4}\boldsymbol{q}^2 l_B^2}e^{ikl_B^2 q_x}c^\dagger_{k-q_y/2}(\tau)\Gamma_{ij}c_{k+q_y/2}(\tau)\right.\\ &\left.\left.+\frac{\widetilde{V}N_\phi}{2}\Pi_{ij}(\boldsymbol{q},\tau)\Pi_{ij}(-\boldsymbol{q},\tau)\right)\right]\end{aligned} \tag{32}$$

To pursue, we note the following:

- We define $\Phi_0 \equiv \frac{V}{N_\phi} = \frac{\phi_0}{B} = 2\pi l_B^2$ where ϕ_0 is the flux quanta, and set the magnetic length $l_B = 1$.

- The saddle point approximation of Eq. 32 (based on the half-filled 4 flavor system) gives $\pi'_i(\mathbf{0},\tau) = 2$ and a fermionic single particle gap of $\Delta_{sp} = 2\widetilde{U}$. Thus we will rescale the scalar field as

$$\pi_i = \frac{1}{2}\pi'_i. \tag{33}$$

  With this rescaling, the modulus of the SO(5) vector is pinned to unity: $\sum_i \pi_i^2 = 1$. We will omit the amplitude fluctuations of this field such that the quadratic term of the π fields are constant and can be neglected.

- We pin one of the five components of the π-vector (say π_5) to a constant in space and time:

$$\begin{aligned}(\pi_1,\pi_2,\pi_3,&\pi_4,\pi_5) \equiv (\widetilde{\pi}_1\sin\nu, \widetilde{\pi}_2\sin\nu,\\ &\widetilde{\pi}_3\sin\nu, \widetilde{\pi}_4\sin\nu, \cos\nu) \quad \nu\to 0\end{aligned} \tag{34}$$

  with the constraint $\widetilde{\pi}_1^2+\widetilde{\pi}_2^2+\widetilde{\pi}_3^2+\widetilde{\pi}_4^2 = 1$. Then, we will only consider 4 out of 10 angular momentum fields in the action: Π_{i5} ($i = 1,2,3,4$) because from what we will show below These angular momentum fields rotate the π-vector around the 5^{th} axis and hence provide contributions of order ν to the bosonic action: $\Pi_{i5}\pi_i$. On the other hand, the contributions from other 6 angular momentum fields ($\Pi_{ij}\pi_i\pi_j$ for $i<j<5$) leads to terms in the action that are of order ν^2.

- From Eq. 32 onwards we omit the repeated flavor indices by using the notation, e.g.

$$c^\dagger_{k-q_y/2}\Gamma^{ij}c_{k+q_y/2} \equiv \sum_{a=1}^{4}\sum_{b=1}^{4} c^\dagger_{a,k-q_y/2}\Gamma^{ij}_{a,b}c_{b,k+q_y/2}. \quad (35)$$

With the above, Eq. 32 simplifies to:

$$\begin{aligned} S = \int d\tau \sum_{\boldsymbol{q}}\sum_{k} \Big[& c^\dagger_k(\tau)\frac{\partial}{\partial\tau}c_k(\tau) - \lambda c^\dagger_k(\tau)O^5 c_k(\tau) \\ & -\lambda\sum_{i=1}^{4}\pi_i(-\boldsymbol{q},\tau)e^{-\frac{1}{4}\boldsymbol{q}^2 l_B^2}e^{il_B^2 kq_x}c^\dagger_{k-q_y/2}(\tau)O^i c_{k+q_y/2}(\tau) \\ & +\frac{\widetilde{V}N_\phi}{2}\sum_{i=1}^{4}\Pi_{i5}(\boldsymbol{q},\tau)\Pi_{i5}(-\boldsymbol{q},\tau) \\ & -\lambda'\sum_{i=1}^{4}\Pi_{i5}(-\boldsymbol{q},\tau)e^{-\frac{1}{4}\boldsymbol{q}^2 l_B^2}e^{il_B^2 kq_x}c^\dagger_{k-q_y/2}(\tau)\Gamma^{i5}c_{k+q_y/2}(\tau)\Big] \end{aligned} \quad (36)$$

where the coupling constants λ and λ' read:

$$\begin{aligned} \lambda &= 2\widetilde{U} \\ \lambda' &= \widetilde{V}. \end{aligned} \quad (37)$$

To proceed we perform a long wave-length and low frequency expansion. We use the Fourier transform conventions:

$$\begin{aligned} c_l(\tau) &= \frac{1}{\beta}\sum_{\omega} e^{-i\omega\tau}c_l(\omega), \\ c_l(\omega) &= \int_0^\beta d\tau e^{i\omega\tau}c_l(\tau), \qquad \omega = \frac{\pi(2n+1)}{\beta} \\ \pi_i(\boldsymbol{x},\tau) &= \frac{1}{\beta}\sum_{\boldsymbol{q}}\sum_{\omega} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}e^{-i\omega\tau}\pi_i(\boldsymbol{q},\omega), \\ \pi_i(\boldsymbol{q},\omega) &= \frac{1}{V}\int d^2\boldsymbol{x}\int_0^\beta e^{i\boldsymbol{q}\cdot\boldsymbol{x}}d\tau e^{i\omega\tau}\pi_i(\boldsymbol{x},\tau), \qquad \omega = \frac{2\pi n}{\beta} \end{aligned} \quad (38)$$

In momentum and frequency space, the action reads:

$$S = S_\Pi + S_c \quad (39)$$

where

$$S_\Pi = \frac{1}{\beta}\frac{\widetilde{V}N_\phi}{2}\sum_{\omega}\sum_{\boldsymbol{q}}\sum_{i=1}^{4}\frac{\Pi_{i5}(\boldsymbol{q},\omega)\Pi_{i5}(-\boldsymbol{q},\omega)}{2} \quad (40)$$

and

$$\begin{aligned} S_c = & \frac{1}{\beta}\sum_{\omega_1}\sum_{\omega_2}\sum_{k_1}\sum_{k_2} c^\dagger_{k_1}(\omega_1)\Big[(i\omega_1 - \lambda O^5)\delta_{k_1,k_2}\delta_{\omega_1,\omega_2} \\ & -\frac{\lambda}{\beta}\Big(\sum_{i=1,4}\sum_{\boldsymbol{q}}\pi_i(-\boldsymbol{q},\omega_1-\omega_2) \\ & \quad e^{-\frac{1}{4}\boldsymbol{q}^2 l_B^2}e^{il_B^2(k_1+q_y/2)\cdot q_x}O^i\delta_{k_1,k_2-q_y}\Big) \\ & -\frac{\lambda'}{\beta}\Big(\sum_{i=1,4}\sum_{\boldsymbol{q}}\Pi_{i5}(-\boldsymbol{q},\omega_1-\omega_2) \\ & \quad e^{-\frac{1}{4}\boldsymbol{q}^2 l_B^2}e^{il_B^2(k_1+q_y/2)\cdot q_x}\Gamma^{i5}\delta_{k_1,k_2-q_y}\Big)\Big]c_{k_2}(\omega_2) \end{aligned} \quad (41)$$

We can now integrate out the fermion degree's of freedom to obtain an action of order parameter and angular momentum fields. Introducing the super-index $\underline{k} \equiv (k,\omega)$ we obtain:

$$\begin{aligned} Z &= \int \prod_{\boldsymbol{q},\tau}\prod_{l}\prod_{i<j} d\pi'_l(\boldsymbol{q},\tau)d\Pi_{ij}(\boldsymbol{q},\tau)\int\prod_{\underline{k}}(Dc^\dagger_{\underline{k}}, Dc_{\underline{k}})e^{-S_c}e^{-S_\Pi} \\ &= \int \prod_{\boldsymbol{q},\tau}\prod_{l}\prod_{i<j} d\pi'_l(\boldsymbol{q},\tau)d\Pi_{ij}(\boldsymbol{q},\tau)\int\prod_{\underline{k}}(Dc^\dagger_{\underline{k}}, Dc_{\underline{k}}) \\ & \quad e^{-\sum_{\underline{k}_1,\underline{k}_2}c^\dagger_{\underline{k}_1}(h_0+h_1+h_2)_{\underline{k}_1,\underline{k}_2}c_{\underline{k}_2}}e^{-S_\Pi} \\ &\equiv \int \prod_{\boldsymbol{q},\tau}\prod_{l}\prod_{i<j} d\pi'_l(\boldsymbol{q},\tau)d\Pi_{ij}(\boldsymbol{q},\tau)e^{-S_{\pi,\Pi}-S_\Pi} \end{aligned} \quad (42)$$

where the matrices are given by:

$$\begin{aligned} [h_0]_{\underline{k}_1,\underline{k}_2} =& \frac{1}{\beta}(i\omega_1 - \lambda O^5)\delta_{\underline{k}_1,\underline{k}_2} \\ [h_1]_{\underline{k}_1,\underline{k}_2} =& -\frac{1}{\beta^2}\lambda\sum_{i=1}^{4}O^i\sum_{\boldsymbol{q}}\pi_i(-\boldsymbol{q},\omega_1-\omega_2) \\ & e^{-\frac{1}{4}\boldsymbol{q}^2 l_B^2}e^{-il_B^2 q_x(k_1+\frac{q_y}{2})}\delta_{k_1,k_2-q_y} \\ [h_2]_{\underline{k}_1,\underline{k}_2} =& -\frac{1}{\beta^2}\lambda'\sum_{i=1}^{4}\Gamma^{i5}\sum_{\boldsymbol{q}}\Pi_{i5}(-\boldsymbol{q},\omega_1-\omega_2) \\ & e^{-\frac{1}{4}\boldsymbol{q}^2 l_B^2}e^{-il_B^2 q_x(k_1+\frac{q_y}{2})}\delta_{k_1,k_2-q_y} \end{aligned} \quad (43)$$

Note that the Kronecker delta function reads $\delta_{\underline{k}_1,\underline{k}_2} \equiv \delta_{k_1,k_2}\delta_{\omega_1,\omega_2}$. Integration over the fermionic degrees of freedom gives:

$$\begin{aligned} S_{\pi,\Pi} =& -\ln\det[(h_0+h_1+h_2)] \\ =& -\mathrm{Tr}\ln[h_0+h_1+h_2] \\ =& -\mathrm{Tr}\{\ln[1+h_0^{-1}h_1+h_0^{-1}h_2]+\ln h_0\} \end{aligned} \quad (44)$$

with

$$h_0^{-1} = \beta\frac{i\omega+\lambda O^5}{(i\omega)^2-\lambda^2}\delta_{\underline{k}_1,\underline{k}_2} \quad (45)$$

Omitting the constant background term ($\ln h_0$), and expanding the above equation up to second order yields:

$$\begin{aligned} S_{eff}(\pi) = & -\operatorname{Tr}[h_0^{-1}h_1 + h_0^{-1}h_2 - \frac{1}{2}h_0^{-1}h_1h_0^{-1}h_1 \\ & -\frac{1}{2}h_0^{-1}h_2h_0^{-1}h_2 - \frac{1}{2}h_0^{-1}h_1h_0^{-1}h_2 - \frac{1}{2}h_0^{-1}h_2h_0^{-1}h_1] \\ & +... \end{aligned} \tag{46}$$

The first order terms vanish since for $i = 1, 2, 3, 4$:

$$\begin{aligned} \operatorname{Tr}^4\{O^i\} = 0 \qquad \operatorname{Tr}^4\{O^5O^i\} = 0 \\ \operatorname{Tr}^4\{\Gamma^{i5}\} = 0 \qquad \operatorname{Tr}^4\{O^5\Gamma^{i5}\} = 0 \end{aligned} \tag{47}$$

Note the difference between Tr and Tr^4. The later denotes the trace over the 4×4 matrices.

In second order we have to evaluate a number of terms. The evaluation of the first term reads:

$$\begin{aligned} & \frac{1}{2}\operatorname{Tr}[h_0^{-1}h_1h_0^{-1}h_1] \\ = & \frac{1}{2}\frac{1}{\beta^2}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{\omega_1,\omega_2,\omega_3,\omega_4}\sum_{k_1,k_2,k_3,k_4}\frac{\lambda^2}{(-\omega_1^2-\lambda^2)(-\omega_3^2-\lambda^2)} \\ & \cdot \operatorname{Tr}^4[(i\omega_1 - \lambda O^5)\delta_{k_1,k_2}\delta(\omega_1-\omega_2)\cdot O^i \\ & \sum_{\boldsymbol{q}}\pi_i(-\boldsymbol{q},\omega_2-\omega_3)e^{-\frac{1}{4}\boldsymbol{q}^2l_B^2}e^{-il_B^2q_x(k_2+\frac{q_y}{2})}\delta_{k_2,k_3-q_y} \\ & (i\omega_3-\lambda O^5)\delta_{k_3,k_4}\delta(\omega_3-\omega_4)\cdot O^j \\ & \sum_{\boldsymbol{p}}\pi_j(-\boldsymbol{p},\omega_4-\omega_1)e^{-\frac{1}{4}\boldsymbol{p}^2l_B^2}e^{-il_B^2p_x(k_4+\frac{p_y}{2})}\delta_{k_4,k_1-p_y}] \\ = & \frac{1}{2}\frac{1}{\beta^2}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{\omega_1}\sum_{\omega_3}\frac{\lambda^2}{(-\omega_1^2-\lambda^2)(-\omega_3^2-\lambda^2)}\cdot \\ & \operatorname{Tr}^4[(i\omega_1-\lambda O^5)(i\omega_3+\lambda O^5)O^iO^j] \\ & \sum_{\boldsymbol{q}}\pi_i(-\boldsymbol{q},\omega_1-\omega_3)e^{-\frac{1}{4}\boldsymbol{q}^2l_B^2} \\ & \sum_{\boldsymbol{p}}\pi_j(-\boldsymbol{p},\omega_3-\omega_1)e^{-\frac{1}{4}\boldsymbol{p}^2l_B^2}\delta(p_x+q_x)\delta(p_y+q_y)N_\phi \\ \approx & \frac{1}{2}\frac{N_\phi}{\beta^2}\sum_{i=1}^{4}\sum_{\omega}\sum_{\Omega}\sum_{\boldsymbol{q}}\pi_i(-\boldsymbol{q},-\Omega)\pi_i(\boldsymbol{q},\Omega)\cdot \\ & [l_B^2(q_x^2+q_y^2)\frac{2\lambda^2}{\omega^2+\lambda^2}+\Omega^2\frac{\lambda^2(-\omega^2+3\lambda^2)}{(\lambda^2+\omega^2)^3}] \\ = & \frac{N_\phi}{\beta}\sum_{i=1}^{4}\sum_{\Omega}\sum_{\boldsymbol{q}}\pi_i(-\boldsymbol{q},-\Omega)\pi_i(\boldsymbol{q},\Omega)\cdot(\rho_s|\boldsymbol{q}|^2+\chi_\perp\Omega^2) \\ = & \frac{1}{\Phi_0}\sum_{i=1}^{4}\int_0^\beta d\tau\int d^2\boldsymbol{x}\{\rho_s[(\frac{\partial}{\partial x^1}\pi_i(\boldsymbol{x},\tau))^2+(\frac{\partial}{\partial x^2}\pi_i(\boldsymbol{x},\tau))^2] \\ & +\chi_\perp(\frac{\partial}{\partial\tau}\pi_i(\boldsymbol{x},\tau))^2 \end{aligned} \tag{48}$$

For simplicity we defined $\omega \equiv (\omega_3+\omega_1)/2$ and $\Omega \equiv \omega_3-\omega_1$ here. The cross terms vanish since $\operatorname{Tr}^4\{O^5O^iO^j\} = 0$ for $i \neq j$. In the third step, we performed the expansion around the long wave length limit, $\Omega \to 0, |\boldsymbol{q}| \to 0$. Again in the third step, we omit the quadratic terms ($\pi^i\pi^i$) due to aforementioned reasons.

In the above,

$$\begin{aligned} \rho_s &= \frac{1}{\beta}l_B^2\sum_{\omega}\frac{\lambda^2}{\omega^2+\lambda^2} \\ &= \frac{1}{2\pi}l_B^2\int_{-\infty}^{\infty}d\omega\frac{\lambda^2}{\omega^2+\lambda^2} = \frac{\lambda}{2}l_B^2 \\ \chi_\perp &= \frac{1}{\beta}\sum_{\omega}\frac{\lambda^2(-\omega^2+3\lambda^2)}{2(\lambda^2+\omega^2)^3} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\frac{\lambda^2(-\omega^2+3\lambda^2)}{2(\lambda^2+\omega^2)^3} = \frac{1}{4\lambda} \end{aligned} \tag{49}$$

The second term gives:

$$\begin{aligned} & \frac{1}{2}\operatorname{Tr}[h_0^{-1}h_2h_0^{-1}h_2] \\ \approx & -\frac{\lambda'^2}{\lambda}\frac{N_\phi}{\beta}\sum_{i=1}^{4}\sum_{\Omega}\sum_{\boldsymbol{q}}\Pi_{i5}(-\boldsymbol{q},-\Omega)\Pi_{i5}(\boldsymbol{q},\Omega) \end{aligned} \tag{50}$$

This term is crucial since it re-scales the prefactor of the quadratic term in Eq. 40. We focus on terms that only contribute to the interaction in the long wave length limit ($|\boldsymbol{q}| \longrightarrow 0$). These contributions originate from the amplitude fluctuations of Π fields such that in this step, we have omitted the higher order terms of angular momentum, $\Pi_{i5}(-\boldsymbol{q},-\Omega)\Pi_{i5}(\boldsymbol{q},\Omega)|\boldsymbol{q}|^2$ and $\Pi_{i5}(-\boldsymbol{q},-\Omega)\Pi_{i5}(\boldsymbol{q},\Omega)\omega^2$.

The third and fourth terms evaluate to:

$$\begin{aligned} & \frac{1}{2}\operatorname{Tr}[h_0^{-1}h_1h_0^{-1}h_2] + \frac{1}{2}\operatorname{Tr}[h_0^{-1}h_1h_0^{-1}h_2] \\ = & \frac{1}{\beta^2}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{\omega_1,\omega_2,\omega_3,\omega_4}\sum_{k_1,k_2,k_3,k_4}\frac{\lambda\lambda'}{(-\omega_1^2-\lambda^2)(-\omega_3^2-\lambda^2)} \\ & \cdot\operatorname{Tr}^4[(i\omega_1-\lambda O^5)\delta_{k_1,k_2}\delta(\omega_1-\omega_2)\cdot O^i \\ & \sum_{\boldsymbol{q}}\pi_i(-\boldsymbol{q},\omega_2-\omega_3)e^{-\frac{1}{4}\boldsymbol{q}^2l_B^2}e^{-il_B^2q_x(k_2+\frac{q_y}{2})}\delta_{k_2,k_3-q_y} \\ & (i\omega_3-\lambda O^5)\delta_{k_3,k_4}\delta(\omega_3-\omega_4)\cdot\Gamma^{j5} \\ & \sum_{\boldsymbol{p}}\Pi_{j5}(-\boldsymbol{p},\omega_4-\omega_1)e^{-\frac{1}{4}\boldsymbol{p}^2l_B^2}e^{-il_B^2p_x(k_4+\frac{p_y}{2})}\delta_{k_4,k_1-p_y}] \end{aligned} \tag{51}$$

$$=\frac{N_\phi}{\beta^2}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{\omega_1,\omega_3}\frac{\lambda\lambda'}{(-\omega_1^2-\lambda^2)(-\omega_3^2-\lambda^2)}$$
$$\cdot\,\mathrm{Tr}^4[(i\omega_1-\lambda O^5)(i\omega_3+\lambda O^5)O^i\Gamma^{j5}]$$
$$\sum_{\boldsymbol{q}}\pi_i(-\boldsymbol{q},\omega_1-\omega_3)e^{-\frac{1}{4}\boldsymbol{q}^2l_B^2}$$
$$\sum_{\boldsymbol{p}}\Pi_{j5}(-\boldsymbol{p},\omega_3-\omega_1)e^{-\frac{1}{4}\boldsymbol{p}^2l_B^2}\delta(p_x+q_x)\delta(p_y+q_y)$$
$$\approx\frac{N_\phi}{\beta}\sum_{i=1}^{4}\sum_{\omega}\sum_{\Omega}\sum_{\boldsymbol{q}}\pi_i(-\boldsymbol{q},-\Omega)\Pi_{i5}(\boldsymbol{q},\Omega)\cdot[i\Omega\frac{4\lambda^2\lambda'}{(\lambda^2+\omega^2)^2}]$$
$$=\frac{N_\phi}{\beta}\sum_{i=1}^{4}\sum_{\Omega}\sum_{\boldsymbol{q}}[\gamma\pi_i(-\boldsymbol{q},-\Omega)\Pi_{i5}(\boldsymbol{q},\Omega)\Omega]$$
$$=\frac{1}{\Phi_0}\sum_{i=1}^{4}\int_0^\beta d\tau\int d^2\boldsymbol{x}[i\gamma(\Pi_{i5}(\boldsymbol{x},\tau)\frac{\partial}{\partial\tau}\pi_i(\boldsymbol{x},\tau))].$$

Here we used the identity, $\mathrm{Tr}^4\{O^5O^i\Gamma^{j5}\}=4i\delta_{ij}$ and

$$\gamma\equiv\frac{1}{\beta}\sum_{\omega}\frac{4\lambda^2\lambda'}{(\lambda^2+\omega^2)^2}=\frac{1}{2\pi}\int d\omega\frac{4\lambda^2\lambda'}{(\lambda^2+\omega^2)^2}=\frac{\lambda'}{\lambda} \tag{52}$$

Adding up all the contributions, we obtain:

$$S_{\pi,\Pi}+S_{\Pi}$$
$$=\frac{1}{\Phi_0}\sum_{i=1}^{4}\int_0^\beta d\tau\int d^2\boldsymbol{x}\{\rho_s[(\frac{\partial}{\partial x^1}\pi_i(\boldsymbol{x},\tau))^2+(\frac{\partial}{\partial x^2}\pi_i(\boldsymbol{x},\tau))^2]$$
$$+\chi_\perp(\frac{\partial}{\partial\tau}\pi_i(\boldsymbol{x},\tau))^2$$
$$+i\gamma[\Pi_{i5}(\boldsymbol{x},\tau)\frac{\partial}{\partial\tau}\pi_i(\boldsymbol{x},\tau)]+s\Pi_{i5}(\boldsymbol{x},\tau)\Pi_{i5}(\boldsymbol{x},\tau)\}. \tag{53}$$

In the above, s stems from both the S_Π term in Eq. 40, and the quadratic term of the fermion integration given in Eq. 50. It reads:

$$s\equiv\frac{\widetilde{V}}{2}+\frac{\lambda'^2}{\lambda}. \tag{54}$$

We can now integrate out the Π fields:

$$e^{-S_{eff}}=\int D\Pi e^{-S_{\pi,\Pi}-S_\Pi}$$
$$=\exp\Big\{-\frac{1}{\Phi_0}\sum_{i=1}^{4}\int_0^\beta d\tau\int d^2\boldsymbol{x}\{\rho_s[(\frac{\partial}{\partial x^1}\pi_i(\boldsymbol{x},\tau))^2$$
$$+(\frac{\partial}{\partial x^2}\pi_i(\boldsymbol{x},\tau))^2]+(\chi_\perp+\frac{\gamma^2}{4s})(\frac{\partial}{\partial\tau}\pi_i(\boldsymbol{x},\tau))^2\Big\}, \tag{55}$$

to obtain out final result: an effective action that only depends on the order parameters:

$$S_{eff}=\frac{1}{\Phi_0}\sum_{i=1}^{4}\int_0^\beta d\tau\int d^dx(\rho_s\left(\nabla\pi_i\right)^2+\chi_0(\partial_\tau\pi_i)^2). \tag{56}$$

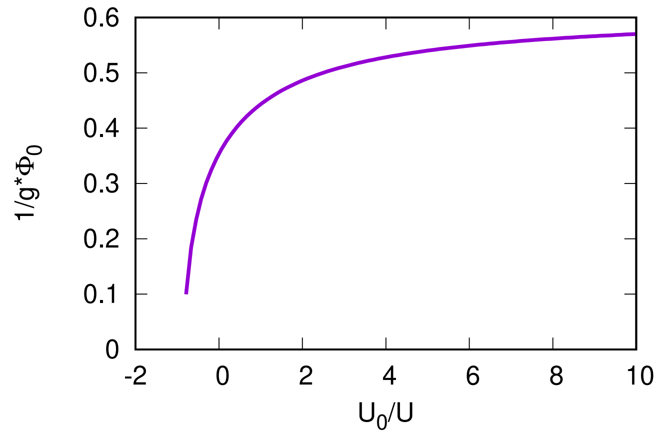

FIG. 4. Coupling strength times unit flux quanta as a function of U_0/U. Our QMC simulation covers the range of $U_0/U=[-1:8]$.

Here

$$\rho_s=\frac{\lambda}{2}l_B^2$$
$$\chi_0=\chi_\perp+\frac{\gamma^2}{4s}=\frac{1}{4\lambda}+\frac{(\lambda'/\lambda)^2}{\widetilde{V}/2+\lambda'^2/\lambda} \tag{57}$$

In the SO(5) symmetrized form the action denotes:

$$S_{eff}=\frac{1}{\Phi_0}\int_0^\beta d\tau\int d^dx(\rho_s(\nabla\vec{\pi})^2+\chi_0(\partial_\tau\vec{\pi})^2) \tag{58}$$

After rescaling the quantum temperature, the coupling strength, $1/g$, reads:

$$\frac{1}{g}=\frac{1}{\Phi_0}\sqrt{\rho_s\chi_0}$$
$$=\frac{l_B}{\Phi_0}\sqrt{\frac{\lambda}{2}(\frac{1}{4\lambda}+\frac{(\lambda'/\lambda)^2}{\widetilde{V}/2+\lambda'^2/\lambda})}$$
$$=\frac{l_B}{\Phi_0}\sqrt{\frac{1}{8}+\frac{\frac{\widetilde{V}}{2\widetilde{U}}}{1+\frac{\widetilde{V}}{\widetilde{U}}}} \tag{59}$$
$$=\frac{l_B}{\Phi_0}\sqrt{\frac{1}{8}+\frac{U_0/U}{10+4U_0/U}}$$
$$=\frac{1}{2\pi l_B}\sqrt{\frac{1}{8}+\frac{U_0/U}{10+4U_0/U}}$$

Thus we obtain $1/g=\sqrt{1/8}$ for $U_0/U=0$ and $1/g=\sqrt{3/8}$ for $U_0/U\longrightarrow\infty$. Interestingly, the above derivation should be valid also in the region $U_0<0$, in the sense that λ' becomes purely imaginary. However, Eq. 59 is only well defined for $U_0/U>-\frac{5}{6}$. We plot the coupling strength as a function of U_0/U in Fig. 4.

We note that the sigma model description will break down in the limit of $U_0/U \longrightarrow -\infty$ since in this limit phase separation sets in. We conjecture that if we push the above calculation to higher precision, the 'critical' value of U_0/U would shift to the phase separation transition point situated at around -3. We now turn our attention to the limit $U_0/U \to \infty$. Let $U_0 = 1$ set the energy scale. At $U = 0$, the Hamiltonian has SU(4) symmetry and using the Fierz identity of Eq. 26 one can conjecture that the ground state is insulating, spontaneously breaks SU(4) symmetry, and has a finite stiffness. Starting from this point a small SO(5) symmetric term will gap out some of the SU(4) Goldstone modes leaving a total of four modes akin to spontaneous breaking of the SO(5) symmetry. We expect this state at $U/U_0 << 1$ or $U_0/U >> 1$ to have a finite stiffness.

Finally, we note that one will obtain the Wess-Zumino-Witten term when expanding the fermion determinant to fourth order : $\frac{1}{4}h_0^{-1}h_1h_0^{-1}h_1h_0^{-1}h_1h_0^{-1}h_1$. We refer the reader to the work of Lee et al.[?] for a detailed overview of this calculation as well as to the next section.

VALIDATION OF THE TOPOLOGICAL TERM IN GLOBAL SENSE

An ambiguity exists in the derivation of Wess-Zumino-Witten (WZW) geometrical term in the work of Lee and Sachdev.[?] In the gradient expansion a polarized direction in the O(n) symmetric order parameter space has to be chosen such that a symmetry broken 'background' term exists with finite fermion one particle gap. The contribution to the bosonic action from the O(n-1) order parameter fluctuations includes a geometrical term which is locally defined. The question is how to restore the 'global' WZW term from this local definition. In this section, we offer our understanding of this issue.

Consider the continuous field configuration

$$\hat{\varphi}_{local}(\boldsymbol{x}) = \left(\pi_1(\boldsymbol{x}), \pi_2(\boldsymbol{x}), \pi_3(\boldsymbol{x}), \pi_4(\boldsymbol{x}), \sqrt{1-\delta^2}\right), \quad \text{with } \delta = 0^+, |\boldsymbol{\pi}|^2 = \delta^2 \tag{60}$$

and $\boldsymbol{x} = (x, y, \tau)$ a space-time coordinate defining the base space, $\mathbb{R}^3$. We will include the point at infinity in Euclidean space, so that the base space is topologically a three-sphere, $S3$. For convenience we parameterize $\mathbb{R}^3$ with spherical coordinates: $U \equiv \{\boldsymbol{x}\}$ for $\tau \in [0, \infty)$, $x \in [0, \pi]$, $y \in [0, 2\pi)$. The aforementioned S^3 topology of $\mathbb{R}^3$ requires for a smooth field:

$$\begin{aligned}
&\widetilde{\varphi}(u, \tau = 0, x, y) \equiv \widetilde{\varphi}(u, \tau = 0, x', y') \quad \forall\, x, x', y, y' \\
&\widetilde{\varphi}(u, \tau = \infty, x, y) \equiv \widetilde{\varphi}(u, \tau = \infty, x', y') \quad \forall\, x, x', y, y' \\
&\widetilde{\varphi}(u, \tau, x, y) \equiv \widetilde{\varphi}(u, \tau, x, y + 2\pi).
\end{aligned} \tag{61}$$

With this compactification of the base space, $\hat{\varphi}_{local}(\boldsymbol{x})$ describes a local closed hyper-path around the north pole of the four-sphere. For this hyper-path, the authors of Ref.[?] derive very clearly, by integrating out fermions in the ZLL basis, the Wess-Zumino-Witten term to leading order in δ:

$$S(\varphi_{\text{local}}) = \frac{3i}{16\pi} \iiint_U d^3\boldsymbol{x} \epsilon^{\alpha\beta\gamma\delta} \pi^\alpha \partial_x \pi^\beta \partial_y \pi^\gamma \partial_\tau \pi^\delta. \tag{62}$$

We again use the notation $\iiint_U$ to specify the integration over $\mathbb{R}^3$. To obtain a geometrical understanding of this term, we can rewrite it in the following way. Consider

$$\tilde{\varphi}_{\text{local}}(\boldsymbol{x}, u) = \left(u\boldsymbol{\pi}(\boldsymbol{x}), u(\sqrt{1-\delta^2} - 1) + 1\right) \tag{63}$$

such that $\tilde{\varphi}_{\text{local}}(\boldsymbol{x}, u = 0)$ corresponds to the north pole of S_4 and $\tilde{\varphi}_{\text{local}}(\boldsymbol{x}, u = 1)$ to the physical local field configuration $\hat{\varphi}_{local}(\boldsymbol{x})$. Then, it can be shown that since $\int_0^1 duu^3 = \frac{1}{4}$,

$$\begin{aligned}
S(\varphi_{\text{local}}) =& \frac{2\pi i}{vol(S_4)} \iiint_U d^3\boldsymbol{x} \int_0^1 du \\
& \epsilon^{abcde} \tilde{\varphi}^a_{\text{local}} \partial_x \tilde{\varphi}^b_{\text{local}} \partial_y \tilde{\varphi}^c_{\text{local}} \partial_\tau \tilde{\varphi}^d_{\text{local}} \partial_u \tilde{\varphi}^e_{\text{local}} \\
=& \frac{3i}{4\pi} \iiint_U d^3\boldsymbol{x} \int_0^1 du \\
& \epsilon^{abcde} \tilde{\varphi}^a_{\text{local}} \partial_x \tilde{\varphi}^b_{\text{local}} \partial_y \tilde{\varphi}^c_{\text{local}} \partial_\tau \tilde{\varphi}^d_{\text{local}} \partial_u \tilde{\varphi}^e_{\text{local}}
\end{aligned} \tag{64}$$

The above is independent on the choice of the north pole $\tilde{\varphi}_{\text{local}}(\boldsymbol{x}, u = 0)$ and on the *smooth* interpolation parametrized by u between the north pole and the physical path. Written in this way the Wess-Zumino-Witten term for local paths has explicit SO(5) symmetry and more importantly has the following geometrical interpretation: its modulus corresponds to the area of the S^4 surface enclosed by the hyper-path $\hat{\varphi}_{\text{local}}(\boldsymbol{x})$ (see Fig. 5).

The key question then is how to extract the Wess-Zumino-Witten term for generic paths on the S^4 sphere from the above derived local information. Consider a hyper-surface S on S^4 with $\partial S = \varphi_{\text{global}}(\boldsymbol{x})$. We can decompose it into a sum of infinitesimal hyper-surface elements $S^{(n)}$ with $\partial S^{(n)} = \varphi^{(n)}_{\text{local}}(\boldsymbol{x})$. With this construction, we can show that:

$$\begin{aligned}
S(\varphi_{\text{global}}(\boldsymbol{x})) = \sum_n S(\varphi^{(n)}_{\text{local}}(\boldsymbol{x})) = \frac{2\pi i}{vol(S_4)} \iiint_U d^3\boldsymbol{x} \int_0^1 du \\
\epsilon^{abcde} \tilde{\varphi}^a_{\text{global}} \partial_x \tilde{\varphi}^b_{\text{global}} \partial_y \tilde{\varphi}^c_{\text{global}} \partial_\tau \tilde{\varphi}^d_{\text{global}} \partial_u \tilde{\varphi}^e_{\text{global}}
\end{aligned} \tag{65}$$

To prove the validity of Eq. 65, we first parameterize the 5 component vector as,

$$\widetilde{\boldsymbol{\varphi}}^T \equiv \begin{pmatrix} \cos\widetilde{\alpha} \\ \sin\widetilde{\alpha}\cos\widetilde{\beta} \\ \sin\widetilde{\alpha}\sin\widetilde{\beta}\cos\widetilde{\theta} \\ \sin\widetilde{\alpha}\sin\widetilde{\beta}\sin\widetilde{\theta}\cos\widetilde{\phi} \\ \sin\widetilde{\alpha}\sin\widetilde{\beta}\sin\widetilde{\theta}\sin\widetilde{\phi} \end{pmatrix} \tag{66}$$

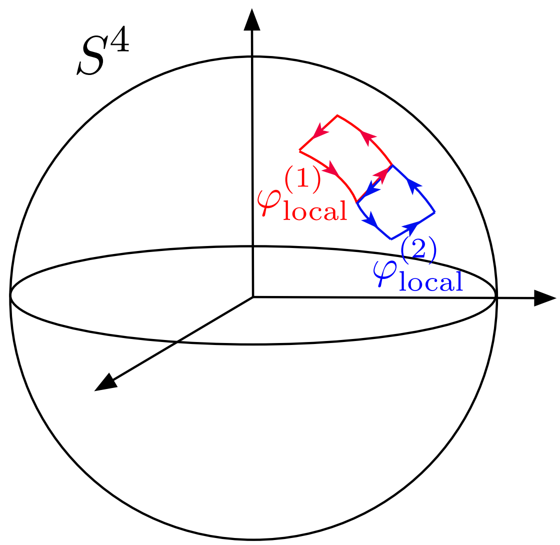

FIG. 5. Two edge sharing closed local hyper-paths on the S^4 sphere. Upon evaluating the Wess-Zumino-Witten term, the contributions from the edge sharing hyper-segments of the hyper-paths cancel due opposite orientations. See text for a precise definition of the orientation of a hyper-path. Hence: $S(\varphi^{(1)}_{\rm local}+\varphi^{(2)}_{\rm local}) = S(\varphi^{(1)}_{\rm local}) + S(\varphi^{(2)}_{\rm local})$.

such that the constraint of unit modulus is automatically satisfied. Here $\widetilde{\alpha},\widetilde{\beta},\widetilde{\theta}$ and $\widetilde{\phi}$ are functions of $u,\boldsymbol{x}$, and their values at $u=1$ are identified as the physical local field configuration α,β,θ and ϕ. At $u=0$ and for simplicity we take

$$\begin{aligned} &\widetilde{\alpha}(u=0,\boldsymbol{x})=0, &&\widetilde{\beta}(u=0,\boldsymbol{x})=\beta(\boldsymbol{x}),\\ &\widetilde{\theta}(u=0,\boldsymbol{x})=\theta(\boldsymbol{x}), &&\widetilde{\phi}(u=0,\boldsymbol{x})=\phi(\boldsymbol{x}) \end{aligned} \tag{67}$$

corresponding to $\widetilde{\boldsymbol{\varphi}}(\boldsymbol{x},u=0)=(1,0,0,0,0)$.

With this parametrization, $S(\varphi_{\rm local})$ defined in Eq. 64 can be reformulated as:

$$\begin{aligned} S(\varphi_{\rm local}) =&\frac{3i}{4\pi}\iiint\limits_U d^3\boldsymbol{x}\int_0^1 du\\ &\epsilon_{abcd}\partial_a\{f(\widetilde{\alpha})\partial_b[g(\widetilde{\beta})\partial_c(h(\widetilde{\theta})\partial_d l(\widetilde{\phi}))]\} \end{aligned} \tag{68}$$

where

$$\begin{aligned} f(\widetilde{\alpha}) &\equiv \frac{1}{3}\cos^3\widetilde{\alpha}-\cos\widetilde{\alpha}\\ g(\widetilde{\beta}) &\equiv \frac{\widetilde{\beta}}{2}-\frac{1}{4}\sin 2\widetilde{\beta}\\ h(\widetilde{\theta}) &\equiv \cos(\widetilde{\theta})\\ l(\widetilde{\phi}) &\equiv \widetilde{\phi}. \end{aligned} \tag{69}$$

In the above the sign of the totally anti-symmetric function ϵ is pinned by setting: $\epsilon_{u\tau xy}=1$.

Hence three out of the four total derivative terms vanish after integration, leaving only the term:

$$\begin{aligned} S(\varphi_{\rm local}) =&\frac{3i}{4\pi}\iiint\limits_U d^3\boldsymbol{x}\\ &\epsilon_{abc}f(\widetilde{\alpha})\partial_a[g(\widetilde{\beta})\partial_b(h(\widetilde{\theta})\partial_c l(\widetilde{\phi}))]|^{u=1}_{u=0}\\ =&\frac{3i}{4\pi}\iiint\limits_U d^3\boldsymbol{x}\\ &\epsilon_{abc}\left(f(\alpha)+\frac{2}{3}\right)\partial_a[g(\beta)\partial_b(h(\theta)\partial_c l(\phi))]. \end{aligned} \tag{70}$$

In the above the sign of the anti-symmetric function ϵ is fixed by: $\epsilon_{\tau xy}=1$.

We will show that for two closed hyper-paths on the S^4 sphere (see Fig. 5), the contributions from their edge sharing segments cancel due to opposite orientations.

First, consider a one-to-one smooth re-parametrization:

$$\tau\equiv T(\tau',x',y') \qquad x\equiv X(\tau',x',y') \qquad y\equiv Y(\tau',x',y') \tag{71}$$

where by definition the Jacobian determinant $\det M_{\boldsymbol{x}}$ with

$$M_{\boldsymbol{x}} \equiv \begin{pmatrix} \partial_{\tau'}T & \partial_{x'}T & \partial_{y'}T\\ \partial_{\tau'}X & \partial_{x'}X & \partial_{y'}X\\ \partial_{\tau'}Y & \partial_{x'}Y & \partial_{y'}Y \end{pmatrix}_{\boldsymbol{x}} . \tag{72}$$

never vanishes and has a fixed sign, positive or negative, in the whole parameter range. The transformation is defined on the initial domain: $U\equiv\{\boldsymbol{x'}\}$ for $\tau'\in[0,\infty)$, $x'\in[0,\pi]$, $y'\in[0,2\pi)$.

Under this transformation:

$$\begin{aligned} \alpha'(\boldsymbol{x}') &\equiv \alpha(T(\boldsymbol{x}'),X(\boldsymbol{x}'),Y(\boldsymbol{x}'))\\ \beta'(\boldsymbol{x}') &\equiv \beta(T(\boldsymbol{x}'),X(\boldsymbol{x}'),Y(\boldsymbol{x}'))\\ \theta'(\boldsymbol{x}') &\equiv \theta(T(\boldsymbol{x}'),X(\boldsymbol{x}'),Y(\boldsymbol{x}'))\\ \phi'(\boldsymbol{x}') &\equiv \phi(T(\boldsymbol{x}'),X(\boldsymbol{x}'),Y(\boldsymbol{x}')) \end{aligned} \tag{73}$$

now $S(\varphi_{\rm local})$ retains the same functional form up to a sign ambiguity:

$$\begin{aligned} S(\varphi_{\rm local}) =&\eta\frac{3i}{4\pi}\iiint\limits_U d^3\boldsymbol{x}'\\ &\epsilon_{a'b'c'}(f(\alpha')+\frac{2}{3})\partial_{a'}[g(\beta')\partial_{b'}(h(\theta')\partial_{c'}l(\phi'))]\\ =&\eta\frac{3i}{4\pi}\iiint\limits_U d^3\boldsymbol{x}\\ &\epsilon_{abc}(f(\alpha')+\frac{2}{3})\partial_a[g(\beta')\partial_b(h(\theta')\partial_c l(\phi'))]. \end{aligned} \tag{74}$$

This follows from the fact that

$$\begin{aligned} &d^3\boldsymbol{x}=d^3\boldsymbol{x}'|\det M_{\boldsymbol{x}}|\\ &\epsilon_{abc}\partial_a[g(\beta)\partial_b(h(\theta)\partial_c l(\phi))]\\ =&\epsilon_{a'b'c'}\partial_{a'}[g(\beta')\partial_{b'}(h(\theta')\partial_{c'}l(\phi'))]\det{}^{-1}M_{\boldsymbol{x}}. \end{aligned} \tag{75}$$

The sign pre-factor in Eq. 74 is given by:

$$\eta = \text{sign}[\det M_{\boldsymbol{x}}] \qquad \forall \boldsymbol{x} \tag{76}$$

and defines the *direction* of a hyper-path in Fig. 5.

Consider two continuous closed S^3 paths on the S^4 sphere:

$$\begin{array}{ll} \text{Path1} & \varphi_1(\boldsymbol{x}) \equiv (\alpha_1(\boldsymbol{x}),\ \beta_1(\boldsymbol{x}),\ \phi_1(\boldsymbol{x}),\ \theta_1(\boldsymbol{x})) \\ \text{Path2} & \varphi_2(\boldsymbol{x}) \equiv (\alpha_2(\boldsymbol{x}),\ \beta_2(\boldsymbol{x}),\ \phi_2(\boldsymbol{x}),\ \theta_2(\boldsymbol{x})) \end{array} \tag{77}$$

with corresponding local action:

$$\begin{aligned} S_1 \equiv & \frac{3i}{4\pi} \iiint_U d^3\boldsymbol{x} \\ & \epsilon_{abc}(f(\alpha_1) + \frac{2}{3})\partial_a[g(\beta_1)\partial_b(h(\theta_1)\partial_c l(\phi_1))] \\ S_2 \equiv & \frac{3i}{4\pi} \iiint_U d^3\boldsymbol{x} \\ & \epsilon_{abc}(f(\alpha_2) + \frac{2}{3})\partial_a[g(\beta_2)\partial_b(h(\theta_2)\partial_c l(\phi_2))]. \end{aligned} \tag{78}$$

Assume that there exits a one-to-one smooth re-parametrization:

$$\begin{array}{l} \text{Path1} \\ \tau \equiv T_1(\tau', x', y') \quad x \equiv X_1(\tau', x', y') \quad y \equiv Y_1(\tau', x', y') \\ \text{sign}[\det M_{\boldsymbol{x}}] > 0 \\ \\ \text{Path2} \\ \tau \equiv T_2(\tau', x', y') \quad x \equiv X_2(\tau', x', y') \quad y \equiv Y_2(\tau', x', y') \\ \text{sign}[\det M_{\boldsymbol{x}}] < 0 \end{array} \tag{79}$$

that leads to:

$$\begin{array}{l} \varphi_1'(\boldsymbol{x'}) \equiv (\alpha_1'(\boldsymbol{x'}),\ \beta_1'(\boldsymbol{x'}),\ \phi_1'(\boldsymbol{x'}),\ \theta_1'(\boldsymbol{x'})) \\ \varphi_2'(\boldsymbol{x'}) \equiv (\alpha_2'(\boldsymbol{x'}),\ \beta_2'(\boldsymbol{x'}),\ \phi_2'(\boldsymbol{x'}),\ \theta_2'(\boldsymbol{x'})) \end{array} \tag{80}$$

with the following identity:

$$\varphi_1'(\boldsymbol{x'}) = \varphi_2'(\boldsymbol{x'}) \tag{81}$$

within a simply connected segment A. The boundary of the shared domain, ∂A, lives on a S^2 sphere. Let us furthermore define the complement of A as $A^c \equiv U - A$. Therefore:

$$\begin{aligned} & S_1 + S_2 \\ =& \frac{3i}{4\pi} \iiint_U d^3\boldsymbol{x} \epsilon_{abc}(f(\alpha_1) + \frac{2}{3})\partial_a[g(\beta_1)\partial_b(h(\theta_1)\partial_c l(\phi_1))] \\ +& \frac{3i}{4\pi} \iiint_U d^3\boldsymbol{x} \epsilon_{abc}(f(\alpha_2) + \frac{2}{3})\partial_a[g(\beta_2)\partial_b(h(\theta_2)\partial_c l(\phi_2))] \\ =& \frac{3i}{4\pi} \iiint_U d^3\boldsymbol{x} \epsilon_{abc}(f(\alpha_1') + \frac{2}{3})\partial_a[g(\beta_1')\partial_b(h(\theta_1')\partial_c l(\phi_1'))] \\ -& \frac{3i}{4\pi} \iiint_U d^3\boldsymbol{x} \epsilon_{abc}(f(\alpha_2') + \frac{2}{3})\partial_a[g(\beta_2')\partial_b(h(\theta_2')\partial_c l(\phi_2'))] \\ =& \frac{3i}{4\pi} \iiint_{A^c} d^3\boldsymbol{x} \epsilon_{abc}(f(\alpha_1') + \frac{2}{3})\partial_a[g(\beta_1')\partial_b(h(\theta_1')\partial_c l(\phi_1'))] \\ -& \frac{3i}{4\pi} \iiint_{A^c} d^3\boldsymbol{x} \epsilon_{abc}(f(\alpha_2') + \frac{2}{3})\partial_a[g(\beta_2')\partial_b(h(\theta_2')\partial_c l(\phi_2'))] \\ \equiv& \frac{3i}{4\pi} \iiint_{\widetilde{A}^c} d^3\boldsymbol{x} \epsilon_{abc}(f(\alpha_1'') + \frac{2}{3})\partial_a[g(\beta_1'')\partial_b(h(\theta_1'')\partial_c l(\phi_1''))] \\ -& \frac{3i}{4\pi} \iiint_{\widetilde{A}^c} d^3\boldsymbol{x} \epsilon_{abc}(f(\alpha_2'') + \frac{2}{3})\partial_a[g(\beta_2'')\partial_b(h(\theta_2'')\partial_c l(\phi_2''))] \end{aligned} \tag{82}$$

In the last step, we consider a re-parametrization of both paths

$$\begin{array}{l} \varphi_1''(\boldsymbol{x'}) \equiv \varphi_1'(\boldsymbol{x}(\boldsymbol{x'})) \\ \varphi_2''(\boldsymbol{x'}) \equiv \varphi_2'(\boldsymbol{x}(\boldsymbol{x'})) \end{array} \tag{83}$$

with

$$\begin{array}{l} \tau \equiv \widetilde{T}(\tau', x', y') \quad x \equiv \widetilde{X}(\tau', x', y') \quad y \equiv \widetilde{Y}(\tau', x', y') \\ \text{sign}[\det M_{\boldsymbol{x}}] > 0 \end{array} \tag{84}$$

such that $\widetilde{A} \equiv \boldsymbol{x}$ for $\tau \in [0, \Lambda_\tau), x \in [0, \pi], y \in [0, 2\pi)$, which implies that $\widetilde{A}_c \equiv \boldsymbol{x}$ for $\tau \in [\Lambda_\tau, \infty), x \in [0, \pi], y \in [0, 2\pi)$. This step is valid since A and A_c are simply connected. Both fields smoothly extrapolate to the S_2 boundary of $\widetilde{A_c}$:

$$\phi_1''(\boldsymbol{x}) = \phi_2''(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \partial\widetilde{A_c} \tag{85}$$

In the following step we will re-parametrization hyper-path 2:

$$\varphi_2'''(\boldsymbol{x'}) \equiv \varphi_2''(\boldsymbol{x}(\boldsymbol{x'})) \tag{86}$$

with

$$\tau \equiv \cot(\frac{\pi/2 - \arctan\Lambda_\tau}{\arctan\Lambda_\tau} \arctan\tau') \quad x \equiv x' \quad y \equiv y'. \tag{87}$$

This corresponds to a map from $\widetilde{\Lambda}_c$ to $\widetilde{\Lambda}$ with the property that at the boundary,

$$\boldsymbol{x} = \boldsymbol{x'} \quad \text{for } \boldsymbol{x} \in \partial\widetilde{A_c}\ (\boldsymbol{x'} \in \partial\widetilde{A}). \tag{88}$$

Importantly this map has the property that: sign[det$M_{\boldsymbol{x}}$] < 0. Note that we do not have to worry about the compactness of $\widetilde{A}$ (or $\widetilde{A_c}$) since under the integration, these differences amont to a zero measure domain and has no effect for smooth fields.

Hence,

$$
\begin{aligned}
&S_1 + S_2 = \\
&\frac{3i}{4\pi}\iiint\limits_{\widetilde{A}^c} d^3\boldsymbol{x}\epsilon_{abc}(f(\alpha_1'') + \frac{2}{3})\partial_a[g(\beta_1'')\partial_b(h(\theta_1'')\partial_c l(\phi_1''))] \\
&-\frac{3i}{4\pi}\iiint\limits_{\widetilde{A}^c} d^3\boldsymbol{x}\epsilon_{abc}(f(\alpha_2'') + \frac{2}{3})\partial_a[g(\beta_2'')\partial_b(h(\theta_2'')\partial_c l(\phi_2''))] \\
&\equiv\frac{3i}{4\pi}\iiint\limits_{\widetilde{A}^c} d^3\boldsymbol{x}\epsilon_{abc}(f(\alpha_1'') + \frac{2}{3})\partial_a[g(\beta_1'')\partial_b(h(\theta_1'')\partial_c l(\phi_1''))] \\
&+\frac{3i}{4\pi}\iiint\limits_{\widetilde{A}} d^3\boldsymbol{x}\epsilon_{abc}(f(\alpha_2''') + \frac{2}{3})\partial_a[g(\beta_2''')\partial_b(h(\theta_2''')\partial_c l(\phi_2'''))] \\
&\equiv\frac{3i}{4\pi}\iiint\limits_{U} d^3\boldsymbol{x}\epsilon_{abc} \\
&\quad(f(\alpha^{12}) + \frac{2}{3})\partial_a[g(\beta^{12})\partial_b(h(\theta^{12})\partial_c l(\phi^{12}))]
\end{aligned}
\tag{89}
$$

in the last step we defined new variable $\varphi^{12} \equiv (\alpha^{12}, \beta^{12}, \theta^{12}, \phi^{12}\ )$ as:

$$
\begin{aligned}
\varphi^{12}(\boldsymbol{x}) &\equiv \varphi_1''(\boldsymbol{x}) \qquad \boldsymbol{x} \in \widetilde{A^c} \\
\varphi^{12}(\boldsymbol{x}) &\equiv \varphi_2'''(\boldsymbol{x}) \qquad \boldsymbol{x} \in \widetilde{A}
\end{aligned}
\tag{90}
$$

Hence, we've shown that for two closed hyper-paths with a common segment, the following identity

$$
S({\varphi^{12}}_{\text{local}}) = S(\varphi^{(1)}_{\text{local}}) + S(\varphi^{(2)}_{\text{local}}) \tag{91}
$$

holds. Hence the construction defined in Eq. 65 is well defined.