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Model anisotropic quantum Hall states

R.-Z. Qiu[1], F. D. M. Haldane[2], Xin Wan[3], Kun Yang[4], and Su Yi[1]
[1] *State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*
[2] *Department of Physics, Princeton University, Princeton NJ 08544-0708*
[3] *Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, China and*
[4] *National High Magnetic Field Laboratory and Department of Physics, Florida State University, Tallahassee, Florida 32306, USA*

Model quantum Hall states including Laughlin, Moore-Read and Read-Rezayi states are generalized into appropriate anisotropic form. The generalized states are exact zero-energy eigenstates of corresponding *anisotropic* two- or multi-body Hamiltonians, and explicitly illustrate the existence of geometric degree of freedom in the fractional quantum Hall effect. These generalized model quantum Hall states can provide a good description of the quantum Hall system with anisotropic interactions. Some numeric results of these anisotropic quantum Hall states are also presented.

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I. INTRODUCTION

Variational wave functions play a fundamental role in our understanding of fractional quantum Hall (FQH) effect, with the Laughlin wave function[1] being the most prominent example. New classes of quantum Hall trial wave functions have been proposed as correlation functions in various conformal field theories,[2,3] by generalizing the clustering properties of the wave functions,[4,5] or as Jack polynomials with a negative parameter and a matching root configuration.[6] For a long time it has been understood that the Laughlin wave function, as well as other FQH trial wave functions, contains *no* variational parameter. This understanding is the consequence of our interest in searching for *topologically distinct* quantum Hall wave functions; geometry, as oppose to topology, is believed to be redundant. Recently one of us[7] pointed out that such topological description of FQH wave functions is not complete; in his geometrical description the original Laughlin wave function is simply a member of a family of Laughlin states, parameterized by a hidden (continuous) geometrical degree of freedom. The family of the Laughlin states, with the geometrical factor as a variational parameter, should provide a better description of the FQH effect in the presence of either anisotropic effective mass or anisotropic interaction, which are present in real materials.

The anisotropic FQH states are of present interest, both theoretically[8–10] and experimentally.[11] In Ref. 9, some of us studied the quantum Hall effects in a fast rotating quasi-two-dimensional gas of polarized fermionic dipoles. The fast rotation is equivalent to the high magnetic field according to the Larmor theorem. And since p-wave interaction for the polarized fermions is typically very small unless in the resonance regime, the only significant interaction is the dipole-dipole interaction that could be tuned by adjusting the applied electric and magnetic fields. By tuning the direction of the dipole moment with respect to the z axis, the dipole-dipole interaction becomes anisotropic in the x-y plane. Specifically, the system contain two parameters: the confining strength determined by the rotation frequency and the tilt angle determined by the applied field. Thus such systems have highly tunable anisotropic interactions, and are ideal for studies of anisotropic FQH states. Results in Ref. 9 clearly indicate the *inadequacy* of the known variational wave functions in the description of such states.

As a matter of fact, consideration of possible anisotropic FQH states has a long history,[12–14] with variational wave functions constructed that are straightforward modifications of the original Laughlin wave function.[12,13] Unlike the original Laughlin wave function and the states to be discussed here however, these earlier anisotropic FQH states are *not* exact eigenstates of special 2-body Hamiltonians. Interest in such states was partially motivated by the observation of *compressible* states with strong anisotropic transport properties in high Landau levels (LLs).[15,16] In all these states, the rotation symmetry is broken *spontaneously*. Very recently, an anisotropic FQH state has been observed at $\nu = 7/3$.[11] In this case the rotation symmetry is broken *explicitly* by an in-plane magnetic field, whose direction dictates the transport anisotropy. Generalizations of the FQH states to spin models yield the supersymmetric valence bond solid states, [17] and to lattice models with zero net magnetic field and full lattice translation symmetry are also called for to describe fractional quantum anomalous Hall states and fractional topological insulators.[18]

Ref. 7 pointed out the existence of a family of Laughlin states that are zero energy ground states of a family of Hamiltonians consisting of projection operators, which depend on a parameter called guiding center metric. However the states themselves were not constructed explicitly, and only some of their qualitative properties were mentioned briefly. The main motivation of this work is to explore the construction of a family of wave functions *in closed form* with numerical comparisons to facilitate the study of the geometrical aspects of anisotropic FQH states as a result of anisotropic interaction in the planer geometry.

The rest of the paper is organized as follows. In Sec. II we introduce the anisotropic LL basis states using the unimodular transformation. In Sec. III we focus on the recipe to generate the anisotropic many-particle states due to the unimodular transformation. Section. IV covers the numerical studies on various properties of the family of Laughlin states, including the density profile, pair correlation function, and variational energy. We summarize the paper in Sec. V.

II. ANISOTROPIC LANDAU LEVEL BASIS STATES

The original Laughlin wave function was most easily written down on a disc, using the symmetric gauge in which the single particle basis states are angular momentum eigenstates. The key to explicitly constructing the anisotropic Laughlin states proposed in Ref. 7 (on a disc) is the usage of a set of *anisotropic* LL basis states. We will, however, continue to use the symmetric gauge, in which the lowest LL (LLL) wave functions are holomorphic. The same set of basis states appeared earlier in the consideration of deformation of shape of quantum Hall liquids in the context of Hall viscosity[19], although the wave functions were not explicitly given. In the following we generate them explicitly. In addition, we also use them to construct the corresponding integer quantum Hall wave function, as a warm-up for their later application to FQH states.

Let us start from the eigenvalues problem of a two-dimensional charged particle subjected into a uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. The one-body Hamiltonian is given by

$$H_0 = \tfrac{1}{2}(m^{-1})^{ab}\pi_a\pi_b, \quad \boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}, \tag{1}$$

where we used the Einstein convention, m_{ab} is the cyclotron effective mass tensor, $e > 0$ is the charge of the particle, $\boldsymbol{A}$ is the vector potential of the uniform magnetic field $\boldsymbol{B}$, and the dynamical momentum $\boldsymbol{\pi}$ satisfies

$$[\pi_a, \pi_b] = i\epsilon_{ab}\hbar^2\ell^{-2} \tag{2}$$

with $\epsilon_{ab} = \epsilon^{ab}$ the 2D antisymmetric Levi-Civita symbol and $\ell = \sqrt{\hbar/(eB)}$ the magnetic length. In terms of a complex vector $\boldsymbol{\omega}$, the effective mass tensor can be written as

$$m_{ab} = m\,(\omega_a^*\omega_b + \omega_b^*\omega_a), \tag{3}$$
$$(m^{-1})^{ab} = m^{-1}(\omega^{a*}\omega^b + \omega^{b*}\omega^a). \tag{4}$$

We note that the assumption that it is isotropic in the (x, y) Cartesian coordinate system means

$$(\omega_x, \omega_y) = (\omega^x, \omega^y) = \frac{1}{\sqrt{2}}(1, i). \tag{5}$$

The one-body Hamiltonian can now be expressed as

$$H_0 = \tfrac{1}{2}\hbar\omega_c\left(b^\dagger b + bb^\dagger\right). \tag{6}$$

where $\hbar\omega_c = \hbar^2/m\ell^2$ is the cyclotron energy and the Landau orbit ladder operators are given by

$$b = \hbar^{-1}\ell\,(\boldsymbol{\omega}\cdot\boldsymbol{\pi}), \quad b^\dagger = \hbar^{-1}\ell\,(\boldsymbol{\omega}^*\cdot\boldsymbol{\pi}), \tag{7}$$

with $[b, b^\dagger] = 1$. By introducing the "guiding center" coordinates $\boldsymbol{R}$,

$$R^a = r^a + \hbar^{-1}\ell^2\epsilon^{ab}\pi_b, \tag{8}$$

which satisfy

$$[R^a, R^b] = -i\epsilon^{ab}\ell^2, \quad [R^a, \pi_b] = 0, \tag{9}$$

we could define a second set of "guiding center" ladder operators, a and $a^\dagger$ that commute with b and $b^\dagger$ (and hence with H_0)

$$a = \ell^{-1}\,(\boldsymbol{\omega}^*\cdot\boldsymbol{R}), \quad a^\dagger = \ell^{-1}\,(\boldsymbol{\omega}\cdot\boldsymbol{R}), \quad [a, a^\dagger] = 1. \tag{10}$$

To proceed further, let us choose the vector potential of the uniform magnetic field in the symmetric gauge,

$$\boldsymbol{A} = \tfrac{1}{2}\boldsymbol{B}\times\boldsymbol{r} = \tfrac{1}{2}B(-y, x), \tag{11}$$

and define the complex coordinate,

$$z = \frac{\boldsymbol{\omega}\cdot\boldsymbol{r}}{\ell}. \tag{12}$$

Let us note that for the case of isotropic mass the complex coordinate is explicitly given by

$$z = \frac{x + iy}{\sqrt{2}\ell}, \tag{13}$$

which is not the most standard definition in the literature. Then we could express the Landau orbit ladder operators as (neglecting the trivial $-i$ and i),

$$b = \tfrac{1}{2}z + \partial_{z*}, \quad b^\dagger = \tfrac{1}{2}z^* - \partial_z, \tag{14}$$

where $\partial_z f(z, z^*)$ is the partial derivative $\partial f/\partial z|_{z^*}$, etc, and the "guiding center" ladder operators as

$$a = \tfrac{1}{2}z^* + \partial_z, \quad a^\dagger = \tfrac{1}{2}z - \partial_{z^*}. \tag{15}$$

The Hamiltonian H_0 has a rotational symmetry generated by

$$L_0 = a^\dagger a - b^\dagger b, \quad [L_0, H_0] = 0, \tag{16}$$

where

$$[L_0, x] = iy, \quad [L_0, y] = -ix, \tag{17}$$
$$[L_0, p_x] = ip_y, \quad [L_0, p_y] = -ip_x. \tag{18}$$

Simultaneous diagonalization of H_0 and L_0 allows a complete orthonormal basis of one particle states $|\psi_{nm}\rangle$ which used to be constructed as

$$H_0|\psi_{nm}\rangle = (n + \tfrac{1}{2})\hbar\omega_c|\psi_{nm}\rangle, \tag{19}$$
$$L_0|\psi_{nm}\rangle = (m - n)|\psi_{nm}\rangle, \tag{20}$$
$$|\psi_{nm}\rangle = \frac{(a^\dagger)^m(b^\dagger)^n}{\sqrt{m!n!}}|\psi_{00}\rangle, \tag{21}$$
$$a|\psi_{00}\rangle = b|\psi_{00}\rangle = 0. \tag{22}$$

This is, of course, the commonly used basis states associated with the symmetric gauge.

However, it is possible to construct a set of closely related but *different* set of basis states. First let us write

$$L_0 = \bar{L} - L, \tag{23}$$
$$L = \tfrac{1}{2}(b^\dagger b + b b^\dagger), \quad H_0 = \hbar\omega_c L, \tag{24}$$
$$\bar{L} = \tfrac{1}{2}(a^\dagger a + a a^\dagger). \tag{25}$$

The conventional basis is then just the set of the simultaneous eigenstates of H_0 and $\bar{L}$,

$$\bar{L}|\psi_{nm}\rangle = (m + \tfrac{1}{2})|\psi_{nm}\rangle. \tag{26}$$

We may write $\bar{L} = \bar{L}(g^0)$, where

$$\bar{L}(g) = \frac{1}{2\ell^2} g_{ab} R^a R^b, \tag{27}$$

and g_{ab} is a positive-definite Euclidean metric tensor with $\det g = 1$. Note that $\bar{L}(g^0)$ uses the "Galilean metric" derived from the effective mass tensor,

$$m_{ab} = m(\omega_a^*\omega_b + \omega_b^*\omega_a) = m g^0_{ab}. \tag{28}$$

However, as has been stressed by one of us[7,20], that before specifying the two-body interactions, there is no fundamental reason to choose a guiding-center basis that is adapted to the shape of the Landau orbits. There is a more general family of bases parameterized by a unimodular positive-definite guiding center metric.[7]

This metric can be factorized by a complex vector $\bar{\boldsymbol{\omega}}$[7]

$$g_{ab} = \bar{\omega}_a^*\bar{\omega}_b + \bar{\omega}_a\bar{\omega}_b^*,$$

where the unimodular property that $\bar{\omega}_x^*\bar{\omega}_y - \bar{\omega}_x\bar{\omega}_y^* = i$ is the only constraint. Generally, the complex vector may be written as

$$\bar{\omega}_a = u\omega_a + v\omega_a^*, \quad uu^* - vv^* = 1.$$

We shall define $\gamma^* = v/u$ with $|\gamma| < 1$ and make the gauge choice that u is real positive. Now the complex vector becomes

$$\bar{\omega}_a = (1 - |\gamma|^2)^{-1/2}(\omega_a + \gamma^*\omega_a^*), \tag{29}$$

that determines a unimodular metric $g_{ab}(\gamma)$,

$$\frac{1}{1-\gamma^*\gamma}\begin{pmatrix} (1+\gamma)(1+\gamma^*) & i(\gamma-\gamma^*) \\ i(\gamma-\gamma^*) & (1-\gamma)(1-\gamma^*) \end{pmatrix}$$

according to

$$g_{ab}(\gamma) = \bar{\omega}_a^*\bar{\omega}_b + \bar{\omega}_a\bar{\omega}_b^*. \tag{30}$$

We can then construct a basis of eigenstates $|\psi_{nm}(\gamma)\rangle$ of H_0 and $\bar{L}(\gamma) \equiv \bar{L}(g(\gamma))$, which is accomplished by defining a new set of "guiding center" ladder operators, also considered in Ref. 19,

$$a_\gamma = \ell^{-1}(\bar{\boldsymbol{\omega}}^* \cdot \boldsymbol{R}), \quad a_\gamma^\dagger = \ell^{-1}(\bar{\boldsymbol{\omega}} \cdot \boldsymbol{R}), \tag{31}$$

and explicitly as

$$\begin{pmatrix} a_\gamma \\ a_\gamma^\dagger \end{pmatrix} = \frac{1}{\sqrt{1-\gamma^*\gamma}}\begin{pmatrix} 1 & \gamma \\ \gamma^* & 1 \end{pmatrix}\begin{pmatrix} a \\ a^\dagger \end{pmatrix}. \tag{32}$$

This is a Bogoliubov (or "squeezing") transformation that preserves the commutation relation $[a_\gamma, a_\gamma^\dagger] = 1$. Then the new simultaneous eigenstates of H_0 and $\bar{L}(\gamma)$ are given by

$$|\psi_{nm}(\gamma)\rangle = \frac{(b^\dagger)^n (a_\gamma^\dagger)^m}{\sqrt{n!m!}}|\psi_{00}(\gamma)\rangle, \tag{33}$$
$$a_\gamma|\psi_{00}(\gamma)\rangle = b|\psi_{00}(\gamma)\rangle = 0. \tag{34}$$

Note that $|\psi_{nm}(0)\rangle$ is the conventional basis $|\psi_{nm}\rangle$ in Eq. (21).

Since the model quantum Hall states are often defined in the lowest Landau level (LLL) ($n = 0$), it is useful to examine the wave functions $\psi_{0m}(\gamma, \boldsymbol{r}) = \langle \boldsymbol{r}|\psi_{0m}(\gamma)\rangle$. First of all, from the LLL property $b\psi_{0m}(\gamma, \boldsymbol{r}) = 0$, we have

$$\psi_{0m}(\gamma, \boldsymbol{r}) = f(z)\psi_{00}(0, \boldsymbol{r}) \tag{35}$$

with

$$\psi_{00}(0, \boldsymbol{r}) = (2\pi)^{-1/2} e^{-|z|^2/2}. \tag{36}$$

Second from $a_\gamma\psi_{00}(\gamma, \boldsymbol{r}) = 0$, we immediately obtain

$$\psi_{00}(\gamma, \boldsymbol{r}) = \lambda^{1/2} e^{-\frac{1}{2}\gamma z^2}\psi_{00}(0, \boldsymbol{r}) \tag{37}$$

with

$$\lambda = \sqrt{1 - |\gamma|^2}. \tag{38}$$

Next let us note that $\psi_{0m}(\gamma, \boldsymbol{r})$,

$$\frac{\left[\frac{1}{2}z - \partial_{z^*} + \gamma^*\left(\frac{1}{2}z^* + \partial_z\right)\right]^m}{\lambda^m\sqrt{m!}}\psi_{00}(\gamma, \boldsymbol{r}),$$

could be simplified by virtue of the property that $b\psi_{00}(0, \boldsymbol{r}) = 0$, or

$$\partial_{z^*}\left[\psi_{00}(0, \boldsymbol{r})f(z)\right] = (-\tfrac{1}{2}z)\psi_{00}(0, \boldsymbol{r})f(z) \tag{39}$$

and $a\psi_{00}(0, \boldsymbol{r}) = 0$, or

$$\partial_z\left[\psi_{00}(0, \boldsymbol{r})f(z)\right] = \psi_{00}(0, \boldsymbol{r})(\partial_z - \tfrac{1}{2}z^*)f(z) \tag{40}$$

and

$$\partial_z\left[e^{-\frac{1}{2}\gamma z^2}f(z)\right] = e^{-\frac{1}{2}\gamma z^2}(\partial_z - \gamma z)f(z). \tag{41}$$

Thus we obtain

$$\psi_{0m}(\gamma, \boldsymbol{r}) = \psi_{00}(\gamma, \boldsymbol{r})\frac{\lambda^m}{\sqrt{m!}}(z + z_0^2\partial_z)^m 1 \tag{42}$$
$$= \psi_{00}(\gamma, \boldsymbol{r})\frac{\lambda^m}{\sqrt{m!}}W_m(z, z_0^2) \tag{43}$$

with

$$z_0^2 = \gamma^*/\lambda^2 \tag{44}$$

and $W_m(z, z_0^2)$ are the polynomials defined by $W_0(z, z_0^2) = 1$, and the recursion relations,

$$W_{m+1}(z, z_0^2) = (z + z_0^2\partial_z)W_m(z, z_0^2), \tag{45}$$
$$W_m(z, z_0^2) = \partial_z W_{m+1}(z, z_0^2)/(m+1). \tag{46}$$

The first few of $W_m(z, z_0^2)$ are given by

$$\begin{aligned} W_0(z, z_0^2) &= 1, \\ W_1(z, z_0^2) &= z, \\ W_2(z, z_0^2) &= z^2 + z_0^2, \\ W_3(z, z_0^2) &= z^3 + 3z_0^2 z, \end{aligned}$$

and the general solutions are given by

$$W_{2k}(z, z_0^2) = k!(2z_0^2)^k L_k^{(-\frac{1}{2})}(-z^2/2z_0^2), \tag{47}$$
$$W_{2k+1}(z, z_0^2) = k!(2z_0^2)^k z L_k^{(\frac{1}{2})}(-z^2/2z_0^2), \tag{48}$$

where $L_k^{(\alpha)}(x)$ are generalized Laguerre polynomials defined by

$$L_k^{(\alpha)}(x) = \sum_{i=0}^{k} \frac{\Gamma(\alpha + k + 1)}{\Gamma(\alpha + i + 1)\Gamma(k - i + 1)} \frac{(-x)^i}{i!}. \tag{49}$$

These expressions do satisfy the reclusion relations (45,46), which could be proved by using the following useful identities for $L_k^{(\alpha)}(x)$,

$$L_k^{(\alpha)}(x) = L_k^{(\alpha+1)}(x) - L_{k-1}^{(\alpha+1)}(x), \tag{50}$$
$$\frac{d}{dx} L_k^{(\alpha)}(x) = -L_{k-1}^{(\alpha+1)}(x), \tag{51}$$
$$xL_{k-1}^{(\alpha+1)}(x) = (k+\alpha)L_{k-1}^{(\alpha)}(x) - kL_k^{(\alpha)}(x). \tag{52}$$

Note that for $|z|^2 \gg |z_0^2|$, $W_m(z, z_0^2) \to z^m$.

The densities $|\psi_{0m}(\gamma, \boldsymbol{r})|^2$ defined by the single-particle orbitals have some noteworthy features. First, the density profile is anisotropic for $\gamma \neq 0$. Take a simple example like $\psi_{00}(|\gamma|, \boldsymbol{r})$,

$$|\psi_{00}(|\gamma|, \boldsymbol{r})|^2 \propto \exp\left[-\frac{(1+|\gamma|)x^2 + (1-|\gamma|)y^2}{2\ell^2}\right].$$

For this Gaussian wave packet, the width along y-axis is larger than the width along x-axis, so non-zero $|\gamma|$ causes stretching the wave function along some direction. Second, $|\psi_{0m}(\gamma, \boldsymbol{r})|^2$ has m zeros as can be seen from Fig. 1 in which we plot the density profiles $|\psi_{0m}(1/2, \boldsymbol{r})|^2$ for $m = 0, 1, 2, 3$. These zeros are roots of $W_m(z, z_0^2)$, and they are aligned along the stretched direction. In the limit of $\gamma \to 0$, they collapse to a multiple root at the origin. Third, the stretching direction is determined by the phase of the complex number γ, $\arg(\gamma)$, which is easy to understand by noting that $W_3(z, z_0^2) = 0$ gives rise to three roots, 0 and $\pm i\sqrt{3}z_0$. In Fig. 2, we plot $|\psi_{03}(e^{i\phi}/2, \boldsymbol{r})|^2$ with $\phi = 0, \pi/2, \pi$ and $3\pi/2$. Thus we realize that the density profile $|\psi_{0m}(\gamma, \boldsymbol{r})|^2$ is equivalent to rotating clockwise the density profile $|\psi_{0m}(|\gamma|, \boldsymbol{r})|^2$ by $\arg(\gamma)/2$.

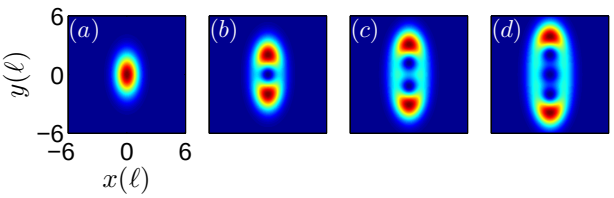

FIG. 1: (Color online) The density profiles of the generalized LLL wave functions $\psi_{0m}(\gamma = 1/2, \boldsymbol{r})$ with $m = 0$ (a), $m = 1$ (b), $m = 2$ (c) and $m = 3$ (d).

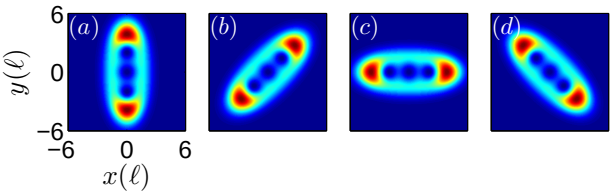

FIG. 2: (Color online) The density profiles of $\psi_{03}(\gamma, \boldsymbol{r})$ with $\gamma = e^{i\phi}/2$ for $\phi = 0$ (a), $\phi = \pi/2$ (b), $\phi = \pi$ (c) and $\phi = 3\pi/2$ (d). Note that the zeros are localized at $z = 0$ and $\pm i\sqrt{3}z_0$.

It is clear that the single particle basis states are anisotropic, with the degree of anisotropy controlled by γ. In addition, the parameter z_0 defined by (44) determine the splitting of the multi-fold zeros in the conventional single-particle basis states, which would be reflected in the multi-body correlation function of many-body states. Finally, as we will demonstrate below, it is very natural to use these basis states to construct many-body states with anisotropic two- and multi-body correlations, which can have lower variational energy than their isotropic counterparts when interactions are anisotropic.

To proceed further, let us define here the non-interacting basis states to prepare for the many-body problem. For bosonic system, the non-interacting basis state in the generalized LLL is given by

$$\mathcal{M}_\mu(\gamma) \propto \mathrm{perm}\left[\psi_{0\mu}(\gamma, \boldsymbol{r})\right], \tag{53}$$

where $\mu = [\mu_1, \mu_2, \ldots, \mu_N]$ is a non-decreasing partition and $\mathrm{perm}[\psi_{0\mu}(\gamma, \boldsymbol{r})]$ is the permanent of the square matrix whose matrix elements are $\psi_{\mu_i}(\gamma, \boldsymbol{r}_j)$ $(i, j = 1, \ldots, N)$. For the fermionic system, the non-interacting basis state is defined by a Slater determinant,

$$\mathfrak{sl}_\mu(\gamma) = \det\left[\psi_{0\mu}(\gamma, \boldsymbol{r})\right], \tag{54}$$

where μ is a decreasing partition and det denotes the matrix determinant.

As a special example, the Slater determinant $\mathfrak{sl}_{\mu_0}(\gamma)$ with the partition $\mu_0 = [N-1, N-2, \ldots, 1, 0]$ is the

generalized N-particle IQH state, whose (unnormalized) expression could be simplified as

$$\Psi_I(\gamma) = \prod_{i<j}(z_i - z_j)\prod_i \psi_{00}(\gamma, \boldsymbol{r}_i), \tag{55}$$

as lower-degree polynomials must vanish due to the Pauli exclusion principle (this is generally not the case for other partitions, which is the key to understanding the anisotropic FQH states).

The generalized IQH state can also be viewed as a coherent superposition of the isotropic IQH state (or simple Vandermonde determinant state) and its edge states with angular momentum differences being multiples of $2\hbar$. This is achieved by expanding the exponential factor in Eq. (55). From the viewpoint of the superposition, this is consistent with the numerical result in Ref. 9.

III. ANISOTROPIC MANY-PARTICLE STATES

In this section, we first consider the two-particle problem and then simplify the interaction Hamiltonian projected into the LLL in Subsec. III A. More importantly, we present the recipe to find the anisotropic counterpart of a LLL state and elaborate it in Subsec. III B. We specifically consider the anisotropic counterpart of the prominent FQH states such as the Laughlin state in Subsec. III C.

A. Two-particle problem and the projected two-body interaction Hamiltonian

If the interaction is dominated by the energy gap $\hbar\omega_c$ between the lowest and second Landau level, the interaction Hamiltonian could be projected into the lowest Landau level. In terms of non-commuting "guiding center" coordinates $\boldsymbol{R}_i$, the projected two-body interaction Hamiltonian we consider here is given by[7]

$$H_{\rm int}(\{V_m\}, \gamma) = \sum_m V_m h_m(\gamma), \tag{56}$$

where V_m is the anisotropic Haldane's pseudopotential and the projection operator $h_m(\gamma)$ is given by

$$h_m(\gamma) = \int \frac{d^2\boldsymbol{q}\ell^2}{\pi} v_m(\boldsymbol{q}, g(\gamma)) \sum_{i<j} e^{i\boldsymbol{q}\cdot(\boldsymbol{R}_i - \boldsymbol{R}_j)}, \tag{57}$$

$$v_m(\boldsymbol{q}, g) = L_m(g^{ab}q_a q_b \ell^2)\exp(-\tfrac{1}{2}g^{ab}q_a q_b \ell^2), \tag{58}$$

where $L_m(x) = L_m^{(\alpha=0)}(x)$ are the Laguerre polynomials. In addition, the total angular momentum associated with the guiding center is given by

$$\bar{L}(\gamma) = \frac{1}{2\ell^2}\sum_i g_{ab} R^{ia} R^{ib}. \tag{59}$$

For two-particle, the LLL states that simultaneously diagonalize $\bar{L}(\gamma)$ and the interaction $H_{\rm int}(\{V_k\}, \gamma)$ are completely fixed by symmetry. To clarify this, let us note that

$$\begin{aligned}\bar{L}(\gamma) &= a^\dagger_{\gamma,1}a_{\gamma,1} + a^\dagger_{\gamma,2}a_{\gamma,2} + 1 \\ &= c^\dagger_\gamma c_\gamma + C^\dagger_\gamma C_\gamma + 1\end{aligned} \tag{60}$$

with $c_\gamma = (a_{1\gamma} - a_{2\gamma})/\sqrt{2}$ and $C_\gamma = (a_{1\gamma} + a_{2\gamma})/\sqrt{2}$, and the only term containing operator in the interaction Hamiltonian, $\boldsymbol{R}_1 - \boldsymbol{R}_2$, contains only c_γ and $c^\dagger_\gamma$. Therefore, we could expand $\bar{L}(\gamma)$ and $H_{\rm int}(\{V_k\}, \gamma)$ in the orthonormal basis

$$|Mm\rangle_\gamma = \frac{(C^\dagger_\gamma)^M (c^\dagger_\gamma)^m}{\sqrt{M!m!}}|00\rangle_\gamma, \tag{61}$$

$$a_{\gamma,1}|00\rangle_\gamma = a_{\gamma,2}|00\rangle_\gamma = 0, \tag{62}$$

and find that

$$H_{\rm int}(\{V_k\}, \gamma)|Mm\rangle_\gamma = V_m|Mm\rangle_\gamma, \tag{63}$$

$$\bar{L}(\gamma)|Mm\rangle_\gamma = (M + m + 1)|Mm\rangle_\gamma. \tag{64}$$

Thus the projected operator $h_m(\gamma)$ for many-particle system could be rewritten as

$$h_m(\gamma) = \sum_{i<j}\left(\sum_M |Mm\rangle_{\gamma\gamma}\langle Mm|\right)(i,j) \tag{65}$$

and in second quantized form as

$$\frac{1}{2}\sum_M \sum_{m_1 m_2 m_3 m_4} {}_\gamma\langle m_1, m_2|Mm\rangle_{\gamma\gamma}\langle Mm|m_3, m_4\rangle_\gamma \; g^\dagger_{m_1} g^\dagger_{m_2} g_{m_4} g_{m_3}, \tag{66}$$

with $|m_1, m_2\rangle_\gamma = |\psi_{0,m_1}(\gamma)\rangle \otimes |\psi_{0,m_2}(\gamma)\rangle$ and $g^\dagger_m$ being the creation operator that creates a particle in state $|\psi_{0m}(\gamma)\rangle$.

For the rotationally-invariant case, $\gamma = 0$,

$$\Psi_{Mm}(0, \boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{(z_1 + z_2)^M}{\sqrt{2^M M!}}\frac{(z_1 - z_2)^m}{\sqrt{2^m m!}}\prod_{i=1}^2 \psi_{00}(0, \boldsymbol{r}_i)$$

and the translation into the Heisenberg state is

$$|Mm\rangle_\gamma = \frac{(a^\dagger_1 + a^\dagger_2)^M}{\sqrt{2^M M!}}\frac{(a^\dagger_1 - a^\dagger_2)^m}{\sqrt{2^m m!}}|00\rangle_\gamma. \tag{67}$$

Therefore we find the anisotropic deformation is now easily achieved by the simple replacement

$$a^\dagger_i \to a^\dagger_{\gamma,i}, \tag{68}$$

and the vacuum state $|\Psi_0(0)\rangle \to |\Psi_0(\gamma)\rangle$ that satisfy $a_{\gamma,i}|\Psi_0(\gamma)\rangle = 0$. In the next subsection, we shall utilize this idea to consider the anisotropic deformation of a general LLL wave function.

B. Anisotropic many-particle states

The general form of a rotationally-invariant N-particle LLL wave function in the symmetric gauge is

$$\Psi(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N) = F(z_1,\ldots,z_N)\prod_i \psi_{00}(\boldsymbol{r}_i), \tag{69}$$

where $F(z_1,\ldots,z_N)$ is a homogeneous multivariate polynomial with $F(\lambda z_1,\ldots\lambda z_N) = \lambda^M F(z_1,\ldots,z_N)$ and $\bar{L}(0)|\Psi\rangle = (M+\frac{1}{2}N)|\Psi\rangle$. The translation to the Heisenberg picture yields

$$\begin{aligned} &|\Psi(\gamma=0)\rangle = F(a_1^\dagger,\ldots,a_N^\dagger)|\Psi_0(\gamma=0)\rangle,\\ &a_i|\Psi_0(\gamma=0)\rangle = b_i|\Psi_0(\gamma=0)\rangle = 0. \end{aligned} \tag{70}$$

The deformation of this Heisenberg state becomes

$$\begin{aligned} &|\Psi(\gamma)\rangle = F(a_{\gamma,1}^\dagger,\ldots,a_{\gamma,N}^\dagger)|\Psi_0(\gamma)\rangle,\\ &a_{\gamma,i}|\Psi_0(\gamma)\rangle = b_i|\Psi_0(\gamma)\rangle = 0. \end{aligned} \tag{71}$$

The Schrodinger wave function for this state $|\Psi(\gamma)\rangle$ will have the form,

$$\Psi(\gamma,\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N) = F_\gamma(z_1,\ldots z_N)\prod_i \psi_{00}(\boldsymbol{r}_i). \tag{72}$$

We now need the formal construction of $F_\gamma(\{z_i\})$, which is still holomorphic, but no longer a polynomial. Using the homogeneity of the polynomial F, the wave function is given by

$$\lambda^{-M}F(\{\tfrac{1}{2}z_i - \partial_{z_i^*} + \gamma^*(\tfrac{1}{2}z_i^* + \partial_{z_i})\})\prod_i \lambda^{\frac{1}{2}}e^{-\frac{1}{2}\gamma z_i^2}\psi_{00}(\boldsymbol{r}_i).$$

Using the property $a_i\psi_{00}(\boldsymbol{r}_i) = 0$, or

$$\partial_z\psi_{00}(\boldsymbol{r})f(z,z^*) = \psi_{00}(\boldsymbol{r})(\partial_z - \tfrac{1}{2}z^*)f(z,z^*), \tag{73}$$

and $b_i\psi_{00}(\boldsymbol{r}_i) = 0$, or

$$\partial_{z*}\psi_{00}(\boldsymbol{r})f(z,z^*) = \psi_{00}(\boldsymbol{r})(\partial_{z*} - \tfrac{1}{2}z)f(z,z^*), \tag{74}$$

the holomorphic part $F_\gamma(\{z_i\})$ of the wave function becomes

$$\lambda^{-M}F(\{z_i + \gamma^*\partial_{z_i}\})\prod_i \lambda^{\frac{1}{2}}e^{-\frac{1}{2}\gamma z_i^2}.$$

Using the following equality,

$$\partial_z\left(e^{-\frac{1}{2}\gamma z_i^2}f(z)\right) = e^{-\frac{1}{2}\gamma z_i^2}\left(\partial_z - \gamma z\right)f(z), \tag{75}$$

F_γ turns into,

$$\left(\prod_i \lambda^{\frac{1}{2}}e^{-\frac{1}{2}\gamma z_i^2}\right)\lambda^M F(\{z_i + z_0^2\partial_{z_i}\})1,$$

which is easy to understand by noting Eq. (42). Using the homogeneity of F, we obtain the final result

$$F_\gamma(\{\lambda^{-1}z_i\}) = \left(\textstyle\prod_i \lambda^{\frac{1}{2}}e^{-\frac{1}{2}\gamma z_i^2}\right)F(\{z_i + z_0^2\partial_{z_i}\})1. \tag{76}$$

where λ and z_0^2 are fixed by γ through Eqs. (38) and (44). This expression no longer requires homogeneity of $F(\{z_i\})$, and is quite general for the deformation of the holomorphic part of a LLL wave function.

Specially, the anisotropic counterpart of the rotationally-invariant non-interacting basis states (53,54)

$$\mathcal{M}_\mu(\gamma=0), \quad \mathfrak{sl}_\mu(\gamma=0), \tag{77}$$

are

$$\mathcal{M}_\mu(\gamma), \quad \mathfrak{sl}_\mu(\gamma). \tag{78}$$

Therefore, if we could expand the many-body state $\Psi(\gamma=0,\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)$ in the non-interacting basis states $\mathcal{M}_\mu(0)$ (53) or $\mathfrak{sl}_\mu(0)$ (54),

$$\sum_\mu c_\mu^{(b)}\mathcal{M}_\mu(0), \quad \sum_\mu c_\mu\mathfrak{sl}_\mu(0), \tag{79}$$

its anisotropic counterpart $\Psi(\gamma,\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N)$ would be

$$\sum_\mu c_\mu^{(b)}\mathcal{M}_\mu(\gamma), \quad \sum_\mu c_\mu\mathfrak{sl}_\mu(\gamma). \tag{80}$$

Here we find the coefficients c_μ and $c_\mu^{(b)}$ are independent of γ.

The γ-independence of the coefficients results in γ-independence of the occupation number for each orbital

$$n_m = \langle\Psi_L^q(\gamma)|g_m^\dagger g_m|\Psi_L^q(\gamma)\rangle.$$

This also suggests that the entanglement spectrum,[21] which encodes the topological properties of the state, is also invariant, if the appropriate cut in the space of the total angular momentum associated with guiding center is chosen. We see that topological properties of a FQH state are built into the coefficients (relative weight), which are manifested, e.g., in the entanglement spectrum calculation. On the other hand, geometrical properties of the anisotropic FQH states are encoded into the deformation of the non-interacting basis. Nevertheless, geometrical information can be revealed by exploring the properties of the family of variational wave functions and the corresponding variational energies; this point will be illustrated by examples in Sec. IV.

C. Anisotropic FQH states

In this subsection, let us consider some prominent variational functions such as the Laughlin wave function, Moore-Read state and so on.

The $\nu = 1/q$ Laughlin wave function is given by

$$\Psi_L^q(\gamma=0) = \prod_{i<j}(z_i - z_j)^q\prod_i \psi_{00}(0,\boldsymbol{r}_i), \tag{81}$$

and its anisotropic counterpart is given by

$$\Psi_L^q(\gamma) = \prod_i \psi_{00}(\gamma, \boldsymbol{r}_i) \prod_{i<j} \left[z_i - z_j + z_0^2(\partial_{z_i} - \partial_{z_j})\right]^q 1. \tag{82}$$

The anisotropic Laughlin states $\Psi_L^q(\gamma)$ are the (minimum total $\bar{L}(\gamma)$) zero-energy ground state of the Haldane's pseudopotential Hamiltonian $H_{\rm int}(\{V_m\}, \gamma)$ with

$$V_{m<q} > 0 \text{ and } V_{m\geq q} = 0. \tag{83}$$

To prove this, we may consider the diagonalization of the projected interaction Hamiltonian matrix and note the γ-independence of the matrix elements

$$\langle \mathfrak{sl}_\mu(\gamma)|H_{\rm int}(\{V_m\}, \gamma)|\mathfrak{sl}_{\mu'}(\gamma)\rangle, \text{ or}$$
$$\langle \mathcal{M}_\mu(\gamma)|H_{\rm int}(\{V_m\}, \gamma)|\mathcal{M}_{\mu'}(\gamma)\rangle,$$

which results from the γ-independence of ${}_\gamma\langle m_1, m_2|mM\rangle_\gamma$.

For two particles, the anisotropic counterpart of the $\nu = 1/q$ Laughlin state

$$\Psi_{0q}(0, \boldsymbol{r}_1, \boldsymbol{r}_2) = (z_1 - z_2)^q \Psi_{00}(0, \boldsymbol{r}_1, \boldsymbol{r}_2) \tag{84}$$

is given by

$$\begin{aligned} & \Psi_{0q}(\gamma, \boldsymbol{r}_1, \boldsymbol{r}_2) \\ = \; & \psi_{00}(\gamma, \boldsymbol{r}_1)\psi_{00}(\gamma, \boldsymbol{r}_2) \left[z_1 - z_2 + z_0^2(\partial_{z_1} - \partial_{z_2})\right]^q 1, \\ = \; & \psi_{00}(\gamma, \boldsymbol{r}_1)\psi_{00}(\gamma, \boldsymbol{r}_2)(z_{12} + 2z_0^2\partial_{z_{12}})^q 1, \\ = \; & \psi_{00}(\gamma, \boldsymbol{r}_1)\psi_{00}(\gamma, \boldsymbol{r}_2) W_q(z_{12}, 2z_0^2), \end{aligned} \tag{85}$$

with $z_{12} = z_1 - z_2$ and $\partial_{z_{12}} = \frac{1}{2}(\partial_{z_1} - \partial_{z_2})$. Among these states, the straightforward (anisotropic) generalization of the $\nu = 1/2$ bosonic Laughlin state is $\Psi_{02}(\gamma, \boldsymbol{r}_1, \boldsymbol{r}_2)$, in which the double zero of the undeformed state at $z_1 = z_2$ is split into zeroes at $z_1 - z_2 = \pm i\sqrt{2}z_0$. While for the $\nu = 1/3$ fermionic Laughlin state $\Psi_{03}(\gamma, \boldsymbol{r}_1, \boldsymbol{r}_2)$, the triple zero at $z_1 = z_2$ is split into a single zero at $z_1 = z_2$, and displaced zeroes at $z_1 - z_2 = \pm i\sqrt{6}z_0$. This suggests the splitting in the pattern of zeros can be used as a tool to study the anisotropic deformation of a quantum Hall wave function.

The other FQH state, such as the Moore-Read state[2], the Read-Rezayi state[3], the Haldane-Rezayi state[22], etc., could be generalized in this way, as long as the corresponding wave functions could be expanded in the non-interacting LLL basis. In addition, we also conclude that the anisotropic Z_k parafermion states[3] are also the (minimum total generalized angular momentum) zero-energy ground state of some special $(k+1)$-body interaction. This kind of interaction could be expressed in terms of the generalized $(k+1)$-body pseudopotentials[23] since the Z_k parafermion states are the (minimum total angular momentum) zero-energy ground state of $(k+1)$-body short-range interaction[3,24].

For example, the $\nu = 1/q$ anisotropic Moore-Read states are given by

$$\begin{aligned} \Psi_{MR}^q(\gamma) \; = \; & \prod_i \psi_{00}(\gamma, \boldsymbol{r}_i) \mathrm{Pf}\left(\frac{1}{z_i - z_j + z_0^2(\partial_{z_i} - \partial_{z_j})}\right) \\ & \prod_{i<j} \left[z_i - z_j + z_0^2(\partial_{z_i} - \partial_{z_j})\right]^q 1, \end{aligned} \tag{86}$$

where Pf means the Pfaffian, the square root of the determinant. Among these states, the anisotropic $\nu = 1$ bosonic Moore-Read states $\Psi_{MR}^{q=1}(\gamma)$ is the (minimum total $\bar{L}(\gamma)$) zero-energy ground state of the anisotropic 3-body pseudopotential Hamiltonian,

$$H_{\rm int}(\gamma) = V_0 h_0(\gamma), \quad V_0 > 0, \tag{87}$$

where the projection operator $h_0(\gamma)$ is given by

$$\sum_{M=0}^{\infty} \sum_{m_1 m_2 m_3 m_4 m_5 m_6} {}_\gamma\langle m_1, m_2, m_3|\Phi_M\rangle_\gamma \; {}_\gamma\langle \Phi_M|m_4, m_5, m_6\rangle_\gamma g_{m_1}^\dagger g_{m_2}^\dagger g_{m_3}^\dagger g_{m_6} g_{m_5} g_{m_4}$$

with the 3-body basis state $|m_1, m_2, m_3\rangle_\gamma = |\psi_{0,m_1}(\gamma)\rangle \otimes |\psi_{0,m_2}(\gamma)\rangle \otimes |\psi_{0,m_3}(\gamma)\rangle$ and

$$|\Phi_M\rangle_\gamma = \frac{\left(a_{\gamma_1}^\dagger + a_{\gamma_2}^\dagger + a_{\gamma_3}^\dagger\right)^M}{\sqrt{3^M M!}}|0,0,0\rangle_\gamma$$

And the anisotropic $\nu = 1/2$ fermionic Moore-Read states $\Psi_{MR}^{q=2}(\gamma)$ is the (minimum total $\bar{L}(\gamma)$) zero-energy ground state of the following anisotropic 3-body pseudopotential Hamiltonian,

$$H_{\rm int}(\gamma) = V_1 h_1(\gamma), \quad V_1 > 0, \tag{88}$$

where the projection operator $h_1(\gamma)$ is given by

$$\sum_{M=0}^{\infty} \sum_{m_1 m_2 m_3 m_4 m_5 m_6} {}_\gamma\langle m_1, m_2, m_3|\Phi_{3,M-3}\rangle_\gamma \; {}_\gamma\langle \Phi_{3,M-3}|m_4, m_5, m_6\rangle_\gamma g_{m_1}^\dagger g_{m_2}^\dagger g_{m_3}^\dagger g_{m_6} g_{m_5} g_{m_4}.$$

with the three-body state $|\Phi_{3,M-3}\rangle_\gamma$ being

$$\begin{aligned} |\Phi_{3,M-3}\rangle_\gamma \; = \; & B_M(a_{\gamma_1}^\dagger - a_{\gamma_2}^\dagger)(a_{\gamma_1}^\dagger - a_{\gamma_3}^\dagger)(a_{\gamma_2}^\dagger - a_{\gamma_3}^\dagger) \\ & \left(a_{\gamma_1}^\dagger + a_{\gamma_2}^\dagger + a_{\gamma_3}^\dagger\right)^{M-3}|0,0,0\rangle_\gamma \end{aligned}$$

and B_M being the appropriate normalization factor.

IV. NUMERICAL RESULTS FOR THE ANISOTROPIC FQH STATES

In this section we numerically study some properties of the the anisotropic Laughlin states $\Psi_L^q(\gamma)$ and its applicability in a system with dipole-dipole interaction. We first

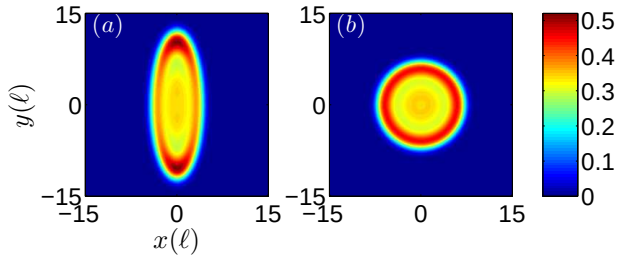

FIG. 3: (Color online) The density profiles (in units of $1/(2\pi\ell^2)$) of the $\nu = 1/3$ anisotropic Laughlin state with $\gamma = 1/2$ (a) and $\gamma = 0$ (b) for an $N = 10$ system. Note the yellow bulk whose value approximates 1/3.

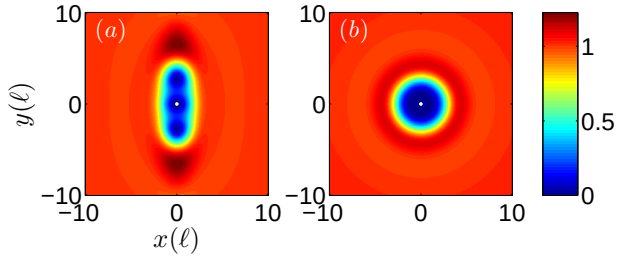

FIG. 4: (Color online) The pair correlation function for $\nu = 1/3$ Laughlin state with $\gamma = 1/2$ (a) and $\gamma = 0$ (b) of an $N = 10$ system.

demonstrate the deformation of the FQH droplet in density profile. The anisotropy leads also to the deformation of the correlation hole, which can be understood by the split of the third-order zero to three adjacent first-order zeros. As a trivial example we show that the isotropic Laughlin states, among the family of generalized states, is the ground state in the variational sense for isotropic hard-core interaction. On the other hand, the anisotropic dipole-dipole interaction picks a variational ground state with a finite γ parameter as expected.

A. Density profiles

The density profile for the anisotropic Laughlin state is most easily calculated using the Jack polynomials[6,25]. Explicitly for a finite number of particles, we can write

$$\varrho(\gamma, \boldsymbol{r}) = \langle \Psi_L^q(\gamma)|\hat{\Psi}^\dagger(\boldsymbol{r})\hat{\Psi}(\boldsymbol{r})|\Psi_L^q(\gamma)\rangle, \tag{89}$$

where $\hat{\Psi}(\boldsymbol{r})$ is the field operator. In Fig. 3, we plot the density profile $\varrho(\gamma, z)$ with $\gamma = 1/2$ for an $N = 10$ fermionic system at $\nu = 1/3$ and compare with that of the isotropic Laughlin wave function. Roughly speaking, in the anisotropic case the FQH droplet is stretched along the y-direction, while it maintains its value around 1/3, in units of $1/(2\pi\ell^2)$ in the bulk (indicated by the yellow color in the color plot).

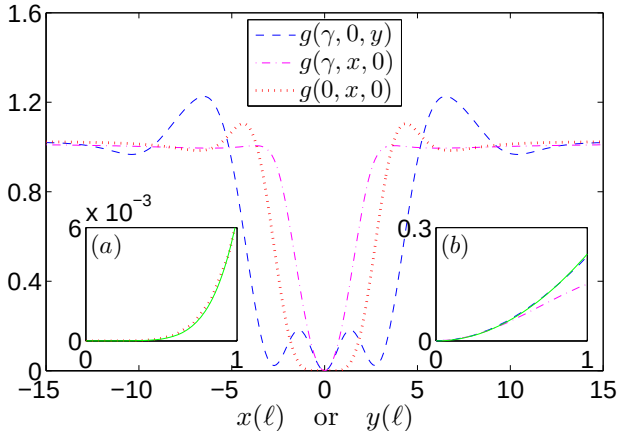

FIG. 5: (Color online) The pair correlation function $g(\gamma, 0, y)$, $g(\gamma, x, 0)$ with $\gamma = 1/2$ and $g(0, x, 0)$ of an $N = 10$ anisotropic Laughlin state. The inset (a) shows the asymptotic (green solid) line of $g(0, x, 0)$, which is proportional to x^6. And the inset (b) shows the asymptotic (green solid) line of $g(\gamma, x, 0)$ ($g(\gamma, 0, y)$), which is proportional to x^2 (y^2).

B. Pair correlation function

The density-density correlation function represents the conditional probability of find one particle at $\boldsymbol{r}$ when another is simultaneously at $\boldsymbol{r}'$. For the anisotropic Laughlin state $\Psi_L^q(\gamma)$, the density-density correlation function is defined by

$$G^{(2)}(\gamma, \boldsymbol{r}, \boldsymbol{r}') = \langle \Psi_L^q(\gamma)|\hat{\Psi}^\dagger(\boldsymbol{r})\hat{\Psi}^\dagger(\boldsymbol{r}')\hat{\Psi}(\boldsymbol{r}')\hat{\Psi}(\boldsymbol{r})|\Psi_L^q(\gamma).$$

Without loss of generality, we consider the pair correlation function with $\boldsymbol{r}'$ fixed at the origin,

$$g(\gamma, \boldsymbol{r}) = \frac{G^{(2)}(\gamma, \boldsymbol{r}, 0)}{\varrho(\gamma, \boldsymbol{r})\varrho(\gamma, 0)}. \tag{90}$$

In Fig. 4, we compare $g(\gamma, \boldsymbol{r})$ of an $N = 10$ Laughlin state at $\nu = 1/3$ with $\gamma = 1/2$ and 0. The two cases are clearly distinguishable, as the correlation hole for $\gamma = 1/2$ is stretched *non-monotonically* along the y-direction, along which the density profile is stretched. The center of the hole is strictly zero but two other minima developed in the y-direction are not. To explore more details we analyze $g(\gamma, x, y)$ as a function of x (or y) along the y-axis (x-axis) in Fig. 5. The comparison shows that the asymptotic behavior of $g(\gamma \neq 0, \boldsymbol{r} \to 0)$ is very different from that of $g(\gamma = 0, \boldsymbol{r} \to 0)$. The difference roots in the two-particle Laughlin states $\Psi_{0q}(\gamma, \boldsymbol{r}_1, \boldsymbol{r}_2)$ in Subsec. III C, from which we expect that $g(\gamma \neq 0, \boldsymbol{r} \to 0)$ vanishes as $|\boldsymbol{r}|^2$, while $g(\gamma = 0, \boldsymbol{r} \to 0)$ as $|\boldsymbol{r}|^6$, as indicated by Eqs. (84) and (85). The insets (a) and (b) in Fig. 5 confirm the asymptotic behavior.

We emphasize that in the anisotropic case the pair correlation function is isotropic for small enough $|\boldsymbol{r}|$, reflecting the first-order zero in the fermionic wave function enforced by the Pauli exclusion principle. The two-particle

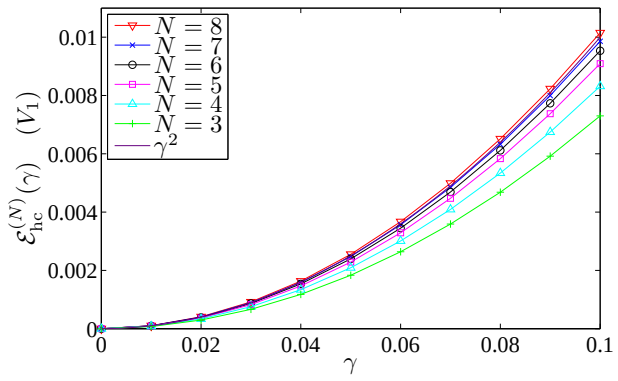

FIG. 6: (Color online) The hard-core interaction energy per particle $\mathcal{E}_{\rm hc}^{(N)}(\gamma)$ with respect to the distort parameter γ for $N = 3-8$ system. Here we also plot γ^2 (solid line) for comparison.

wave function can be regarded as the asymptotic wave function when two particles are significantly closer than their distance to any other particles. For $|\boldsymbol{r}|$ comparable to $|\gamma|$, the pair correlation function becomes anisotropic. The observed anisotropy at short distances encodes the geometrical deformation of the Laughlin state. When two particles are close to each other, each particle sees q zeros (for the Laughlin state at $\nu = 1/q$); the spatial spread of them reflects the extent of the deformation. Topological properties, manifested in the isotropic case by the qth-order zero, can be identified when one look not too close ($|\boldsymbol{r}| \gg |\gamma|$).

C. Variational energy

Given the family of Laughlin states $\Psi_L^q(\gamma)$, we test the variational principle on a trivial Hamiltonian with the isotropic hard-core interaction, which renders the isotropic Laughlin state as its exact zero-energy ground state, and a Hamiltonian with anisotropic dipolar-dipolar interaction. In both cases the expectation value of the Hamiltonian develops a minimum, which may be identified as the variational ground state energy. The minimum occurs at $\gamma = 0$ for the isotropic interaction and a nonzero γ for the anisotropic interaction.

Because the Laughlin wave function is the zero-energy ground state of the (isotropic) hard-core interaction, $\gamma = 0$ is naturally the minimum for the energy expectation value. The variational energy per particle is expected to increase as $\beta|\gamma|^2$ for small $|\gamma|$, as can be quickly understood, e.g., from the expansion of $\Psi_L^q(\gamma)$ with respect to small $|\gamma|$,

$$\Psi_L^q(\gamma) = \left[1 - \tfrac{1}{2}\gamma \textstyle\sum_i z_i^2 + \tfrac{1}{2}\gamma^* \sum_i \partial_{z_i}^2 + \ldots\right] \Psi_L^q(0). \quad (91)$$

Here β, like stiffness, quantifies the energy cost of the fluctuation of metric. In Fig. 6 we plot the hard-core interaction energy per particle for the anisotropic fermionic Laughlin state $\Psi_L^{q=3}(\gamma)$

$$\mathcal{E}_{\rm hc}^{(N)}(\gamma) = \frac{\langle \Psi_L^q(\gamma)|H_{\rm int}(\{V_m\}, \gamma = 0)|\Psi_L^q(\gamma)\rangle}{N}, \quad (92)$$

with $V_m = V_1\delta_{m,1}$. For the set of finite-size systems with $N = 3$-8, we confirm that the minimum of the hard-core interaction energy occurs identically at $\gamma = 0$. Therefore, the (isotropic) Laughlin wave function is indeed the optimal state for the hard-core interaction.

We now turn to the dipole-dipole interaction, which some of us studied in Ref. 9 by exact diagonalization. For the dipole-dipole interaction with the dipole moments being polarized in the x-z plane, the potential in the x-y plane is given by

$$V(\theta, x, y) = \int d\xi \frac{e^{-\xi^2/2d^2}}{\sqrt{2\pi d^2}} \frac{c_d}{(x^2+y^2+\xi^2)^{5/2}} \left\{x^2+y^2+\xi^2 - 3(\xi\cos\theta + x\sin\theta)^2\right\},$$

where c_d is the interaction strength and the motion of all the particles along the z-axis is frozen to the ground state of the axial harmonic oscillator $\pi^{-1/4}d^{-1/2}e^{-z^2/2d^2}$ with $d \ll \ell$ being the axial oscillator length. Here the tilt angle θ is introduced to tune the dipole-dipole interaction such that $V(\theta, z)$ is isotropic (anisotropic) on x-y plane for $\theta = 0$ ($\theta \neq 0$). The dipole-dipole interaction energy per particle in the anisotropic fermionic Laughlin state $\Psi_L^{q=3}(\gamma)$ is

$$\mathcal{E}_{\rm dd}^{(N)}(\theta, \gamma) = \frac{\langle \Psi_L^q(\gamma)| \sum_{i<j} V(\theta, z_i - z_j)|\Psi_L^q(\gamma)\rangle}{N}. \quad (93)$$

In Fig. 7, we plot $\mathcal{E}_{\rm dd}^{(N)}(\theta = 30°, \gamma)$ with respect to γ for finite-size system with $N = 3-8$. We found that the lowest ground state energy of the dipole-dipole interaction occurs for the anisotropic Laughlin state with a nonzero γ. The further quantitative comparison between the model anisotropic quantum Hall state and the exact ground state of the anisotropic dipole-dipole interaction will be given elsewhere.

V. SUMMARY

In this work we have explicitly constructed families of anisotropic fractional quantum Hall states, which are exact zero energy ground states of appropriate anisotropic short-range two- or multi-particle interactions. These families generalize the celebrated Laughlin, Moore-Read and Read-Rezayi states. Each family is parameterized by a single geometric parameter that describes the distortion of correlation hole in the density-density correlation functions. These states thus explicitly illustrate the existence of geometric degree of freedom in fractional quantum Hall effect, as recently demonstrated in Ref. 7. Application of these states in studies of systems with realistic anisotropic interactions, like dipole-dipole interaction, will be pursued in the near future and reported elsewhere.

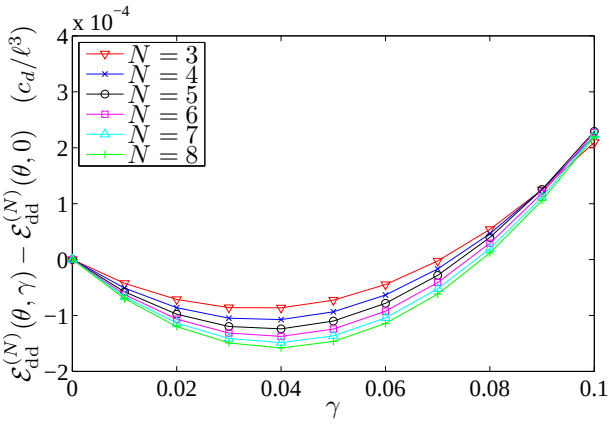

FIG. 7: (Color online) The dipole-dipole interaction energy per particle $\mathcal{E}_{\rm dd}^{(N)}(\theta=30^\circ,\gamma)$ with respect to the distort parameter γ for $N=3-8$ system. Here axial oscillator length d is set as 0.01ℓ.

Acknowledgments

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[1] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[2] G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
[3] N. Read and E. Rezayi, Phys. Rev. B **59**, 8084 (1999).
[4] S. H. Simon, E. H. Rezayi and N. R. Cooper, Phys. Rev. B **75**, 075318 (2007).
[5] X.-G. Wen and Z. Wang, Phys. Rev. B **77**, 235108 (2008).
[6] B. A. Bernevig and F. D. M. Haldane, Phys. Rev. Lett. **100**, 246802 (2008).
[7] F. D. M. Haldane, Phys. Rev. Lett. **107**, 116801 (2011).
[8] M. Mulligan, C. Nayak and S. Kachru, Phys. Rev. B **82**, 085102 (2010); **84**, 195124 (2011).
[9] R.-Z. Qiu, S.-P. Kou, Z.-X. Hu, X. Wan and S. Yi, Phys. Rev. A. **83**, 063633 (2011).
[10] B. Yang, Z. Papic, E. H. Rezayi, R. N. Bhatt and F. D. M. Haldane, arXiv:1202.5586 (to be published).
[11] J. Xia, J. P. Eisenstein, L. N. Pfeiffer and K. W. West, Nature Phys. **7**, 845 (2011).
[12] K. Musaelian and R. Joynt, J. Phys.: Condens. Matter **8**, L105 (1996).
[13] O. Ciftja and C. Wexler, Phys. Rev. B **65**, 045306 (2001).
[14] M. M Fogler, Europhys. Lett. **66**, 572 (2004).
[15] M. P. Lilly, K. B. Cooper, J. P. Eisenstein, L. N. Pfeiffer and K. W. West, Phys. Rev. Lett. **82**, 394 (1999).
[16] R. R. Du, D. C. Tsui, H. L. Stormer, L. N. Pfeiffer, K. W. Baldwin and K. W. West, Solid State Commun. **109**, 389 (1999).
[17] D. P. Arovas, K. Hasebe, X.-L. Qi and S.-C Zhang, Phys. Rev. B **79**, 224404 (2009).
[18] X.-L. Qi, Phys. Rev. Lett. **107**, 126803 (2011).
[19] N. Read and E. H. Rezayi, Phys. Rev. B **84**, 085316 (2011).
[20] F. D. M. Haldane, arXiv:0906.1854 (unpublished).
[21] H. Li and F. D. M. Haldane, Phys. Rev. Lett. **101**, 010504 (2008).
[22] F. D. M. Haldane and E. H. Rezayi, Phys. Rev. Lett. **60**, 956 (1988).
[23] S. H. Simon, E. H. Rezayi and N. R. Cooper, Phys. Rev. B **75**, 195306 (2007).
[24] M. Greiter, X.-G. Wen and F. Wilczek, Phys. Rev. Lett. **66**, 3205 (1991).
[25] B. A. Bernevig and N. Regnault, Phys. Rev. Lett. **103**, 206801 (2009).