Noncommutative geometry, extended W_∞ algebra, and Grassmannian solitons in multicomponent quantum Hall systems

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Noncommutative geometry governs the physics of quantum Hall (QH) effects. We introduce the Weyl ordering of the second quantized density operator to explore the dynamics of electrons in the lowest Landau level. We analyze QH systems made of N-component electrons at the integer filling factor $\nu=k\leqslant N$. The basic algebra is the SU(N)-extended W_∞. A specific feature is that noncommutative geometry leads to a spontaneous development of SU(N) quantum coherence by generating the exchange Coulomb interaction. The effective Hamiltonian is the Grassmannian $G_{N,k}$ sigma model, and the dynamical field is the Grassmannian $G_{N,k}$ field, describing $k(N-k)$ complex Goldstone modes and one kind of topological solitons (Grassmannian solitons).

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I. INTRODUCTION

Noncommutative geometry[1] has received growing attention in field theory and superstring theory.[2–4] However, physical evidence of it is still very rare. The quantum Hall (QH) effect provides a rare evidence,[5] where all physics results from the noncommutative geometry in the plane:

$$[X,Y]=-il_B^2. \tag{1.1}$$

Here, (X,Y) describes the position of the planar electron confined to the lowest Landau level (LLL), and l_B is the magnetic length proving the scale. As discussed extensively,[6–8] the QH system is characterized by the W_∞ algebra.

An aspect of QH systems[5,9] is quantum coherence due to the spin degree of freedom, which is also a consequence of noncommutativity (1.1). Electron spins are polarized spontaneously rather than compulsively by the Zeeman effect: Hence the system is called a QH ferromagnet. The basic algebraic structure is the SU(2)-extended W_∞.[10] Topological solitons (CP^1 solitons) arise as coherent excitations,[11] which have been observed experimentally.[12–14] Much more interesting phenomena occur in bilayer QH systems. For instance, an anomalous tunneling current has been observed[15] between the two layers at zero bias voltage. This may well be a manifestation of the Josephson-like phenomena predicted a decade ago.[16] It occurs due to a quantum coherence developed spontaneously across the layers.[10,17] QH effects present experimental tests of various ideas inherent to noncommutative geometry.

In this paper we investigate the algebraic structure of an N-component QH system subject to noncommutativity (1.1). We then analyze the spontaneous symmetry breaking at the filling factor $\nu=k$, $k=1,2,\ldots,N$, and show that the Goldstone modes and topological solitons are described by the Grassmannian $G_{N,k}$ field. Here the $G_{N,k}$ field is the one that takes values on the Grassmannian $G_{N,k}$ manifold. Note that the $G_{N,1}$ manifold is equal to the CP^{N-1} manifold.

Such multicomponent QH systems may be realized approximately by constructing N'-layer QH systems, where $N=2N'$ with the spin degree of freedom and $N=N'$ without it. We note that the symmetries do not hold exactly in these instances for $N>2$. Nevertheless, by regarding the symmetries to be broken explicitly but only softly, it would reveal physics essential to the noncommutative geometry [Eq. (1.1)]. Indeed, we know many examples where the study of toy models traces out the basic physics of complicated realistic ones. It should also be mentioned that, as far as we are aware, this is the first established system where the Grassmannian $G_{N,k}$ field plays a key role in physics. See Ref. 18 for a specific application to bilayer QH ferromagnets with $N=4$, where some of the theoretical consequences have been compared with experimental data.

In Secs. II and III we review the noncommutative planar system and the LLL projection, respectively. We make a proposition for the Weyl ordering of the second quantized density operator. In Sec. IV we derive the SU(N)-extended W_∞ as the algebraic structure of multicomponent electrons in the noncommutative planar system. In Sec. V, employing an algebraic method, we represent the Coulomb potential so that the exchange interaction effect is made manifest. The exchange Coulomb interaction is the key to quantum coherence. We also stress that it is necessary to use the Weyl-ordered density operators rather than the projected ones to describe physics in the LLL. In Sec. VI, we make a derivative expansion and derive the SU(N) nonlinear sigma model as an effective Hamiltonian. It yields the Grassmannian $G_{N,k}$ sigma model for the QH system at $\nu=k$. In Sec. VII we show that the dynamic field is the $G_{N,k}$ field describing $k(N-k)$ complex Goldstone modes. In Sec. VIII we construct Grassmannian $G_{N,k}$ solitons as topological objects. In Sec. IX, by re-examining the LLL projection, we discuss what quasiparticles we expect to observe in QH systems. In Sec. X we make a brief application of our results to realistic QH systems by including the Zeeman and tunneling interactions. Note that all Goldstone modes are made massive due to these interactions. Section XI is devoted to discussions.

II. NONCOMMUTATIVE PLANAR SYSTEMS

The position of an electron confined to the lowest Landau level is specified by the guiding center $\mathbf{X}=(X,Y)$ subject to noncommutativity (1.1). There exists a procedure (the Weyl prescription[19,20]) to construct a noncommutative theory with the coordinate $\mathbf{X}=(X,Y)$ from a commutative theory with the coordinate $\mathbf{x}=(x,y)$. From a function $f(\mathbf{x})$ in the commutative space, we construct

$$\hat{O}_f=\frac{1}{(2\pi)^2}\int d^2q d^2x\, e^{-i\mathbf{q}(\mathbf{x}-\mathbf{X})}f(\mathbf{x}). \tag{2.1}$$

Taking the plane wave $f(\mathbf{x})=e^{i\mathbf{p}\mathbf{x}}$ in Eq. (2.1), we find

$$\hat{O}_f=e^{i\mathbf{p}\mathbf{X}}. \tag{2.2}$$

We call this a Weyl-ordered plane wave. It approaches the plane wave $e^{i\mathbf{p}\mathbf{x}}$ in the commutative limit ($l_B\to 0$).

It is convenient to construct a Fock representation of algebra (1.1) by way of the operators

$$b=\frac{1}{\sqrt{2}l_B}(X-iY),\quad b^\dagger=\frac{1}{\sqrt{2}l_B}(X+iY), \tag{2.3}$$

obeying $[b,b^\dagger]=1$. The Fock space is made of states

$$|n\rangle=\frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle,\quad n=0,1,2,\ldots. \tag{2.4}$$

They are quantum mechanical states inherent to noncommutativity (1.1). We call them Landau sites in QH systems.

Two of the Landau sites are related as

$$|m\rangle=\sqrt{\frac{n!}{m!}}L(m,n)|n\rangle, \tag{2.5}$$

with $L(m,n)=(b^\dagger)^m b^n$. They generate the algebra

$$[L(m,n),L(k,l)]=\sum_{t=1}^{\infty} C^t_{mn;kl}L(m+k-t,n+l-t), \tag{2.6}$$

where the structure constant is

$$C^t_{mn;kl}=\frac{1}{t!}\left(\frac{n!k!}{(n-t)!(k-t)!}-\frac{m!l!}{(m-t)!(l-t)!}\right). \tag{2.7}$$

This is the W_∞ algebra[7,8] characterizing the noncommutative planar system.

The noncommutative coordinate $\mathbf{X}=(X,Y)$ acts on the Landau site as

$$X|n\rangle=\frac{l_B}{\sqrt{2}}[\sqrt{n}|n-1\rangle+\sqrt{n+1}|n+1\rangle],$$

$$Y|n\rangle=\frac{il_B}{\sqrt{2}}[\sqrt{n}|n-1\rangle-\sqrt{n+1}|n+1\rangle]. \tag{2.8}$$

We may represent algebra (1.1) by the differential operators

$$X(\mathbf{x})=\frac{1}{2}x-il_B^2\frac{\partial}{\partial y},\quad Y(\mathbf{x})=\frac{1}{2}y+il_B^2\frac{\partial}{\partial x}. \tag{2.9}$$

Then, $\hat{O}_f$ acts on the Fock space via Eq. (2.8), and it is represented by a differential operator

$$\langle\mathbf{x}|\hat{O}_f|n\rangle=\hat{O}_f(\mathbf{x})\langle\mathbf{x}|n\rangle \tag{2.10}$$

acting on the wave function. In this representation the wave function reads

$$\mathfrak{S}_n(\mathbf{x})=\langle\mathbf{x}|n\rangle=\sqrt{\frac{1}{2^{n+1}\pi l_B^2 n!}}\left(\frac{z}{l_B}\right)^n e^{-\mathbf{x}^2/4l_B^2}, \tag{2.11}$$

with $z=x+iy$. It is seen from this wave function that each Landau site $|n\rangle$ occupies an area $2\pi l_B^2$.

The Weyl-ordered plane wave $e^{i\mathbf{p}\mathbf{X}}$ generates the projective translation group

$$e^{i\mathbf{p}\mathbf{X}}e^{i\mathbf{q}\mathbf{X}}=e^{i(\mathbf{p}+\mathbf{q})\mathbf{X}}\exp\left[\frac{i}{2}l_B^2\mathbf{p}\wedge\mathbf{q}\right], \tag{2.12}$$

with $\mathbf{p}\wedge\mathbf{q}=p_xq_y-p_yq_x$, as follows from noncommutativity (1.1). We present two important relations,

$$\mathrm{Tr}[e^{i\mathbf{p}\mathbf{X}}]\equiv\sum_{n=0}^{\infty}\langle n|e^{i\mathbf{p}\mathbf{X}}|n\rangle=\frac{2\pi}{l_B^2}\delta(\mathbf{p}) \tag{2.13}$$

and

$$\int d^2p\langle m|e^{-i\mathbf{p}\mathbf{X}}|n\rangle\langle i|e^{i\mathbf{p}\mathbf{X}}|j\rangle=\frac{2\pi}{l_B^2}\delta_{ni}\delta_{mj}. \tag{2.14}$$

They are proved in Appendix A.

We derive the inversion relation of the Weyl ordering [Eq. (2.1)]. We evaluate

$$\int d^2p\,\mathrm{Tr}[\hat{O}_f e^{i\mathbf{p}(\mathbf{x}-\mathbf{X})}]$$

$$=\frac{1}{(2\pi)^2}\int d^2p d^2q d^2y\,\mathrm{Tr}[e^{i\mathbf{q}\mathbf{X}}e^{-i\mathbf{p}\mathbf{X}}]e^{-\mathbf{q}\mathbf{y}+i\mathbf{p}\mathbf{x}}f(\mathbf{y}). \tag{2.15}$$

Using Eqs. (2.12) and (2.13), we obtain

$$f(\mathbf{x})=\frac{l_B^2}{2\pi}\int d^2p\, e^{i\mathbf{p}\mathbf{x}}\mathrm{Tr}[\hat{O}_f e^{-i\mathbf{p}\mathbf{X}}]. \tag{2.16}$$

This is the inversion relation of the Weyl ordering. An interesting relation follows trivially:

$$\mathrm{Tr}[\hat{O}_f]=\frac{1}{2\pi l_B^2}\int d^2x\, f(\mathbf{x}). \tag{2.17}$$

We may regard this as a generalization of Eq. (2.13), which is reproduced by setting $f(\mathbf{x})=e^{i\mathbf{p}\mathbf{x}}$ and $\hat{O}_f=e^{i\mathbf{p}\mathbf{X}}$.

In this paper we deal with the second-quantized density operator. It is necessary to define the Weyl ordering of such an operator. As a standard procedure we proceed from *clas-*

sical mechanics to *quantum mechanics* and then to *field theory*.[21] The classical density is

$$\rho_{\mathbf{r}}^{\mathrm{CM}}(\mathbf{x})=\delta(\mathbf{x}-\mathbf{r}), \tag{2.18}$$

where $\mathbf{x}$ denotes the particle coordinate, $\mathbf{x}=\mathbf{x}(t)$. On the other hand, $\mathbf{r}$ is the variable parametrizing the plane, which remains commutative after the first quantization as well as the second quantization. Passing to quantum mechanics we replace the *c*-number coordinate $\mathbf{x}$ by the corresponding operator $\mathbf{X}$. Every quantity $f(\mathbf{x})$ is to be replaced by the Weyl-ordered one $\hat{O}_f$ according to Eq. (2.1). Setting $f(\mathbf{x})=\rho_{\mathbf{r}}^{\mathrm{CM}}(\mathbf{x})=\delta(\mathbf{x}-\mathbf{r})$, we find

$$\rho_{\mathbf{r}}^{\mathrm{QM}}\equiv\hat{O}_f=\frac{1}{(2\pi)^2}\int d^2q d^2x\, e^{-i\mathbf{q}(\mathbf{x}-\mathbf{X})}\delta(\mathbf{x}-\mathbf{r})$$

$$=\frac{1}{(2\pi)^2}\int d^2q\, e^{-i\mathbf{q}(\mathbf{r}-\mathbf{X})}. \tag{2.19}$$

Here no requirement has yet been made on the commutativity of the operator $\mathbf{X}=(X,Y)$.

In passing to field theory, denoting by $|n\rangle$ the quantum-mechanical one-body state, we introduce the second-quantized fermion operator $c(n)$ with $\{c(n),c^{\dagger}(m)\}=\delta_{nm}$. We define

$$|\Psi\rangle=\sum_n |n\rangle c(n), \tag{2.20}$$

so that the field operator is $\Psi(\mathbf{x})=\langle\mathbf{x}|\Psi\rangle$. The field theoretical density operator is

$$\rho^{\mathrm{FT}}(\mathbf{r})\equiv\langle\Psi|\rho_{\mathbf{r}}^{\mathrm{QM}}|\Psi\rangle=\frac{1}{(2\pi)^2}\int d^2q\, e^{-i\mathbf{q}\mathbf{r}}\langle\Psi|e^{i\mathbf{q}\mathbf{X}}|\Psi\rangle$$

$$=\frac{1}{(2\pi)^2}\int d^2q d^2x d^2y\, e^{-i\mathbf{q}\mathbf{r}}\langle\Psi|\mathbf{x}\rangle\langle\mathbf{x}|e^{i\mathbf{q}\mathbf{X}}|\mathbf{y}\rangle\langle\mathbf{y}|\Psi\rangle. \tag{2.21}$$

This is the standard procedure for second quantization, where the first-quantized operator is sandwiched between $\Psi^{\dagger}(\mathbf{x})$ and $\Psi(\mathbf{x})$.

In order to show that formula (2.21) is the general one, we first apply it to the commutative theory with $[X,Y]=0$. It is represented by $\mathbf{X}|\mathbf{x}\rangle=\mathbf{x}|\mathbf{x}\rangle$ with $\langle\mathbf{x}|\mathbf{y}\rangle=\delta(\mathbf{x}-\mathbf{y})$. Using this in Eq. (2.21) we find $\rho^{\mathrm{FT}}(\mathbf{r})=\rho(\mathbf{r})$, with

$$\rho(\mathbf{r})=\Psi^{\dagger}(\mathbf{r})\Psi(\mathbf{r}). \tag{2.22}$$

This is the well-known result in the commutative theory.

We proceed to discuss the noncommutative theory with $[X,Y]=-il_B^2$. It is represented by the Fock space made of Eq. (2.4). Substituting expansion (2.20) into Eq. (2.21), we obtain $\rho^{\mathrm{FT}}(\mathbf{r})=\hat{\rho}(\mathbf{r})$, with

$$\hat{\rho}(\mathbf{r})=\frac{1}{(2\pi)^2}\int d^2q\, e^{-i\mathbf{q}\mathbf{r}}\left[\sum_{mn}\langle m|e^{i\mathbf{q}\mathbf{X}}|n\rangle\rho(m,n)\right], \tag{2.23}$$

where $\rho(m,n)\equiv c^{\dagger}(m)c(n)$. This approaches the ordinary density $\rho(\mathbf{r})$ in the commutative limit ($l_B\to 0$). The Fourier transformation is

$$\hat{\rho}(\mathbf{q})=\frac{1}{2\pi}\sum_{mn}\langle m|e^{-i\mathbf{q}\mathbf{X}}|n\rangle\rho(m,n). \tag{2.24}$$

We propose $\hat{\rho}(\mathbf{r})$ as the Weyl-ordered density operator. We interpret that the matrix element $\langle\hat{\rho}(\mathbf{r})\rangle$ is the classical density measured at the point $\mathbf{r}$ in the commutative space. The commutative space is the one from which the noncommutative space is constructed by deforming the commutation relation: It is used for representation (2.9) of noncommutativity (1.1).

III. LOWEST LANDAU LEVEL PROJECTION

We proceed to discuss QH systems. Electrons make cyclotron motions under a perpendicular magnetic field *B*, and their energies are quantized into Landau levels. The number density of magnetic flux quanta is $\rho_{\Phi}\equiv B/\Phi_{\mathrm{D}}$, with $\Phi_{\mathrm{D}}=2\pi\hbar/e$ the flux unit, which is equal to the number density of Landau sites. One electron occupies area $2\pi l_B^2$ with $l_B=\sqrt{\hbar/eB}$ the magnetic length. The filling factor is $\nu=\rho_0/\rho_{\Phi}$ with ρ_0 the electron number density. At $\nu=k$ (integer), one Landau site accommodates *k* electrons with different isospins due to the Pauli exclusion principle.

We first review the one-body property of electrons in strong magnetic field. The electron coordinate $\mathbf{x}=(x,y)$ is decomposed as $\mathbf{x}=\mathbf{X}+\mathbf{R}$, where $\mathbf{X}=(X,Y)$ is the guiding center and $\mathbf{R}=(-P_y,P_x)/eB$ is the relative coordinate. From them we construct two sets of harmonic-oscillator operators,

$$a\equiv\frac{l_B}{\sqrt{2}\hbar}(P_x+iP_y),\quad a^{\dagger}\equiv\frac{l_B}{\sqrt{2}\hbar}(P_x-iP_y), \tag{3.1a}$$

$$b\equiv\frac{1}{\sqrt{2}l_B}(X-iY),\quad b^{\dagger}\equiv\frac{1}{\sqrt{2}l_B}(X+iY), \tag{3.1b}$$

obeying

$$[a,a^{\dagger}]=[b,b^{\dagger}]=1,\quad [a,b]=[a^{\dagger},b]=0. \tag{3.2}$$

The kinetic Hamiltonian is

$$H_{\mathrm{K}}=\left(a^{\dagger}a+\frac{1}{2}\right)\hbar\omega_{\mathrm{c}} \tag{3.3}$$

with $\hbar\omega_{\mathrm{c}}$ the cyclotron energy. When the cyclotron gap is large enough, thermal excitations across Landau levels are practically impossible. Hence it is a good approximation to neglect all those excitations by requiring the confinement of electrons to the LLL. The guiding center is the noncommutative coordinate.

We make the LLL projection in a systematic way.[5] We decompose the coordinate $\mathbf{x}$ into the relative coordinate $\mathbf{R}$ and the guiding center $\mathbf{X}$. The relative coordinate $\mathbf{R}$ is frozen

when the electron is confined to the LLL. We denote the LLL projection of the c-number function $f(\mathbf{x})$ as $\langle\langle f\rangle\rangle$. In particular, we have[5]

$$\langle\langle f\rangle\rangle = e^{-l_B^2\mathbf{p}^2/4}e^{i\mathbf{pX}} \tag{3.4}$$

for the plane wave $f(x)=e^{i\mathbf{px}}$. The suppression factor $e^{-l_B^2\mathbf{p}^2/4}$ arises due to the LLL projection of the relative coordinate. In general, we have

$$\langle\langle f\rangle\rangle = \frac{1}{(2\pi)^2}\int d^2q d^2x\, e^{-l_B^2\mathbf{q}^2/4}e^{-i\mathbf{q}(\mathbf{x}-\mathbf{X})}f(\mathbf{x}). \tag{3.5}$$

Consequently, the LLL projection is equivalent to Weyl ordering (2.1) but for the suppression factor.

In the field theoretical framework kinetic Hamiltonian (3.3) reads

$$H_{\mathrm{K}} = \frac{1}{2M}\int d^2x \psi^\dagger(\mathbf{x})(P_x - iP_y)(P_x + iP_y)\psi(\mathbf{x}), \tag{3.6}$$

apart from the cyclotron energy $\hbar\omega_c/2$ per electron, where $P_k = -i\hbar\partial_k + eA_k$. We assume the electron field ψ to possess N isospin components ψ_σ. We introduce the field operator describing electrons confined to the LLL. It is determined so as to satisfy the LLL condition

$$(P_x + iP_y)\psi_\sigma(\mathbf{x})|\mathfrak{S}\rangle = 0, \tag{3.7}$$

implying that kinetic Hamiltonian (3.6) is quenched. Solving this equation we find the projected field to be

$$\overline{\psi}_\sigma(\mathbf{x}) = \sum_{n=0}^{\infty} c_\sigma(n)\langle\mathbf{x}|n\rangle, \tag{3.8}$$

where $\langle\mathbf{x}|n\rangle$ is the one-body wave function [Eq. (2.11)] and $c_\sigma(n)$ is the annihilation operator of the electron with isospin σ at the Landau site n:

$$\{c_\sigma(n), c_\tau^\dagger(m)\} = \delta_{nm}\delta_{\sigma\tau},$$

$$\{c_\sigma(n), c_\tau(m)\} = \{c_\sigma^\dagger(n), c_\tau^\dagger(m)\} = 0. \tag{3.9}$$

The Hilbert space $\mathbb{H}_{\mathrm{LLL}}$ is made of the Fock spaces $\mathbb{H}_n$ of electrons in all Landau sites, $\mathbb{H}_{\mathrm{LLL}} = \otimes_n \mathbb{H}_n$.

The LLL projection of the density operator $\rho(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{x})$ is given by[7,8]

$$\overline{\rho}(\mathbf{x}) = \overline{\psi}^\dagger(\mathbf{x})\overline{\psi}(\mathbf{x}). \tag{3.10}$$

Substituting expansion (3.8) into this, we find

$$\overline{\rho}(\mathbf{x}) = \sum_{mn}\langle m|\mathbf{x}\rangle\langle\mathbf{x}|n\rangle\rho(m,n), \tag{3.11}$$

where

$$\rho(m,n) \equiv \sum_\sigma c_\sigma^\dagger(m)c_\sigma(n). \tag{3.12}$$

The Fourier transformation of $\overline{\rho}(\mathbf{x})$ is

$$\overline{\rho}(\mathbf{q}) = \sum_{mn}^{\infty}\rho(m,n)\int\frac{d^2x}{2\pi}e^{-i\mathbf{qx}}\langle m|\mathbf{x}\rangle\langle\mathbf{x}|n\rangle. \tag{3.13}$$

Since $e^{-i\mathbf{px}}$ is just a c number, this is moved into the matrix element. We replace $\mathbf{x}$ with the position operator acting on the state $|\mathbf{x}\rangle$, and separate it into the guiding center $\mathbf{X}$ and the relative coordinate $\mathbf{R}$. The relative coordinate being frozen, the plane wave $e^{-i\mathbf{qx}}$ is projected as in Eq. (3.4),

$$\int d^2x\, e^{-i\mathbf{qx}}\langle m|\mathbf{x}\rangle\langle\mathbf{x}|n\rangle = e^{-l_B^2\mathbf{q}^2/4}\langle m|e^{-i\mathbf{qX}}|n\rangle, \tag{3.14}$$

where we have used $\int d^2x|\mathbf{x}\rangle\langle\mathbf{x}| = 1$. Hence we find

$$\overline{\rho}(\mathbf{q}) = \frac{1}{2\pi}e^{-l_B^2\mathbf{q}^2/4}\sum_{mn}^{\infty}\langle m|e^{-i\mathbf{qX}}|n\rangle\rho(m,n). \tag{3.15}$$

This is equivalent to the Weyl-ordered density operator $\hat{\rho}(\mathbf{q})$ given by Eq. (2.24), but for the suppression factor. This is what we have expected from Eqs. (2.1) and (3.5).

Let us reexamine the LLL projection of the density operator in a spirit of the basic formula [Eq. (3.4)]. We start with the Fourier transformation of the unprojected density:

$$\rho(\mathbf{q}) = \int\frac{d^2x}{2\pi}e^{-i\mathbf{qx}}\psi_\sigma^\dagger(\mathbf{x})\psi_\sigma(\mathbf{x}) \tag{3.16a}$$

$$= \int\frac{d^2x}{2\pi}\int d^2y\, \psi_\sigma^\dagger(\mathbf{x})\langle\mathbf{x}|e^{-i\mathbf{qx}}|\mathbf{y}\rangle\psi_\sigma(\mathbf{y}). \tag{3.16b}$$

Here we project the plane wave according to Eq. (3.4),

$$\langle\langle\rho(\mathbf{q})\rangle\rangle = e^{-l_B^2\mathbf{q}^2/4}\int\frac{d^2xd^2y}{2\pi}\langle\mathbf{x}|e^{-i\mathbf{qX}}|\mathbf{y}\rangle\psi_\sigma^\dagger(\mathbf{x})\psi_\sigma(\mathbf{y}), \tag{3.17}$$

as is done by substituting the completeness condition in the Hilbert space $\mathbb{H}_{\mathrm{LLL}}$, $\Sigma_{m=0}^{\infty}|m\rangle\langle m| = 1$. Then it is easy to see that formula (3.17) is reduced to Eq. (3.15).

A comment is in order. When we take the LLL projection of the plane wave naively in Eq. (3.16a), we would obtain

$$\langle\langle\rho(\mathbf{q})\rangle\rangle = e^{-l_B^2\mathbf{q}^2/4}\int\frac{d^2x}{2\pi}e^{-i\mathbf{qX}}\rho(\mathbf{x}). \tag{3.18}$$

This is the formula given in Ref. 5. It should be understood as a symbolic notation of Eq. (3.17).

IV. NONCOMMUTATIVE ALGEBRA

The kinetic Hamiltonian [Eq. (3.6)] possesses the global symmetry $\mathrm{U(N)} = \mathrm{U(1)}\otimes\mathrm{SU(N)}$, whose generators are

$$\rho(\mathbf{x}) = \psi^\dagger(\mathbf{x})\psi(\mathbf{x}), \quad S^A(\mathbf{x}) = \frac{1}{2}\psi^\dagger(\mathbf{x})\lambda^A\psi(\mathbf{x}), \tag{4.1}$$

where λ^A are the generating matrices,

$$[\lambda^A,\lambda^B]=2if^{ABC}\lambda^C,$$

$$\{\lambda^A,\lambda^B\}=2d^{ABC}\lambda^C+\frac{4}{N}\delta^{AB}, \tag{4.2}$$

with f^{ABC} and d^{ABC} the structure constants of SU(N).

We investigate how the symmetry is modified for those electrons in the noncommutative plane. In the momentum space the Weyl-ordered generators are given by Eq. (2.24), or

$$\hat{\rho}(\mathbf{q})=\frac{1}{2\pi}\sum_{mn}^{\infty}\langle m|e^{-i\mathbf{q}\mathbf{X}}|n\rangle\rho(m,n), \tag{4.3}$$

$$\hat{S}^A(\mathbf{q})=\frac{1}{2\pi}\sum_{mn}^{\infty}\langle m|e^{-i\mathbf{q}\mathbf{X}}|n\rangle S^A(m,n), \tag{4.4}$$

with

$$\rho(m,n)\equiv\sum_{\sigma}c_\sigma^\dagger(m)c_\sigma(n), \tag{4.5}$$

$$S^A(m,n)\equiv\frac{1}{2}\sum_{\sigma\tau}c_\sigma^\dagger(m)\lambda^A_{\sigma\tau}c_\tau(n). \tag{4.6}$$

Taking the Fourier transformation we have

$$\hat{\rho}(\mathbf{x})=\int\frac{d^2q}{2\pi}e^{i\mathbf{q}\mathbf{x}}\hat{\rho}(\mathbf{q}), \tag{4.7}$$

$$\hat{S}^A(\mathbf{x})=\int\frac{d^2q}{2\pi}e^{i\mathbf{q}\mathbf{x}}\hat{S}^A(\mathbf{q}). \tag{4.8}$$

The inversions of Eqs. (4.3) and (4.4) are

$$\rho(m,n)=l_B^2\int d^2q\langle n|e^{i\mathbf{q}\mathbf{X}}|m\rangle\hat{\rho}(\mathbf{q}), \tag{4.9}$$

$$S^A(m,n)=l_B^2\int d^2q\langle n|e^{i\mathbf{q}\mathbf{X}}|m\rangle\hat{S}^A(\mathbf{q}), \tag{4.10}$$

as is verified with the use of Eq. (2.14).

The operators $\rho(m,n)$ generate the algebra

$$[\rho(m,n),\rho(j,k)]=\delta_{jn}\rho(m,k)-\delta_{mk}\rho(j,n), \tag{4.11}$$

as follows from anticommutation relation (3.9) of $c_\sigma(m)$. This is closely related to W_∞ algebra (2.6). It is easy to show that the element

$$\mathcal{L}(m,n)=\int d^2x\,\overline{\psi}^\dagger(\mathbf{x})L(m,n)\overline{\psi}(\mathbf{x}) \tag{4.12}$$

generates the algebra isomorphic to Eq. (2.6). On the other hand, they span the same linear space as $\rho(m,n)$ span: $\mathcal{L}(0,0)=\Sigma_m\rho(m,m)$, $\mathcal{L}(0,1)=\Sigma_m\sqrt{m+1}\rho(m,m+1)$, $\mathcal{L}(1,1)=\Sigma_m m\rho(m,m)$, and so on. Hence $\mathcal{L}(m,n)$ and $\rho(m,n)$ give the same Fock representation of W_∞.

The set of $\rho(m,n)$ and $S^A(m,n)$ generates an extended algebra,

$$[\rho(m,n),\rho(j,k)]=\delta_{jn}\rho(m,k)-\delta_{mk}\rho(j,n),$$

$$[\rho(m,n),S^A(j,k)]=\delta_{jn}S^A(m,k)-\delta_{mk}S^A(j,n),$$

$$[S^A(m,n),S^B(j,k)]=\frac{i}{2}f^{ABC}[\delta_{jn}S^C(m,k)+\delta_{mk}S^C(j,n)]+\frac{1}{2}d^{ABC}[\delta_{jn}S^C(m,k)-\delta_{mk}S^C(j,n)]+\frac{1}{2N}\delta^{AB}[\delta_{jn}\rho(m,k)-\delta_{mk}\rho(j,n)], \tag{4.13}$$

as follows from anticommutation relations (3.9) of $c_\sigma(m)$. We reformulate it in terms of the electron density $\hat{\rho}(\mathbf{p})$ and the isospin density $\hat{S}^A(\mathbf{p})$;

$$[\hat{\rho}(\mathbf{p}),\hat{\rho}(\mathbf{q})]=\frac{i}{\pi}\hat{\rho}(\mathbf{p}+\mathbf{q})\sin\left[\frac{1}{2}l_B^2\mathbf{p}\wedge\mathbf{q}\right],$$

$$[\hat{S}^A(\mathbf{p}),\hat{\rho}(\mathbf{q})]=\frac{i}{\pi}\hat{S}^A(\mathbf{p}+\mathbf{q})\sin\left[\frac{1}{2}l_B^2\mathbf{p}\wedge\mathbf{q}\right],$$

$$[\hat{S}^A(\mathbf{p}),\hat{S}^B(\mathbf{q})]=\frac{i}{2\pi}f^{ABC}\hat{S}^C(\mathbf{p}+\mathbf{q})\cos\left[\frac{1}{2}l_B^2\mathbf{p}\wedge\mathbf{q}\right]+\frac{i}{2\pi}d^{ABC}\hat{S}^C(\mathbf{p}+\mathbf{q})\sin\left[\frac{1}{2}l_B^2\mathbf{p}\wedge\mathbf{q}\right]+\frac{i}{2\pi N}\delta^{AB}\hat{\rho}(\mathbf{p}+\mathbf{q})\sin\left[\frac{1}{2}l_B^2\mathbf{p}\wedge\mathbf{q}\right]. \tag{4.14}$$

These are easily derived with the use of Eqs. (4.3), (4.4), (4.13), and (2.12). We call Eq. (4.13), or equivalently Eq. (4.14) the W_∞(N) algebra since it is the SU(N) extension of W_∞.

In the coordinate space the commutation relations read

$$[\hat{\rho}(\mathbf{x}),\hat{\rho}(\mathbf{y})]=\int d^2z[[\delta_{\mathbf{x}}(\mathbf{z}),\delta_{\mathbf{y}}(\mathbf{z})]]\hat{\rho}(\mathbf{z}),$$

$$[\hat{S}^A(\mathbf{x}),\hat{\rho}(\mathbf{y})]=\int d^2z[[\delta_{\mathbf{x}}(\mathbf{z}),\delta_{\mathbf{y}}(\mathbf{z})]]\hat{S}^A(\mathbf{z}),$$

$$[\hat{S}^A(\mathbf{x}),\hat{S}^B(\mathbf{y})]=\frac{i}{2}f^{ABC}\int d^2z\{\{\delta_{\mathbf{x}}(\mathbf{z}),\delta_{\mathbf{y}}(\mathbf{z})\}\}\hat{S}^C(\mathbf{z})+\frac{1}{2}d^{ABC}\int d^2z[[\delta_{\mathbf{x}}(\mathbf{z}),\delta_{\mathbf{y}}(\mathbf{z})]]\hat{S}^C(\mathbf{z})+\frac{1}{2N}\delta^{AB}\int d^2z[[\delta_{\mathbf{x}}(\mathbf{z}),\delta_{\mathbf{y}}(\mathbf{z})]]\hat{\rho}(\mathbf{z}), \tag{4.15}$$

where $[\delta_{\mathbf{x}}(\mathbf{z}),\delta_{\mathbf{y}}(\mathbf{z})]$ is the Moyal bracket,

$$[[\delta_{\mathbf{x}}(\mathbf{z}),\delta_{\mathbf{y}}(\mathbf{z})]]=\delta_{\mathbf{x}}(\mathbf{z})\star\delta_{\mathbf{y}}(\mathbf{z})-\delta_{\mathbf{y}}(\mathbf{z})\star\delta_{\mathbf{x}}(\mathbf{z})$$
$$=\frac{2i}{(2\pi)^4}\int d^2p\,d^2q\,e^{i\mathbf{p}(\mathbf{x}-\mathbf{z})+i\mathbf{q}(\mathbf{y}-\mathbf{z})}$$
$$\times\sin\left[\frac{1}{2}l_B^2\mathbf{p}\wedge\mathbf{q}\right], \tag{4.16}$$

and $\{\delta_{\mathbf{x}}(\mathbf{z}),\delta_{\mathbf{y}}(\mathbf{z})\}$ is the Moyal antibracket:

$$\{\{\delta_{\mathbf{x}}(\mathbf{z}),\delta_{\mathbf{y}}(\mathbf{z})\}\}=\delta_{\mathbf{x}}(\mathbf{z})\star\delta_{\mathbf{y}}(\mathbf{z})+\delta_{\mathbf{y}}(\mathbf{z})\star\delta_{\mathbf{x}}(\mathbf{z})$$
$$=\frac{2}{(2\pi)^4}\int d^2p\,d^2q\,e^{i\mathbf{p}(\mathbf{x}-\mathbf{z})+i\mathbf{q}(\mathbf{y}-\mathbf{z})}$$
$$\times\cos\left[\frac{1}{2}l_B^2\mathbf{p}\wedge\mathbf{q}\right]. \tag{4.17}$$

Here $\delta_{\mathbf{x}}(\mathbf{z})=\delta(\mathbf{x}-\mathbf{z})$, and $\star$ denotes the star product with respect to $\mathbf{z}$. We have adopted the convention

$$f(\mathbf{z})\star g(\mathbf{z})=\exp\left(-\frac{i}{2}l_B^2\nabla_{\mathbf{x}}\wedge\nabla_{\mathbf{y}}\right)f(\mathbf{x})g(\mathbf{y})|_{\mathbf{x}=\mathbf{y}=\mathbf{z}}, \tag{4.18}$$

in accord with noncommutativity (1.1).

The algebra W_∞(N) regresses to the algebra U(N) in the commutative limit ($l_B\to 0$), where

$$[[\delta_{\mathbf{x}}(\mathbf{z}),\delta_{\mathbf{y}}(\mathbf{z})]]\to 0,$$
$$\{\{\delta_{\mathbf{x}}(\mathbf{z}),\delta_{\mathbf{y}}(\mathbf{z})\}\}\to 2\delta(\mathbf{x}-\mathbf{z})\delta(\mathbf{y}-\mathbf{z}). \tag{4.19}$$

In this limit the densities $\hat{\rho}(\mathbf{x})$ and $\hat{S}^A(\mathbf{x})$ are reduced to the physical densities $\rho(\mathbf{x})$ and $S^A(\mathbf{x})$ in the original commutative geometry.

It is instructive to evaluate the commutation relations of the projected densities (3.10). From Eq. (4.14) we obtain

$$[\overline{\rho}(\mathbf{x}),\overline{\rho}(\mathbf{y})]=\int d^2z\,K^-(\mathbf{x},\mathbf{y};\mathbf{z})\overline{\rho}(\mathbf{z}), \tag{4.20}$$

and so on, where the kernel $K^-(\mathbf{x},\mathbf{y};\mathbf{z})$ contains the integration

$$\int d^2p\,d^2q\,e^{i\mathbf{p}(\mathbf{x}-\mathbf{z})+i\mathbf{q}(\mathbf{y}-\mathbf{z})+(il_B^2/2)\mathbf{p}\wedge\mathbf{q}+(l_B^2/2)\mathbf{pq}}. \tag{4.21}$$

It is divergent due to the factor $e^{(l_B^2/2)\mathbf{pq}}$. Similar divergences also appear in $[\overline{S^A}(\mathbf{x}),\overline{\rho}(\mathbf{y})]$ and $[\overline{S^A}(\mathbf{x}),\overline{S^B}(\mathbf{y})]$. The projected densities $\overline{\rho}(\mathbf{x})$ and $\overline{S^A}(\mathbf{x})$ are not good operators with respect to the W_∞(N) algebra.

V. COULOMB INTERACTIONS

Electrons interact with each other via the Coulomb potential,

$$H_{\mathrm{C}}=\frac{1}{2}\int d^2x\,d^2y\,\rho(\mathbf{x})V(\mathbf{x}-\mathbf{y})\rho(\mathbf{y}), \tag{5.1}$$

where $V(\mathbf{x}-\mathbf{y})=e^2/4\pi\varepsilon|\mathbf{x}-\mathbf{y}|$. (We later include the Zeeman and tunneling interactions to discuss realistic systems.) In the previous sections we have projected the states to the LLL. However, even if we start with a state in the LLL, the potential term kicks it out up to higher Landau levels and results in an increase of the kinetic energy. To suppress such excitations we also make LLL projection[6,22] of the potential term.

The projected Coulomb Hamiltonian is given by replacing the density $\rho(\mathbf{x})$ with the projected density $\overline{\rho}(\mathbf{x})$:

$$\hat{H}_{\mathrm{C}}=\frac{1}{2}\int d^2x\,d^2y\,\overline{\rho}(\mathbf{x})V(\mathbf{x}-\mathbf{y})\overline{\rho}(\mathbf{y}). \tag{5.2}$$

We substitute expansion (3.8) of the electron field into the projected density and reduce Eq. (5.2) to

$$\hat{H}_{\mathrm{C}}=\sum_{mnij}\sum_{\sigma\tau}V_{mnij}c_\sigma^\dagger(m)c_\tau^\dagger(i)c_\tau(j)c_\sigma(n), \tag{5.3}$$

where

$$V_{mnij}=\frac{1}{2}\int d^2x\,d^2y\,V(\mathbf{x}-\mathbf{y})\langle m|\mathbf{x}\rangle\langle\mathbf{x}|n\rangle\langle i|\mathbf{y}\rangle\langle\mathbf{y}|j\rangle. \tag{5.4}$$

By using Eq. (3.14) this is rewritten as

$$V_{mnij}=\frac{1}{4\pi}\int d^2k\,e^{-l_B^2\mathbf{k}^2/2}V(\mathbf{k})\langle m|e^{i\mathbf{Xk}}|n\rangle\langle i|e^{-i\mathbf{Xk}}|j\rangle. \tag{5.5}$$

We may rewrite Eq. (5.3) as

$$\hat{H}_{\mathrm{C}}=\sum_{mnij}V_{mnij}\rho(m,n)\rho(i,j) \tag{5.6}$$

with Eq. (4.5).

In terms of the Weyl-ordered density, Eq. (5.2) yields

$$\hat{H}_{\mathrm{C}}=\frac{1}{2}\int d^2x\,d^2y\,V_{\mathrm{D}}(\mathbf{x}-\mathbf{y})\hat{\rho}(\mathbf{x})\hat{\rho}(\mathbf{y}). \tag{5.7}$$

It is derived from the expression in momentum space:

$$\hat{H}_{\mathrm{C}}=\pi\int d^2k\,V_{\mathrm{D}}(\mathbf{k})\hat{\rho}(-\mathbf{k})\hat{\rho}(\mathbf{k}). \tag{5.8}$$

Here we have separated out the suppression factors from the density operator $\overline{\rho}(\mathbf{k})$ in Eq. (3.15) and have attached it to $V(\mathbf{k})$ to construct $V_{\mathrm{D}}(\mathbf{k})$. The potential is given by

$$V_{\mathrm{D}}(\mathbf{k})=e^{-l_B^2\mathbf{k}^2/2}V(\mathbf{k}), \tag{5.9}$$

with $V(\mathbf{k})=e^2/4\pi\varepsilon|\mathbf{k}|$. Its Fourier transformation is

$$V_{\mathrm{D}}(\mathbf{x})=\frac{e^2\sqrt{2\pi}}{8\pi\varepsilon l_B}I_0(\mathbf{x}^2/4l_B^2)e^{-\mathbf{x}^2/4l_B^2}, \tag{5.10}$$

where $I_0(x)$ is the modified Bessel function. It approaches the ordinary Coulomb potential at large distance,

$$V_{\rm D}(\mathbf{x}) \to V(\mathbf{x}) = \frac{e^2}{4\pi\varepsilon|\mathbf{x}|} \quad \text{as} \quad |\mathbf{x}| \to \infty, \tag{5.11}$$

but at short distance it does not diverge in contrast to the ordinary Coulomb potential:

$$V_{\rm D}(\mathbf{x}) \to \frac{e^2\sqrt{2\pi}}{8\pi\varepsilon l_B} \quad \text{as} \quad |\mathbf{x}| \to 0. \tag{5.12}$$

This is physically reasonable because a real electron cannot be localized to a point within the LLL. Regularity (5.12) of the potential is attributed to the exponential suppression factor in Eq. (5.9), whose origin is the suppression factor in LLL projection (3.4).

We consider a local SU(N) isospin rotation of electrons. Without the LLL projection, since the isospin generator commutes with the density operator, it does not affect the Coulomb energy [Eq. (5.1)] but increases a kinetic energy. However, since the kinetic energy is very large in high magnetic field, it is energetically favorable for electrons to stay within the LLL. That is, the confinement of electrons to the LLL occurs dynamically. This dynamical effect is taken into account automatically by making the LLL projection.

With LLL condition (3.7), kinetic Hamiltonian (3.6) is quenched, $H_{\rm K}|\mathfrak{S}\rangle = 0$ for $|\mathfrak{S}\rangle \in \mathbb{H}_{\rm LLL}$. Thus, a local SU(N) isospin rotation takes place without requiring a kinetic energy within the LLL. However, it turns out to increase the Coulomb energy [Eq. (5.7)] because an isospin rotation modulates the electron density according to algebra (4.14). It has been argued[10,17,23] that this results in the increase of the exchange Coulomb energy and leads to a physics associated with quantum coherence. However, the Hamiltonian has not yet been obtained in a closed form, and it would be impossible in previous methods. In one method,[10,17] the effective Hamiltonian was extracted by evaluating the Coulomb energy of a sufficiently smooth spin texture. In another method,[23] it was constructed by taking a continuum limit of the Landau-site Hamiltonian. In these methods it was difficult to calculate higher order corrections systematically. In this section we propose an algebraic analysis to overcome this problem.

The key observation is that the projected Coulomb Hamiltonian [Eq. (5.2)] can be represented in an entirely different form. Making use of the relation

$$\sum_{A}^{N^2-1} \lambda^A_{\sigma\tau}\lambda^A_{\alpha\beta} = 2\left(\delta_{\sigma\beta}\delta_{\tau\alpha} - \frac{1}{N}\delta_{\sigma\tau}\delta_{\alpha\beta}\right), \tag{5.13}$$

we rewrite Eq. (5.3) as

$$\hat{H}_{\rm C} = -2\sum_{mnij} V_{mnij}\left[S^A(m,j)S^A(i,n) + \frac{1}{2N}\rho(m,j)\rho(i,n)\right]. \tag{5.14}$$

We substitute Eqs. (4.9), (4.10) and (5.5) into it, and use the relation

$$\sum_n \langle n|e^{-i\mathbf{X}\mathbf{k}}e^{i\mathbf{X}\mathbf{p}}e^{i\mathbf{X}\mathbf{k}}e^{i\mathbf{X}\mathbf{q}}|n\rangle = \frac{2\pi}{l_B^2}\delta(\mathbf{p}+\mathbf{q})\exp(il_B^2\mathbf{p}\wedge\mathbf{k}), \tag{5.15}$$

which follows from Eqs. (2.12) and (2.13). We obtain

$$\hat{H}_{\rm C} = -\int d^2p\, J(\mathbf{p})\left[\hat{\mathbf{S}}(-\mathbf{p})\hat{\mathbf{S}}(\mathbf{p}) + \frac{1}{2N}\hat{\rho}(-\mathbf{p})\hat{\rho}(\mathbf{p})\right], \tag{5.16}$$

with

$$J(\mathbf{p}) = l_B^2\int d^2k\, e^{-l_B^2k^2/2}V(\mathbf{k})\exp(-il_B^2\mathbf{p}\wedge\mathbf{k}) = \frac{e^2\sqrt{2\pi}l_B}{4\varepsilon}I_0(l_B^2\mathbf{p}^2/4)e^{-l_B^2\mathbf{p}^2/4}. \tag{5.17}$$

We express the Hamiltonian in the coordinate space,

$$\hat{H}_{\rm C} = -\int d^2x d^2y\, V_{\rm X}(\mathbf{x}-\mathbf{y})\left[\hat{\mathbf{S}}(\mathbf{x})\hat{\mathbf{S}}(\mathbf{y}) + \frac{1}{2N}\hat{\rho}(\mathbf{x})\hat{\rho}(\mathbf{y})\right], \tag{5.18}$$

where

$$V_{\rm X}(\mathbf{x}) = \frac{e^2 l_B}{8\pi\sqrt{2\pi}\varepsilon}\int d^2p\, e^{i\mathbf{p}\mathbf{x}}e^{-l_B^2\mathbf{p}^2/4}I_0(l_B^2\mathbf{p}^2/4) = V(\mathbf{x})e^{-\mathbf{x}^2/2l_B^2}. \tag{5.19}$$

This exhibits the short-range property characteristic to the exchange Coulomb interaction.

It is worthwhile to mention that we are unable to write Hamiltonian (5.18) in terms of the projected densities $\overline{\rho}(\mathbf{x})$ and $\overline{S^A}(\mathbf{x})$. If we dare to do so, we would obtain

$$\hat{H}_{\rm C} = -\int d^2x d^2y\, \overline{V}_{\rm X}(\mathbf{x}-\mathbf{y})\left[\overline{\mathbf{S}}(\mathbf{x})\overline{\mathbf{S}}(\mathbf{y}) + \frac{1}{2N}\overline{\rho}(\mathbf{x})\overline{\rho}(\mathbf{y})\right], \tag{5.20}$$

with

$$\overline{V}_{\rm X}(\mathbf{x}) = \frac{e^2 l_B}{8\pi\sqrt{2\pi}\varepsilon}\int d^2p\, e^{i\mathbf{p}\mathbf{x}}e^{l_B^2\mathbf{p}^2/4}I_0(l_B^2\mathbf{p}^2/4). \tag{5.21}$$

However, this is divergent partially due to the factor $e^{l_B^2\mathbf{p}^2/4}$. Thus it is necessary to use the Weyl-ordered density operators rather than the projected ones to describe physics in the LLL.

VI. EFFECTIVE HAMILTONIAN

We have derived two expressions [Eqs. (5.7) and (5.18)] for the same Hamiltonian. They are equivalent but the physical picture is very different. The potential $V_{\rm D}(\mathbf{x})$ in Eq. (5.7) is long-ranged, while $V_{\rm X}(\mathbf{x})$ in Eq. (5.18) is short-ranged.

In this paper we analyze physics and long-distance scale. The long-distance limit corresponds to the limit $l_B \to 0$. We may replace the densities with the corresponding ones in the

commutative limit, $\hat{\rho}(\mathbf{x})\to\rho(\mathbf{x})$ and $\hat{\mathbf{S}}(\mathbf{x})\to\mathbf{S}(\mathbf{x})$, with $\rho(\mathbf{x})$ and $\mathbf{S}(\mathbf{x})$ being the commutative fields. Hamiltonian (5.7) yields $\hat{H}_{\rm C}\to H_{\rm D}^{\rm eff}$, where

$$H_{\rm D}^{\rm eff}=\frac{1}{2}\int d^2x d^2y\, V_{\rm D}(\mathbf{x}-\mathbf{y})\rho(\mathbf{x})\rho(\mathbf{y}), \tag{6.1}$$

with $\rho(\mathbf{x})$ being the commutative field. It represents the direct Coulomb interaction. On the other hand, Hamiltonian (5.18) yield, $\hat{H}_{\rm C}\to H_{\rm X}^{\rm eff}$. We make the derivative expansion for a smooth configuration. The first nontrivial term is

$$H_{\rm X}^{\rm eff}=\frac{2J_s}{\rho_\Phi^2}\int d^2x\left[\partial_k\mathbf{S}(\mathbf{x})\partial_k\mathbf{S}(\mathbf{x})+\frac{1}{2N}\partial_k\rho(\mathbf{x})\partial_k\rho(\mathbf{x})\right], \tag{6.2}$$

where J_s is the stiffness parameter defined by

$$J_s=\frac{1}{16\sqrt{2\pi}}\frac{e^2}{4\pi\varepsilon l_B}. \tag{6.3}$$

We have used the relation $\nu=2\pi l_B^2\rho_0$ for the filling factor with ρ_0 the electron density. It describes the exchange Coulomb interaction.[23] Stiffness (6.3) agrees with the previous result.[10,17]

The two Hamiltonians (5.7) and (5.18) are equivalent when all terms are included. It is intriguing, however, that they have yielded different effective Hamiltonians in the commutative limit. They describe entirely different physical effects and they are complementary. Taking the direct and exchange interactions we construct the full effective theory,[10,17,23]

$$H^{\rm eff}=H_{\rm D}^{\rm eff}+H_{\rm X}^{\rm eff}, \tag{6.4}$$

with Eqs. (6.1) and (6.2).

In the rest of this section we explain why we take sum (6.4) as the effective Hamiltonian. In previous sections we have worked in Fock representation (2.4) of noncommutativity (1.1). As far as we can make an exact analysis, the results are independent of the choice of representation. However, to derive an effective theory, it is necessary to make a judicious choice to reveal the essence of the approximation. For the present purpose it is convenient to adopt the von Neumann lattice representation[23] of noncommutativity (1.1), where the Landau-site index n runs over a lattice with the lattice point being the center of the cyclotron motion.

We introduce an eigenstate of the annihilation operator b given by Eq. (2.3):

$$b|\beta\rangle=\beta|\beta\rangle. \tag{6.5}$$

It is a coherent state by definition,

$$|\beta\rangle\equiv e^{\beta b^\dagger-\beta^* b}|0\rangle=e^{-|\beta|^2/2}e^{\beta b^\dagger}|0\rangle, \tag{6.6}$$

where $|0\rangle$ is the Fock vacuum obeying $b|0\rangle=0$. The wave function $\mathfrak{S}_\beta(\mathbf{x})=\langle\mathbf{x}|\beta\rangle$ is calculated as

$$\mathfrak{S}_\beta(\mathbf{x})=\frac{1}{\sqrt{2\pi l_B^2}}\exp\left(-\frac{|z-\sqrt{2}l_B\beta|^2}{4l_B^2}+\frac{i(y\beta_{\mathfrak{R}}-x\beta_{\mathfrak{I}})}{\sqrt{2}l_B}\right), \tag{6.7}$$

where $\beta=\beta_{\mathfrak{R}}-i\beta_{\mathfrak{I}}$. This describes an electron localized around the point $z=\sqrt{2}l_B\beta$.

The coherent state has a minimum uncertainty subject to the Heisenberg uncertainty associated with noncommutativity (1.1). The state $|\beta\rangle$ corresponds to the classical state describing a cyclotron motion around the point

$$x_\beta=\sqrt{2}l_B\beta_{\mathfrak{R}},\quad y_\beta=\sqrt{2}l_B\beta_{\mathfrak{I}}, \tag{6.8}$$

as follows from Eqs. (2.3) and (6.5). Since each electron occupies an area $2\pi l_B^2$, it is reasonable to choose a lattice with the unit cell area $2\pi l_B^2$. Then there is a one-to-one correspondence between the magnetic flux quantum and the lattice site. Such a lattice is nothing but a von Neumann lattice.[24–27] The states on a von Neumann lattice form a minimum complete set[25,26] in the lowest Landau level. Thus we may expand the electron field in terms of coherent states $\langle\mathbf{x}|\beta_n\rangle$ as in Eq. (3.8), where n runs over all lattice points.

The merit of this representation is that the wave function $\langle\mathbf{x}|\beta_n\rangle$ is nonvanishing only in a tiny region around the lattice point $\mathbf{x}_\beta$ in the limit $l_B\to 0$. The projected density [Eq. (3.11)] is well approximated by

$$\overline{\rho}(\mathbf{x})\simeq\sum_n\langle\beta_n|\mathbf{x}\rangle\langle\mathbf{x}|\beta_n\rangle\rho(\beta_n,\beta_n). \tag{6.9}$$

Consequently, the Weyl-ordered densities [Eqs. (4.3) and (4.4)] are well approximated by

$$\hat{\rho}(\mathbf{q})\simeq\frac{1}{2\pi}\sum_n\langle\beta_n|e^{-i\mathbf{q}\mathbf{X}}|\beta_n\rangle\rho(\beta_n,\beta_n), \tag{6.10}$$

$$\hat{S}^A(\mathbf{q})\simeq\frac{1}{2\pi}\sum_n\langle\beta_n|e^{-i\mathbf{q}\mathbf{X}}|\beta_n\rangle S^A(\beta_n,\beta_n). \tag{6.11}$$

The main contribution to $\hat{\rho}(\mathbf{x})$ and $\hat{S}^A(\mathbf{x})$ come from the electrons in one Landau site $|\beta\rangle$ containing the position $\mathbf{x}$. With this approximation $\hat{\rho}$ and $\hat{S}^A$ satisfy the U(N) algebra rather than the W_∞(N) algebra. Hence, they correspond to the densities in the commutative limit.

We now examine Coulomb energy (5.3), or

$$\hat{H}_{\rm C}=\sum_{mnij}\sum_{\sigma\tau}V_{mnij}c_\sigma^\dagger(\beta_m)c_\tau^\dagger(\beta_i)c_\tau(\beta_j)c_\sigma(\beta_n), \tag{6.12}$$

where the indices m,n,i,j run over the lattice points. In a semiclassical approximation the matrix element matters. It vanishes unless $\beta_m=\beta_n$ and $\beta_i=\beta_j$, or $\beta_m=\beta_j$ and $\beta_i=\beta_n$. These two terms represent the direct and exchange Coulomb interactions, respectively, which are the dominant ones in Eq. (6.12). We may summarize them as[23]

$$\hat{H}_{\rm D}=\sum_{mi}V_{mmii}\rho(\beta_m,\beta_m)\rho(\beta_i,\beta_i) \tag{6.13}$$

and

$$\hat{H}_{\mathrm{X}}=-2\sum_{mi}\ V_{miim}\Big[S^{A}(\beta_m,\beta_m)S^{A}(\beta_i,\beta_i) + \frac{1}{2N}\rho(\beta_m,\beta_m)\rho(\beta_i,\beta_i)\Big], \tag{6.14}$$

which have no parts in common by construction. They are the special parts of the two equivalent and exact Hamiltonians [Eqs. (5.6) and (5.14)]. Furthermore, it is clear from our arguments that they are reduced to Eqs. (6.13) and (6.14) in the commutative limit ($l_B\to 0$). Hence, we take Eq. (6.4) as the effective Hamiltonian in the commutative limit.

VII. GOLDSTONE MODES

It is convenient to study quantum coherence based on Hamiltonian (5.16). It is minimized by the uniform configuration of the isospin as well as the density;

$$\hat{\rho}(\mathbf{p})=2\pi\rho_0\delta(\mathbf{p}),\quad \hat{\mathbf{S}}(\mathbf{p})=2\pi\mathbf{S}_0\delta(\mathbf{p}). \tag{7.1}$$

Consequently, all isospins are spontaneously polarized into one isospin direction. In the zero-momentum sector the W_∞(N) algebra [Eq. (4.14)] is reduced to the U(N) algebra,

$$[\hat{\rho}_0,\hat{\rho}_0]=0,\quad [\hat{S}_0^A,\hat{\rho}_0]=0,\quad [\hat{S}_0^A,\hat{S}_0^B]=\frac{i}{2\pi}f^{ABC}\hat{S}_0^C, \tag{7.2a}$$

where $\hat{\rho}_0=\hat{\rho}(\mathbf{p}=0)$ and $\hat{S}_0^A=\hat{S}^A(\mathbf{p}=0)$.

The ground state is characterized by the algebra U(N) rather than W_∞(N). At $\nu=1$, there are N degenerate isospin states any one of which may be spontaneously filled up to make a ground state. At $\nu=k$, k of the N degenerate states are occupied and $(N-k)$ of them are empty. Hence, the unbroken global symmetry is $\mathrm{SU}(k)\otimes\mathrm{SU}(\mathrm{N}-k)\otimes\mathrm{U}(1)$, implying a spontaneous breaking of the global SU(N) symmetry:

$$\mathrm{SU(N)}\to\mathrm{SU}(k)\otimes\mathrm{SU}(\mathrm{N}-k)\otimes\mathrm{U}(1). \tag{7.3}$$

The target space is the coset space,

$$G_{N,k}=\mathrm{SU(N)}/[\mathrm{SU(k)}\otimes\mathrm{SU}(\mathrm{N}-k)\otimes\mathrm{U}(1)], \tag{7.4}$$

which is known as the Grassmannian $G_{N,k}$ manifold. Its real dimension is $N^2-k^2-(N-k)^2=2k(N-k)$. We expect $k(N-k)$ complex Goldstone modes to appear as a result of this spontaneous symmetry breaking. Note that $G_{N,k}=G_{N,N-k}$ as a manifold. Hence the physics at $\nu=k$ and $\nu=N-k$ is identical. It is enough to study the case for $\nu\leqslant N/2$.

We analyze the Goldstone modes based on effective Hamiltonian (6.4). Because the QH system is incompressible,[5,9] we may set $\rho(\mathbf{x})=\rho_0$ as far as perturbational fluctuations are concerned. When we define the normalized isospin field $\mathcal{S}^A(\mathbf{x})$ by

$$S^A(\mathbf{x})=\rho(\mathbf{x})\mathcal{S}^A(\mathbf{x}), \tag{7.5}$$

Hamiltonian (6.4) yields

$$H^{\mathrm{eff}}=2J_s\sum_A\ \int d^2x[\partial_k\mathcal{S}^A(\mathbf{x})]^2, \tag{7.6}$$

up to the leading order in the derivative expansion. This is the SU(N) nonlinear sigma model.

We first study the filling factor $\nu=1$. It is convenient to use the composite boson (CB) theory of quantum Hall ferromagnets[28] to identify the dynamic degree of freedom. The CB field $\phi^\sigma(\mathbf{x})$ is defined by making a singular phase transformation[29] to the electron field $\psi^\sigma(\mathbf{x})$,

$$\phi^\sigma(\mathbf{x})=e^{-ie\Theta(\mathbf{x})}\psi^\sigma(\mathbf{x}), \tag{7.7}$$

where the phase field $\Theta(\mathbf{x})$ attaches one flux quantum to each electron via the relation

$$\varepsilon_{ij}\partial_i\partial_j\Theta(\mathbf{x})=\Phi_{\mathrm{D}}\rho(\mathbf{x}). \tag{7.8}$$

We then introduce the normalized CB field $n^\sigma(\mathbf{x})$ by

$$\phi^\sigma(\mathbf{x})=\phi(\mathbf{x})n^\sigma(\mathbf{x}), \tag{7.9}$$

where the N-component field $n^\sigma(\mathbf{x})$ obeys the constraint $\Sigma_\sigma n^{\sigma\dagger}(\mathbf{x})n^\sigma(\mathbf{x})=1$: Such a field is the CP^{N-1} field.[30] On the other hand, $\phi^*(\mathbf{x})\phi(\mathbf{x})=\rho(\mathbf{x})$ for the U(1) field $\phi(\mathbf{x})$.

Formula (7.9) is interpreted as a charge-isospin separation. Indeed, by substituting Eq. (7.7) together with Eq. (7.9) into the kinetic Hamiltonian [Eq. (3.6)], the electromagnetic field $A_k(\mathbf{x})$ is found to be coupled only with the U(1) field $\phi(\mathbf{x})$. Thus, the charge is carried by $\phi(\mathbf{x})$, while the isospin is carried by $n^\sigma(\mathbf{x})$.

In terms of the CP^{N-1} field $\mathbf{n}(\mathbf{x})$, the isospin $\mathcal{S}^A(\mathbf{x})$ field reads

$$\mathcal{S}^A(\mathbf{x})=\frac{1}{2}\mathbf{n}^\dagger(\mathbf{x})\lambda^A\mathbf{n}(\mathbf{x}), \tag{7.10}$$

with which Hamiltonian (7.6) is equivalent to

$$H^{\mathrm{eff}}=2J_s\int d^2x(\partial_j\mathbf{n}^\dagger+iK_j\mathbf{n}^\dagger)\cdot(\partial_j\mathbf{n}-iK_j\mathbf{n}), \tag{7.11}$$

where $K_\mu(\mathbf{x})=-i\mathbf{n}^\dagger(\mathbf{x})\partial_\mu\mathbf{n}(\mathbf{x})$. The field K_μ is not a dynamic field because of the absence of the kinetic term. The N-component field $\mathbf{n}(\mathbf{x})$ has $N-1$ independent complex components: They are the Goldstone modes.

There are N degenerate states any one of which can be chosen as the ground state. For definiteness, let us choose

$$\mathbf{n}_{\mathrm{g}}(\mathbf{x})=(1,0,\ldots,0) \tag{7.12}$$

as a ground state. The CP^{N-1} field is parametrized as

$$\mathbf{n}(\mathbf{x})=(1,\eta_1,\ldots,\eta_{N-1}) \tag{7.13}$$

up to the lowest order of perturbation, where η_i are the $N-1$ Goldstone modes.

We next study the case $\nu=k$. To describe k electrons in one Landau site we introduce k normalized CB fields $n_i^\sigma(\mathbf{x})$. They should be orthogonal one to another,

$$\mathbf{n}_i^\dagger(\mathbf{x})\cdot\mathbf{n}_j(\mathbf{x})=\delta_{ij}, \tag{7.14}$$

because they are not ordinary bosons but hard-core bosons. (Two hard-core bosons never occupy a single quantum state, just like electrons subject to the Pauli exclusion principle.) We then construct an $N\times k$ matrix field

$$Z(\mathbf{x})=(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_k), \tag{7.15}$$

using a set of k fields subject to this normalization condition, or

$$Z^\dagger Z=1. \tag{7.16}$$

Though we have introduced k fields $\mathbf{n}_i(\mathbf{x})$, we cannot distinguish them quantum mechanically since they describe k identical electrons in the same Landau site. That is, two fields $Z(\mathbf{x})$ and $Z'(\mathbf{x})$ are indistinguishable physically when they are related by a local U(k) transformation $U(\mathbf{x})$:

$$Z'(\mathbf{x})=Z(\mathbf{x})U(\mathbf{x}). \tag{7.17}$$

By identifying these two fields $Z(\mathbf{x})$ and $Z'(\mathbf{x})$, the $N\times k$ matrix field $Z(\mathbf{x})$ takes values on the Grassmannian manifold $G_{N,k}$ defined by Eq. (7.4). The field $Z(\mathbf{x})$ is no longer a set of k independent CP^{N-1} fields. It is a new object, called the Grassmannian field, carrying $k(N-k)$ complex degrees of freedom.

At $\nu=k$ the isospin field $\mathcal{S}(\mathbf{x})$ is represented in terms of the Grassmannian $G_{N,k}$ field $Z(\mathbf{x})$ as

$$\mathcal{S}^A(\mathbf{x})=\mathrm{Tr}\left[Z^\dagger(\mathbf{x})\frac{\lambda^A}{2}Z(\mathbf{x})\right]=\frac{1}{2}\sum_i \mathbf{n}_i^\dagger(\mathbf{x})\lambda^A\mathbf{n}_i(\mathbf{x}). \tag{7.18}$$

This is a simple sum of isospins of k electrons in one Landau site. With this identification we are able to rewrite the SU(N) sigma-model Hamiltonian [Eq. (7.6)] as

$$\mathcal{H}^{\mathrm{eff}}=2J_s\mathrm{Tr}[(\partial_j Z-iZK_j)^\dagger(\partial_j Z-iZK_j)], \tag{7.19}$$

where

$$K_\mu(\mathbf{x})=-iZ^\dagger(\mathbf{x})\partial_\mu Z(\mathbf{x}). \tag{7.20}$$

This Hamiltonian is known as the Grassmannian sigma-model Hamiltonian.[31] It has the local U(k) gauge symmetry

$$Z(\mathbf{x})\to Z(\mathbf{x})U(\mathbf{x}),$$

$$K_\mu(\mathbf{x})\to U(\mathbf{x})^\dagger K_\mu(\mathbf{x})U(\mathbf{x})-iU(\mathbf{x})^\dagger\partial_\mu U(\mathbf{x}). \tag{7.21}$$

The gauge field K_μ is not a dynamic field because of the absence of the kinetic term.

The $G_{N,k}$ field has $k(N-k)$ independent complex components: They are the Goldstone modes η_{ij} parametrized as

$$Z=\begin{pmatrix} 1 & 0 & \cdots & 0\\ 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 1\\ \eta_{1,1} & \eta_{2,1} & \cdots & \eta_{k,1}\\ \vdots & \vdots & \vdots & \vdots\\ \eta_{1,N-k} & \eta_{2,N-k} & \cdots & \eta_{k,N-k}\end{pmatrix}U(\mathbf{x}), \tag{7.22}$$

up to the lowest order of perturbation. The corresponding ground state Z_{g} is given by setting $\eta_{ij}=0$ in Eq. (7.22). We make a gauge choice such that $U(\mathbf{x})=1$ in Eq. (7.22). Substituting parametrization (7.22) of the Grassmannian field into Hamiltonian (7.19), we expand it up to the second order,

$$\mathcal{H}^{\mathrm{eff}}=2J_s\sum_{s=1}^{k(N-k)}\partial_k\eta_s^\dagger(\mathbf{x})\partial_k\eta_s(\mathbf{x})$$

$$=\frac{J_s}{2}\sum_{s=1}^{k(N-k)}\{(\partial_k\sigma_s)^2+(\partial_k\vartheta_s)^2\}, \tag{7.23}$$

where $\eta_s(\mathbf{x})$ stands for $\eta_{ij}(\mathbf{x})$ and

$$\eta_s(\mathbf{x})=\frac{1}{2}[\sigma_s(\mathbf{x})+i\vartheta_s(\mathbf{x})], \tag{7.24}$$

with

$$[\sigma_s(\mathbf{x}),\vartheta_t(\mathbf{y})]=2i\rho_\Phi^{-1}\delta_{st}\delta(\mathbf{x}-\mathbf{y}). \tag{7.25}$$

Here, $\rho_\Phi=(1/k)\rho_0=1/(2\pi l_B^2)$ is the magnetic flux density, that is, the density of Landau sites. This Hamiltonian realizes the SU(N) symmetry nonlinearly.

VIII. GRASSMANNIAN $G_{N,k}$ SOLITONS

The existence of topological solitons, which we call $G_{N,k}$ solitons, is guaranteed by the homotopy theorem

$$\pi_2(G_{N,k})=\pi_1[\mathrm{U}(1)]=\mathbb{Z}, \tag{8.1}$$

which follows from Eq. (7.4), where we have used $\pi_2(G/H)=\pi_1(H)$ (when G is simply connected) and $\pi_n(G\otimes G')=\pi_n(G)\oplus\pi_n(G')$. The topological charge is defined[31] as a gauge invariant by

$$Q=\frac{i}{2\pi}\int d^2x\,\epsilon_{jk}\mathrm{Tr}[(\partial_j Z-iZK_j)^\dagger(\partial_k Z-iZK_k)]. \tag{8.2}$$

It is a topological invariant since it is the charge of the topological current, $Q=\int d^2x J^0_{\mathrm{sol}}(\mathbf{x})$, with

$$J^\mu_{\mathrm{sol}}(\mathbf{x})=\frac{i}{2\pi}\epsilon^{\mu\nu\lambda}\mathrm{Tr}[(\partial_\nu Z)^\dagger(\partial_\lambda Z)]. \tag{8.3}$$

Based on Eq. (7.15) we rewrite it as

$$J^\mu_{\mathrm{sol}}(\mathbf{x})=\frac{i}{2\pi}\sum_{i=1}^{k}\epsilon^{\mu\nu\lambda}(\partial_\nu\mathbf{n}_i)^\dagger\cdot(\partial_\lambda\mathbf{n}_i). \tag{8.4}$$

It is the sum of the topological charges associated with the k CP^{N-1} fields $\mathbf{n}_i$. Hence the $G_{N,k}$ soliton consists of k CP^{N-1} solitons,

$$n_i^\sigma(\mathbf{x}) = \frac{1}{\sqrt{\sum_{\tau=1}^{N} \omega_i^\tau(z)|}} \omega_i^\sigma(z), \tag{8.5}$$

where $\omega_i^\sigma(z)$ are arbitrary analytic functions.

Grassmannian solitons are constructed as classical configurations, as dictated by the homotopy theorem [Eq. (8.1)]. They are BPS states of the Grassmannian sigma model [Eq. (7.19)]. Indeed, the following inequality holds[31] between the exchange energy [Eq. (7.19)] and the topological charge [Eq. (8.2)],

$$H^{\text{eff}} \geqq 4\pi J_s Q, \tag{8.6}$$

where the equality is achieved by the $G_{N,k}$ soliton.

The simplest soliton would be a set of one CP^3 soliton in one component and the ground state in all others. An example reads

$$Z^1 = \frac{1}{\sqrt{|z|^2+\kappa^2}} \begin{pmatrix} z & 0 & \cdots & 0 \\ 0 & \sqrt{|z|^2+\kappa^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{|z|^2+\kappa^2} \\ \kappa & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{8.7}$$

for which the topological charge [Eq. (8.2)] is $Q=1$. We argue in the next section that the simplest $G_{N,k}$ soliton [Eq. (8.7)] is ruled out since it is not confined to the LLL. As we shall see, the simplest allowed one is a set of kCP^{N-1} solitons with $Q=k$ when $k \leqslant \frac{1}{2}N$. We give an instance of a $G_{5,2}$ soliton,

$$Z^2 = \frac{1}{\sqrt{|z|^2+\kappa^2}} \begin{pmatrix} z & 0 \\ 0 & z \\ \kappa & 0 \\ 0 & \kappa \\ 0 & 0 \end{pmatrix},$$

for which $Q=2$.

IX. CHARGE-ISOSPIN RELATION

We have found topological solitons in effective Hamiltonian (7.6). However, it determines only the isospin part of the excitation in the charge-isospin separation formula [Eq. (7.9)]. It is necessary to analyze how the isospin modulation affects the charge part. This can be done by requiring LLL condition (3.7) on soliton states with a combined charge-isospin modulation.[28] Recall that the effective Hamiltonian is an ordinary local field theory, in which we have made the charge-isospin separation. Hence it is necessary to examine whether these solitons are confined to the LLL.

The LLL condition becomes particularly simple in the dressed-CB picture[5] in the symmetric gauge with the angular-momentum state for the Landau site. We define the dressed-CB field $\varphi^\sigma(\mathbf{x})$ by[28,32]

$$\varphi^\sigma(\mathbf{x}) = e^{-\mathcal{A}(\mathbf{x})}\phi^\sigma(\mathbf{x}) = e^{-\mathcal{A}(\mathbf{x})}\sqrt{\rho(\mathbf{x})}n^\sigma(\mathbf{x}), \tag{9.1}$$

where $\phi^\sigma(\mathbf{x})$ is the CB field [Eq. (7.7)] with the U(1) phase factor removed; the auxiliary field $\mathcal{A}(\mathbf{x})$ is determined by

$$\nabla^2 \mathcal{A}(\mathbf{x}) = 2\pi[\rho(\mathbf{x}) - \rho_0], \tag{9.2}$$

as follows from condition (7.8) on the phase field. It is straightforward to rewrite LLL condition (3.7) as

$$\frac{\partial}{\partial z^*}\varphi^\sigma(\mathbf{x})|\mathfrak{S}\rangle = 0. \tag{9.3}$$

We take a coherent state of $\varphi^\sigma(\mathbf{x})$, for which Eq. (9.3) implies

$$\varphi^\sigma(\mathbf{x})|\mathfrak{S}\rangle = \omega^\sigma(z)|\mathfrak{S}\rangle, \tag{9.4}$$

where $\omega^\sigma(z)$ is an analytic function. The coherent state $|\mathfrak{S}\rangle$ must be an eigenstate of the density operator $\rho(\mathbf{x})$ and a coherent state of the CP^3 field $\mathbf{n}(\mathbf{x})$ since they commute with each other. Hence we have

$$e^{-\mathcal{A}^{\text{cl}}(\mathbf{x})}\sqrt{\rho^{\text{cl}}(\mathbf{x})}n^{\text{cl}(\sigma)}(\mathbf{x}) = \omega^\sigma(z), \tag{9.5}$$

where $\mathcal{A}^{\text{cl}}(\mathbf{x})$, $\rho^{\text{cl}}(\mathbf{x})$, and $n^{\text{cl}(\sigma)}(\mathbf{x})$ are classical fields. The holomorphicity of $\omega^\sigma(z)$ is a consequence of the requirement that the excitation is confined to the LLL. This is the LLL condition for soliton states.

We study the $G_{N,k}$ soliton at $\nu=k$. When the CB field acts on the state at $\nu=k$, it picks up contributions from k electrons at each point,

$$\mathbf{n}(\mathbf{x})|\Phi\rangle = \mathbf{n}^{\text{cl}}(\mathbf{x})|\Phi\rangle = \sum_{i=1}^{k} \mathbf{n}_i^{\text{cl}}(\mathbf{x})|\Phi\rangle, \tag{9.6}$$

together with $\mathbf{n}_i^{\text{cl}}(\mathbf{x}) \cdot \mathbf{n}_j^{\text{cl}}(\mathbf{x}) = \delta_{ij}$ and

$$\rho(\mathbf{x})|\Phi\rangle = \rho^{\text{cl}}(\mathbf{x})|\Phi\rangle = \sum_{i=1}^{k} \rho_i^{\text{cl}}(\mathbf{x})|\Phi\rangle. \tag{9.7}$$

We may solve Eq. (9.5) as

$$n^{\text{cl}(\sigma)}(\mathbf{x}) = \frac{\sqrt{k}}{\sqrt{\sum_{\tau=1}^{N} |\omega^\tau(x)|^2}} \omega^\sigma(z). \tag{9.8}$$

From Eqs. (8.5), (9.6), and (9.8) we find $\omega^\sigma(z) = \Sigma_{i=1}^k \omega_i^\sigma(z)$ and $\Sigma_\sigma|\omega_1^\sigma(z)|^2 = \Sigma_\sigma|\omega_2^\sigma(z)|^2 = \cdots = \Sigma_\sigma|\omega_k^\sigma(z)|^2$. Thus, the LLL condition [Eq. (9.5)] holds for each component,

$$e^{-\mathcal{A}^{\text{cl}}(\mathbf{x})}\sqrt{\rho^{\text{cl}}(\mathbf{x})}n_i^{\text{cl}(\sigma)}(\mathbf{x}) = \omega_i^\sigma(z), \tag{9.9}$$

where $\rho^{\rm cl}(\mathbf{x})$ represents the total density [Eq. (9.7)]. Substituting Eq. (8.5) into Eq. (9.9) and using Eq. (9.2) we derive the soliton equation

$$\frac{1}{4\pi}\nabla^2 \ln \rho^{\rm cl}(\mathbf{x}) - \rho^{\rm cl}(\mathbf{x}) + \rho_0 = j^0_{\rm sol}(\mathbf{x}), \tag{9.10}$$

where

$$j^0_{\rm sol}(\mathbf{x}) = \frac{1}{4\pi}\nabla^2 \ln \sum_{\sigma=1}^{N} |\omega_1^\sigma(z)|^2. \tag{9.11}$$

It is easy to see that the topological charge density [Eq. (8.4)] is given by $J^0_{\rm sol}(\mathbf{x}) = k j^0_{\rm sol}(\mathbf{x})$.

Soliton equation (9.10) is solved iteratively. The density modulation is given by

$$\delta\rho^{\rm cl}(\mathbf{x}) = \rho^{\rm cl}(\mathbf{x}) - \rho_0 = -J^0_{\rm sol}(\mathbf{x}) + \cdots. \tag{9.12}$$

The iteration corresponds to the derivative expansion. The leading term is precisely the topological charge density. The electric charge density modulation is

$$\delta Q_e = -e\,\delta\rho^{\rm cl}(\mathbf{x}).$$

An isospin rotation turns out to induce the density modulation of the electric charge according to this formula. The total change of the charge due to a soliton excitation is given by $\int d^2x\ \delta Q_e = ke$.

We have found that k CP^{N-1} fields [Eq. (8.5)] have the same normalization. That is, among many soliton configurations in the $G_{N,k}$ sigma model, only a special type of configurations are allowed by requiring the LLL condition. The $G_{N,k}$ soliton has a general expression:

$$Z^1 = \frac{1}{\sqrt{\sum_\sigma |\omega_1^\sigma(z)|^2}} \begin{pmatrix} \omega_1^1(z) & \omega_2^1(z) & \cdots & \omega_k^1(z) \\ \omega_1^2(z) & \omega_2^2(z) & \cdots & \omega_k^2(z) \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^k(z) & \omega_2^k(z) & \cdots & \omega_k^k(z) \\ \omega_1^{k+1}(z) & \omega_2^{k+1}(z) & \cdots & \omega_k^{k+1}(z) \\ \vdots & \vdots & \vdots & \vdots \\ \omega_1^N(z) & \omega_2^N(z) & \cdots & \omega_k^N(z) \end{pmatrix}. \tag{9.13}$$

This rules out soliton (8.7) with $Q=1$.

The origin of this peculiarity may be attributed to charge-isospin separation formula (7.9), by way of which the normalized CB field $\mathbf{n}(\mathbf{x})$ is introduced. It is essential that the total electron density $\rho(\mathbf{x})$ is common to all the N components. Otherwise, the symmetry SU(N) is explicitly broken by hand. As a consequence, even if we try to excite a soliton only in one of the components, the density modulation associated with it affects equally electrons in other components to satisfy the LLL condition. It is impossible to have a soliton excitation only in one of the components.

X. APPLICATIONS

We have studied the dynamics of N-component electrons projected to the LLL. We have shown that the long-distance physics is described by the Grassmannian sigma model. Physically it is realized by multilayer QH systems. Various experiments have already been carried out not only on monolayer QH systems with spin ($N=2$) but also on bilayer QH systems with spin ($N=4$).

In the monolayer QH system with spin the effective Hamiltonian consists of the exchange term, the direct term and the Zeeman term. It is well approximated by

$$H^{\rm eff} = 2J_s \sum_{A=x,y,z} \int d^2x [\partial_k \mathcal{S}^A(\mathbf{x})]^2 + \rho_0 \Delta_{\rm Z} \int d^2x\ \mathcal{S}^z(\mathbf{x}) + \frac{1}{2}\int d^2x d^2y\ V_{\rm D}(\mathbf{x}-\mathbf{y})\rho(\mathbf{x})\rho(\mathbf{y}), \tag{10.1}$$

where $S^A(\mathbf{x})$ is the spin SU(2) field, and $\Delta_{\rm Z}$ is the energy gap between the one-particle up- and down-spin states due to the Zeeman effect. The direct Coulomb term is necessary since the soliton modulates the density $\rho(\mathbf{x})$ according to soliton equation (9.10). The system possesses one Goldstone mode, which is made massive by the Zeeman term. The topological soliton is the $G_{2,1}$ (CP^1) soliton, represented by the Grassmannian field [Eq. (9.13)] or

$$Z = \frac{1}{\sqrt{|z|^2+\kappa^2}} \begin{pmatrix} z \\ \kappa \end{pmatrix}, \tag{10.2}$$

where κ represents the size of the soliton. The exchange energy is independent of it. As κ increases, the direct Coulomb energy is decreased while the Zeeman energy increases. Thus it is determined to optimize these two energies.[11]

Let us discuss bilayer QH systems[5,23] somewhat in detail. They are very interesting because they exhibit various novel quantum coherent phenomena due to the rich isospin degree of freedom. An electron in bilayer QH systems is labeled by the spin SU(2) and the pseudospin SU(2) representing the layer degree of freedom. A group that accommodates the spin SU(2) and pseudospin SU(2) is SU(4), which is a good symmetry of the system when the two layers are placed close enough. It reminds us of the grand unified theory (GUT), where the standard-model gauge group SU(3)$\otimes$ SU(2) $\otimes$U(1) is incorporated into a larger group. However, there exists a big difference: In QH systems it is a *global* symmetry which is in problem, while in the GUT it is a *local* (gauge) symmetry. Thus, Goldstone bosons appear in QH systems, while some gauge bosons get massive by eating Goldstone bosons in the GUT.

In bilayer QH systems we introduce four-component spinors, and analyze spontaneous symmetry breaking, Goldstone bosons, and topological solitons. According to our general arguments, at the integer filling factor $\nu=k$, complex $k(4-k)$ Goldstone bosons appear to be accompanied by a

spontaneous breakdown of the global SU(4) symmetry. In actual systems the SU(4) symmetry is broken explicitly but only softly by sufficiently weak Zeeman or tunneling interactions. All Goldstone modes are made massive by these interactions. Topological solitons are Grassmannian $G_{4,k}$ solitons. The topological mapping is determined by the U(1) group, as follows from Eq. (8.1). It reminds us of the U(1) monopole in the context of the GUT, which appears when the GUT gauge group is broken to a subgroup including U(1) group.

The integer filling factor is up to 4 in the LLL. The nontrivial Grassmannian manifold is realized only at $\nu=1$, 2, and 3. We explain what we expect at these filling factors. At $\nu=1$, the breakdown pattern of the SU(4) symmetry is

$$\mathrm{SU}(4)\rightarrow \mathrm{U}(1)\otimes \mathrm{SU}(3). \tag{10.3}$$

There arise three complex pseudo-Goldstone modes. The 15 generators of SU(4) accommodate ''six'' different SU(2) generators. Some internal different SU(2) symmetries also break down when the SU(4) symmetry breaks down. These three Goldstone modes appear due to the symmetry breaking of ''three'' internal SU(2)'s. Topological excitations are $G_{4,1}(CP^3)$ solitons. The mode associated with the pseudospin SU(2) breaking induces the Josephson-like tunneling current,[16] whose signals have been detected experimentally.[15]

At $\nu=2$, the breakdown pattern of the SU(4) symmetry is

$$\mathrm{SU}(4)\rightarrow \mathrm{U}(1)\otimes \mathrm{SU}(2)\otimes \mathrm{SU}(2). \tag{10.4}$$

There arise four complex Goldstone modes. The topological excitations are $G_{4,2}$ solitons. In actual samples the degeneracy is resolved by the Zeeman and tunneling interactions. According to their relative strength we have two phases, where either the spin or the pseudospin is polarized. In the spin-polarized phase, one $G_{4,2}$ soliton flips twice as much spins as one CP^3 soliton does, whose specific features have already been observed experimentally:[33] see Ref. 18 for more details.

We have mentioned the property $G_{N,k}=G_{N,N-k}$ of the Grassmannian manifold. We can explain it based on a concrete example in bilayer QH systems. At $\nu=3$, there are three electrons in one Landau site. We may equivalently rephrase that there is one hole in one Landau site. Hence we may regard the system as a hole system at the hole filling factor $\nu_h=1$. Most discussions in the $\nu=1$ electron system go through to the $\nu_h=1$ hole system with the replacement of electrons by holes. The symmetry breaking pattern is the same as at $\nu=1$ and there arise three complex Goldstone modes. Solitons are $G_{4,1}$ (CP^3) solitons as in the $\nu=1$ case.

We proceed to discuss briefly the N-layer QH system, where tunneling interactions operate between two adjacent layers. It is described by the tunneling term consisting of an $N\times N$ matrix:

$$H_T=-\frac{1}{2}\Delta_{\mathrm{SAS}}\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}. \tag{10.5}$$

Let us first freeze the spin degree of freedom. By diagonalizing this matrix the degeneracies of the N energy levels are found to be resolved. The energy of the jth level is

$$E_j=\Delta_{\mathrm{SAS}}\cos\frac{\pi j}{N+1}, \quad j=1,2,\ldots,N, \tag{10.6}$$

as is shown in Appendix B. The lowest energy level is unique, which gives the ground state at $\nu=1$. At $\nu=k$ the lowest k levels are occupied. A Goldstone mode is a perturbational excitation from one of the k occupied levels to one of the $N-k$ empty levels. Thus the number of Goldstone modes is $k(N-k)$, which is the dimension of the Grassmannian manifold $G_{N,k}$. All Goldstone modes are made massive due to the tunneling interactions. When the spin degree of freedom is included together with the Zeeman interaction, each of these N levels is split into two levels by the Zeeman energy. All Goldstone modes are massive. At $\nu=k$ the lowest k levels are occupied: There arise k phases depending on the relative strength between the Zeeman and tunneling energies.

XI. DISCUSSIONS

We have developed an algebraic method to explore the dynamics of electrons in the noncommutative plane. For this purpose we have introduced the Weyl ordering of the second quantized density operator. By making a LLL projection we have constructed the Hamiltonian for these electrons interacting via the Coulomb potential. It is given by Eq. (5.16) in the momentum space and by Eq. (5.18) in the coordinate space. The density operators make the W_∞(N) algebra [Eq. (4.14)]. Then we have made a derivative expansion of the Coulomb potential and derived the effective Hamiltonian appropriate for a description of long-distance physics of electrons confined to the LLL. It is the SU(N) nonlinear sigma model [Eq. (7.6)].

The SU(N) nonlinear sigma model has arisen solely from the SU(N)-invariant Coulomb interaction [Eq. (5.1)] depending only on the total density $\rho(\mathbf{x})$. That is, a modulation of the isospin $S(\mathbf{x})$ turns out to increase the Coulomb energy by affecting the density $\rho(\mathbf{x})$. The origin of the effective Hamiltonian is noncommutativity (1.1), implying the density and the isospin no longer commute as in Eq. (4.14) when electrons are confined to the LLL. Effective Hamiltonian (7.6) is the leading order term of the underlying noncommutative theory.

The effective Hamiltonian turns out to be the Grassmannian $G_{N,k}$ sigma model at the filling factor $\nu=k$, based upon which we have explored quantum coherence in the N-component QH systems. We have analyzed the Goldstone modes and topological solitons. As is well known, the exis-

tence of massless modes in low-dimensional spaces induces an infrared catastrophe and unstabilizes the system. In QH systems there is no such a catastrophe because all Goldstone modes are made massive due to the Zeeman and tunneling interactions.

It is important to investigate how these perturbational and nonperturbational objects are represented in the original noncommutative field theory. We would like to pursue these problems in a forthcoming paper.

In this paper we have analyzed a simplified multicomponent QH system. In actual systems there are separations between the layers, which breaks the SU(N) symmetry explicitly. The SU(N)-breaking effect would be included as a perturbation. It has been argued[34] in an instance of the spin-frozen bilayer QH system that stiffness parameter (6.3) is then subject to a renormalization. It is interesting how the renormalization effect is formulated in the present framework.

It is also interesting to investigate fractional QH systems. They are mapped to integer QH systems by way of the composite-fermion picture.[35] Topological excitations in fractional QH systems are anyons, which have fractional electric charges and obey fractional statistics.[36] Such anyons have already been observed experimentally in a monolayer $\nu=1/3$ QH system.[37,38] The fractional statistics stems from the statistical transmutations specific to the low-dimensional system, and represent a deep connection between the space-time and particle statistics. The noncommutativity is also a space-time property. Then topological excitations in fractional QH systems are intriguing objects inherent to these two exotic space-time properties: They are noncommutative anyons. The study of the noncommutative anyons may reveal novel structures of the low-dimensional noncommutative space-time. The noncommutative gauge theory has been extensively studied in the context of *D* branes with a *B* field in the string theory. Various concepts cultivated in the *D*-brane analysis can be applied to noncommutative anyons and tested experimentally in QH systems.

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APPENDIX A: WEYL-ORDERED PLANE WAVE

We prove basic formulas (2.13) and (2.14) for the Weyl-ordered plane wave. It is convenient to diagonalize the coordinate X, $X|x\rangle = x|x\rangle$. Noncommutativity (1.1) is represented by setting

$$Y = il_B^2 \frac{\partial}{\partial X}. \tag{A1}$$

The merit of this representation is that the simple orthonormality condition holds within the LLL:

$$\langle x'|x\rangle = \delta(x-x'). \tag{A2}$$

Since Y is a shifting operator it is easy to show that

$$e^{i\mathbf{pX}}|x\rangle = \exp\left[-\frac{i}{2}l_B^2 p_x p_y\right]\exp[ip_x x]|x-l_B^2 p_y\rangle. \tag{A3}$$

Hence

$$\langle x'|e^{i\mathbf{pX}}|x\rangle = \exp\left[ip_x x - \frac{i}{2}l_B^2 p_x p_y\right]\delta(x-x'-l_B^2 p_y) \tag{A4a}$$

$$= \exp\left[ip_x x' + \frac{i}{2}l_B^2 p_x p_y\right]\delta(x-x'-l_B^2 p_y). \tag{A4b}$$

We set $x'=x$ and integrate over it:

$$\int dx\,\langle x|e^{i\mathbf{pX}}|x\rangle = \frac{2\pi}{l_B^2}\delta(\mathbf{p}). \tag{A5}$$

Substituting $\Sigma_n |n\rangle\langle n| = 1$, we obtain

$$\sum_n \langle n|e^{i\mathbf{pX}}|n\rangle = \frac{2\pi}{l_B^2}\delta(\mathbf{p}), \tag{A6}$$

which is Eq. (2.13).

We next study

$$I \equiv \int d^2p \langle m|e^{-i\mathbf{pX}}|n\rangle\langle i|e^{i\mathbf{pX}}|j\rangle. \tag{A7}$$

Substituting $\int dx|x\rangle\langle x| = 1$, we find

$$I \equiv \int dx_m dx_n dx_i dx_j \int d^2p \langle m|x_m\rangle\langle x_n|n\rangle \times\langle i|x_i\rangle\langle x_j|j\rangle\langle x_m|e^{-i\mathbf{pX}}|x_n\rangle\langle x_i|e^{i\mathbf{pX}}|x_j\rangle. \tag{A8}$$

We use Eqs. (A4a) and (A4b),

$$I \equiv \int dx_m dx_n dx_i dx_j \int dp_x dp_y \langle m|x_m\rangle\langle x_n|n\rangle\langle i|x_i\rangle \times\langle x_j|j\rangle\exp[ip_x(x_j-x_m)] \times\delta(x_m-x_n+l_B^2 p_y)\,\delta(x_i-x_j-l_B^2 p_y)$$

$$= \frac{2\pi}{l_B^2}\int dx_m dx_n \langle m|x_m\rangle\langle x_m|j\rangle\langle i|x_n\rangle\langle x_n|n\rangle$$

$$= \frac{2\pi}{l_B^2}\delta_{ni}\delta_{mj}, \tag{A9}$$

which is Eq. (2.14).

For the sake of completeness let us prove Eq. (2.13) based on representation (2.9). Here both X and Y are shifting operators:

$$e^{i\mathbf{pX}}|x,y\rangle = e^{i\mathbf{px}/2}|x-l_B^2 p_y, y+l_B^2 p_x\rangle. \tag{A10}$$

Thus

$$\langle x',y'|e^{i\mathbf{pX}}|x,y\rangle = e^{i\mathbf{px}/2}\langle x',y'|x-l_B^2 p_y, y+l_B^2 p_x\rangle_{\mathrm{LLL}}. \tag{A11}$$

Here it is necessary to evaluate the scalar product within the LLL. Using wave function (2.11) we find

$$\langle \mathbf{x}|\mathbf{y}\rangle_{\mathrm{LLL}} = \sum_{n=0}^{\infty}\langle \mathbf{x}|n\rangle\langle n|\mathbf{y}\rangle = \frac{1}{2\pi l_B^2}\exp\left(i\frac{\mathbf{x}\wedge\mathbf{y}}{2l_B^2}\right)\exp\left(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4l_B^2}\right). \tag{A12}$$

From these we derive Eq. (A5) or (2.13). We can prove Eq. (2.14) also in this representation though slightly complicated.

APPENDIX B: TUNNELING MATRIX

We diagonalize tunneling matrix (10.5), by solving the secular equation

$$\det(H_{\mathrm{T}} - \lambda I) = 0.$$

Here H_{T} is given by Eq. (10.5) and I is the $N\times N$ identity matrix. This equation leads to $D_N(x)=0$, where

$$D_N(x) = \det\begin{pmatrix} x & 1 & 0 & \cdots & 0 & 0 \\ 1 & x & 1 & \cdots & 0 & 0 \\ 0 & 1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & 1 \\ 0 & 0 & 0 & \cdots & 1 & x \end{pmatrix}, \tag{B1}$$

with $x = 2\lambda/\Delta_{\mathrm{SAS}}$. Expanding this with respect to the first column we obtain the recurrence relation

$$D_N(x) = xD_{N-1}(x) - D_{N-2}(x). \tag{B2}$$

Evidently, $D_1(x)=x$ and $D_2(x)=x^2-1$, and we can define $D_0(x)=1$ from this relation.

It is crucial to recall that the Chebyshev polynomial $S_N(x)$ satisfies the same recurrence relation (see Eq. 22.7.6 in Ref. 39). We use Eqs. 22.5.13 and 22.5.48 in this reference to find that

$$S_N(x) = U_N\left(\frac{x}{2}\right) = (N+1)F\left(-N, N+2; \frac{3}{2}; \frac{2-x}{4}\right), \tag{B3}$$

where $F(a,b;c;z)$ is the hypergeometric function. Since $a=-N$, the hypergeometric function becomes truncated. We can easily check that $S_0(x)=1$ and $S_1(x)=x$.

Therefore, we may identify

$$D_N(x) = U_N\left(\frac{x}{2}\right) = U_N\left(\frac{\lambda}{\Delta_{\mathrm{SAS}}}\right). \tag{B4}$$

According to Eq. 22.16.5 in the same reference we get the following set of roots;

$$\lambda_j(N) = \Delta_{\mathrm{SAS}}\cos\frac{\pi j}{N+1}, \quad j = 1,2,\ldots,N. \tag{B5}$$

We note that $\lambda_j = -\lambda_{N-j+1}$. For $N=2K$ the roots with $j=1,\ldots,K$ are positive. The ones with $j=K+1,\ldots,2K$ are of the same magnitude but negative. We have no zero root. For $N=2K+1$ there is one zero root corresponding to $j=K+1$, the ones with $j=1,\ldots,K$ are positive, while those with $j=K+2,\ldots,2K$ are negative with the same magnitude. The lowest root is

$$\lambda_N = -\Delta_{SAS}\cos\frac{\pi}{N+1}, \tag{B6}$$

and the next root is

$$\lambda_{N-1} = -\Delta_{SAS}\cos\frac{2\pi}{N+1}. \tag{B7}$$

There is no degeneracy among N energy levels.

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