Feynman path integral formulation of the Bell-Clauser-Horne-Shimony-Holt inequality in quantum field theory

G. Peruzzo [1,*] and S. P. Sorella [2,†]

[1] *Instituto de Física, Universidade Federal Fluminense, Campus da Praia Vermelha, Avenida Litorânea s/n, 24210-346, Niterói, Rio de Janeiro, Brazil*

[2] *UERJ—Universidade do Estado do Rio de Janeiro, Instituto de Física—Departamento de Física Teórica—Rua São Francisco Xavier 524, 20550-013, Maracanã, Rio de Janeiro, Brazil*

(Received 23 March 2023; accepted 11 April 2023; published 1 May 2023)

By employing a free scalar quantum field theory model previously introduced [G. Peruzzo and S. P. Sorella, Phys. Rev. D **106**, 125020 (2022)], we attempt to formulate the Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) inequality within the Feynman path integral. This possibility relies on the observation that the Bell-CHSH inequality exhibits a natural extension to quantum field theory in such a way that it is compatible with the time ordering T. By treating the Feynman propagator as a distribution and by introducing a suitable localizing set of compact support smooth test functions, we work out the path integral setup for the Bell-CHSH inequality, recovering the same results of the canonical quantization.

DOI: 10.1103/PhysRevD.107.105001

I. INTRODUCTION

The study of the Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) [1–6] inequality is one of the cornerstones of the physics of the entanglement, as documented by the large literature available on the subject in quantum mechanics.

From the theoretical point of view of relativistic quantum field theory, it seems fair to state that the study of the Bell-CHSH inequality is yet to be considered at its beginning. Let us mention that the topic has a great phenomenological interest in view of the future experiments at LHC, see [7] and references therein.

The study of the Bell-CHSH inequality in quantum field theory started with the pioneering work of Summers and co-workers [8–12], who, making use of algebraic quantum field theory [13], showed that even free fields lead to a violation of the Bell-CHSH inequality. This result highlights the strength of entanglement in quantum field theory [14]. Though, many aspects remain still to be unraveled. Let us quote, for example, the general treatment of interacting quantum field theories as well as the construction of a Becchi-Rouet-Stora-Tyutin (BRST) invariant setup for the Bell-CHSH inequality in Abelian and non-Abelian gauge theories.

More recently, following [8–12], we have constructed an explicit quantum field theory model built out by means of a massive free scalar field and of a suitable Bell-CHSH operator exhibiting a violation of the Bell-CHSH inequality at the quantum level [15]. The results obtained in [15] relied on the use of the canonical quantization.

The aim of the present work is that of pursuing the investigation of the Bell-CHSH inequality in quantum field theory. More precisely, we shall attempt to formulate the Bell-CHSH inequality within the framework of the Feynman path integral, a topic which, to our knowledge, has not yet been addressed. Needless to say, the path integral formulation will enable us to study the Bell-CHSH inequality for interacting field theories by employing the dictionary of the Feynman diagrams, including the BRST invariant formulation of Abelian and non-Abelian gauge theories.

Several issues arise when trying to achieve the path integral formulation of the Bell-CHSH inequality. Willing to present them briefly, we might start by mentioning that the Feynman path integral is intrinsically related to the chronological time ordering T. A second issue concerns the complex character of the Feynman propagator, i.e.,

$$\Delta_F(x-x') = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-x')}}{p^2 - m^2 + i\varepsilon} \neq (\Delta_F(x-x'))^{\dagger}. \quad (1)$$

Both aspects have to be properly addressed when comparing the Hermitian expression of the Bell-CHSH correlator obtained via canonical quantization with the corresponding quantum correlator evaluated with the Feynman path integral.

As we shall see in the following, these issues can be faced by making use of smeared fields, namely,

*gperuzzofisica@gmail.com
†silvio.sorella@gmail.com

$$\varphi(f) = \int_{\Omega} d^4x \varphi(x) f(x), \tag{2}$$

where $f(x)$ is a test function with compact support Ω, belonging to the space $\mathcal{C}_0^\infty(\mathbb{R}^4)$, i.e., to the space of smooth infinitely differentiable functions decreasing as well as their derivatives faster than any power of $(x) \in \mathbb{R}^4$ in any direction [13]. As it is apparent from (2), the introduction of the test function $f(x)$ has the effect of localizing the field $\varphi(x)$ in the region Ω. Working with smeared fields has many advantages. First, the use of a suitable set of test functions will enable us to introduce a Bell-CHSH correlator compatible with the time ordering T, a basic requirement in order to have a path integral formulation. Second, the analytic properties of the Fourier transform of test functions belonging to $\mathcal{C}_0^\infty(\mathbb{R}^4)$, see [16], allow us to handle the Feynman $i\varepsilon$ prescription by the usual Cauchy theorem, so as to recover exactly the result of the canonical setup.

Moreover, in addition to the pure mathematical aspects related to the introduction of the test functions, we would like to point out that, in the case of the study of the violation of the Bell-CHSH inequality, the smearing procedure acquires a rather clear and simple physical meaning. Looking at the details of one of the most recent experiments 17]], one realizes that issues like the so-called causality loophole, i.e., the effective experimental implementation of the spacelike separation between the two polarizers, namely, Alice and Bob's devices, is very carefully handled. Both polarizers are randomly rotating while the pair of entangled photons emitted by the source are flying toward them, so that it turns out to be impossible for the photons to communicate with each other about the direction in which their respective polarization is being measured. This very sophisticated setup has the practical effect of closing the causality loophole. Willing thus to achieve a relativistic quantum field theory framework for the Bell-CHSH inequality, it seems to us very helpful to employ a clear localization procedure that stays as close as possible with the experiments. This is precisely the role played by the introduction of the test functions, namely, allowing for a well-defined localization procedure in space-time.

The paper is organized as follows. In Sec. II, we elaborate on the comparison between the Bell-CHSH operator of quantum mechanics and that of quantum field theory, pointing out the very basic requirement of compatibility with the time ordering T. This section contains a detailed construction of the quantum field theory operators entering the Bell-CHSH correlator. Here, we shall rely on a field theory model built out with a pair of free scalar fields. As we shall see, this model enable us to make use of a squeezed state, allowing for the maximal violation of the Bell-CHSH inequality. Section III is devoted to the Feynman path integral formulation of the Bell-CHSH inequality by establishing the equivalence with the canonical formalism. In Sec. IV, we collect our conclusion.

Overall, for the benefit of the reader, we attempted to present the various topics in a self-contained way.

II. BELL-CHSH INEQUALITY IN QUANTUM MECHANICS AND IN RELATIVISTIC QUANTUM FIELD THEORY: SMEARING AND COMPATIBILITY WITH THE TIME ORDERING T

A. Bell-CHSH inequality in quantum mechanics: A short reminder

Let us begin by reminding the reader of the construction of the Bell-CHSH operator in quantum mechanics, as presented in textbooks, see, for example, [18–20]. One starts by introducing a two spin 1/2 operator

$$\begin{aligned} \mathcal{C}_{\mathrm{CHSH}} = [(\vec{\alpha}\cdot\vec{\sigma}_A + \vec{\alpha}'\cdot\vec{\sigma}_A) \otimes \vec{\beta}\cdot\vec{\sigma}_B \\ + (\vec{\alpha}\cdot\vec{\sigma}_A - \vec{\alpha}'\cdot\vec{\sigma}_A) \otimes \vec{\beta}'\cdot\vec{\sigma}_B], \end{aligned} \tag{3}$$

where (A, B) refer to Alice and Bob, $\vec{\sigma}$ are the spin 1/2 Pauli matrices, and $(\vec{\alpha}, \vec{\alpha}', \vec{\beta}, \vec{\beta}')$ are four arbitrary unit vectors.[1] The operator (3) has the renowned form

$$\mathcal{C}_{\mathrm{CHSH}} = (A + A')B + (A - A')B', \tag{4}$$

with (A, A') and (B, B') denoting the Alice and Bob spin operators

$$A = \vec{\alpha}\cdot\vec{\sigma}_A, \quad A' = \vec{\alpha}'\cdot\vec{\sigma}_A, \quad B = \vec{\beta}\cdot\vec{\sigma}_B, \quad B' = \vec{\beta}'\cdot\vec{\sigma}_B, \tag{5}$$

fulfilling the following commutation relations:

$$[A, B] = 0, \quad [A, B'] = 0, \quad [A', B] = 0, \quad [A', B'] = 0. \tag{6}$$

Moreover, (A, A') and (B, B') are all Hermitian, with eigenvalues ± 1.

On the basis of the so-called local realism of hidden variables [21], one expects that

$$|\mathcal{C}_{\mathrm{CHSH}}| \leq 2, \tag{7}$$

for any possible choice of the unit vectors $(\vec{\alpha}, \vec{\alpha}', \vec{\beta}, \vec{\beta}')$.

Nevertheless, it turns out that this inequality is violated by quantum mechanics, due to entanglement. In fact, when evaluating the Bell-CHSH correlator in quantum mechanics, i.e., $\langle\psi|\mathcal{C}_{\mathrm{CHSH}}|\psi\rangle$, where $|\psi\rangle$ is an entangled state as, for example, the Bell singlet, one gets

[1]Because of $\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$, it follows that $(\vec{n}\cdot\vec{\sigma})^2 = 1$ for any unit vector $|\vec{n}| = 1$.

$$|\langle\psi|\mathcal{C}_{\mathrm{CHSH}}|\psi\rangle| = 2\sqrt{2},$$
$$|\psi\rangle = \frac{|+\rangle_A \otimes |-\rangle_B - |-\rangle_A \otimes |+\rangle_B}{\sqrt{2}}. \quad (8)$$

The bound $2\sqrt{2}$ is known as Tsirelson's bound [22,23], providing the maximum violation of the CHSH inequality (7). The experiments carried out over the last decades, see [17] and references therein, have largely confirmed the violation of the Bell-CHSH inequality, being in very good agreement with the bound $2\sqrt{2}$.

B. Construction of the Bell-CHSH in quantum field theory: Localization and compatibility with the time ordering *T*

1. Basic features of the canonical quantization

In order to address the issue of the construction of the analog of the Bell-CHSH operator (3) in quantum field theory, it is useful to recall here a few basic properties of the canonical quantization of a free massive scalar field [13],

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\varphi\partial_\mu\varphi - m^2\varphi^2). \quad (9)$$

Expanding φ in terms of annihilation and creation operators, one gets

$$\varphi(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}(e^{-ikx}a_k + e^{ikx}a_k^\dagger),$$
$$k^0 = \omega(k,m) = \sqrt{\vec{k}^2 + m^2}, \quad (10)$$

where

$$[a_k, a_q^\dagger] = (2\pi)^3 2\omega(k,m)\delta^3(\vec{k}-\vec{q}),$$
$$[a_k, a_q] = 0, \qquad [a_k^\dagger, a_q^\dagger] = 0 \quad (11)$$

are the canonical commutation relations. A quick computation shows that

$$[\varphi(x),\varphi(y)] = i\Delta_{\mathrm{PJ}}(x-y) = 0 \quad \text{for } (x-y)^2 < 0, \quad (12)$$

where $\Delta_{\mathrm{PJ}}(x-y)$ is the Lorentz invariant causal Pauli-Jordan function, encoding the principle of relativistic causality

$$\Delta_{\mathrm{PJ}}(x-y) = \frac{1}{i}\int \frac{d^4k}{(2\pi)^3}(\theta(k^0) - \theta(-k^0))\delta(k^2 - m^2) \times e^{-ik(x-y)}, \quad (13)$$

$$\Delta_{\mathrm{PJ}}(x-y) = -\Delta_{\mathrm{PJ}}(y-x), \qquad (\partial_x^2 + m^2)\Delta_{\mathrm{PJ}}(x-y) = 0, \quad (14)$$

$$\Delta_{\mathrm{PJ}}(x-y) = \left(\frac{\theta(x^0-y^0) - \theta(y^0-x^0)}{2\pi}\right)\left(-\delta((x-y)^2) + m\frac{\theta((x-y)^2)J_1(m\sqrt{(x-y)^2})}{2\sqrt{(x-y)^2}}\right), \quad (15)$$

where J_1 is the Bessel function.

It is known that expression (10) is a too singular object, being in fact an operator valued distribution [13]. To give a well-defined meaning to Eq. (10), one introduces the smeared field

$$\varphi(h) = \int d^4x\varphi(x)h(x), \quad (16)$$

where $h(x)$ is a test function belonging to the space of compactly supported smooth functions $\mathcal{C}_0^\infty(\mathbb{R}^4)$. The support of $h(x)$, $supp_h$, is the region in which the test function $h(x)$ is nonvanishing. Moving to the Fourier space,

$$\hat{h}(p) = \int d^4x e^{ipx}h(x), \quad (17)$$

expression (16) becomes

$$\varphi(h) = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}\left(\hat{h}^*(\omega(k,m),\vec{k})a_k + \hat{h}(\omega(k,m),\vec{k})a_k^\dagger\right) = a_h + a_h^\dagger, \quad (18)$$

where $(a_h, a_h^\dagger)$ stand for

$$a_h = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}\hat{h}^*(\omega(k,m),\vec{k})a_k,$$
$$a_h^\dagger = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}\hat{h}(\omega(k,m),\vec{k})a_k^\dagger. \quad (19)$$

One sees that the smearing procedure has turned the too singular object $\varphi(x)$, Eq. (10), into an operator acting on the Hilbert space of the system, Eq. (18). When rewritten in terms of the operators $(a_f, a_g^\dagger)$, the canonical commutation relations (11) read

$$[a_h, a_{h'}^\dagger] = \langle h|h'\rangle, \quad (20)$$

where $\langle h|h'\rangle$ denotes the Lorentz invariant scalar product between the test functions h and h', i.e.,

$$\langle h|h'\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}\hat{h}^*(\omega(k,m),\vec{k})\hat{h}'(\omega(k,m),\vec{k}) = \int \frac{d^4\vec{k}}{(2\pi)^4}2\pi\theta(k^0)\delta(k^2-m^2)\hat{h}^*(k)\hat{h}'(k). \quad (21)$$

The scalar product (21) can be recast in configuration space. Taking the Fourier transform, one has

$$\langle h|h'\rangle = \int d^4x d^4x' h(x)\mathcal{D}(x-x')h'(x'), \tag{22}$$

where $\mathcal{D}(x-x')$ is the so-called Wightman function

$$\mathcal{D}(x-x') = \langle 0|\varphi(x)\varphi(x')|0\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}e^{-ik(x-x')},\\ k^0 = \omega(k,m), \tag{23}$$

which can be decomposed as

$$\mathcal{D}(x-x') = \frac{i}{2}\Delta_{\rm PJ}(x-x') + H(x-x'), \tag{24}$$

where $H(x-x') = H(x'-x)$ is the real symmetric quantity [24]

$$H(x-x') = \frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m)}(e^{-ik(x-x')} + e^{ik(x-x')})\\ k^0 = \omega(k,m). \tag{25}$$

Finally, the commutation relation (12) can be expressed in terms of smeared fields as

$$[\varphi(h),\varphi(h')] = i\Delta_{\rm PJ}(h,h'), \tag{26}$$

where h, h' are test functions and

$$\Delta_{\rm PJ}(h,h') = \int d^4x d^4x' h(x)\Delta_{\rm PJ}(x-x')h'(x'). \tag{27}$$

Therefore, the causality condition in terms of smeared fields becomes

$$[\varphi(h),\varphi(h')] = 0, \tag{28}$$

if $supp_h$ and $supp_{h'}$ are spacelike.

2. Weyl operators

For further use, let us present here the so-called Weyl operators. The Weyl operators are bounded unitary operators built out by exponentiating the smeared field, namely,

$$\mathcal{A}_h = e^{i\varphi(h)}, \tag{29}$$

where $\varphi(h)$ is the smeared field defined in Eq. (16). Making use of the following relation:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}, \tag{30}$$

valid for two operators (A,B) commuting with $[A,B]$, one immediately checks that the Weyl operators give rise to the following algebraic structure:

$$\mathcal{A}_h\mathcal{A}_{h}' = e^{-\frac{1}{2}[\varphi(h),\varphi(h')]}\mathcal{A}_{(h+h')} = e^{-\frac{i}{2}\Delta_{\rm PJ}(h,h')}\mathcal{A}_{(h+h')},\\ \mathcal{A}_h^\dagger = \mathcal{A}_{(-h)}, \tag{31}$$

where $\Delta_{\rm PJ}(h,h')$ is defined in Eq. (27). Also, using the canonical commutation relations written in the form (20), for the vacuum expectation value of $\mathcal{A}_h$, one gets

$$\langle 0|\mathcal{A}_h|0\rangle = e^{-\frac{1}{2}||h||^2}, \tag{32}$$

where the vacuum state $|0\rangle$ is the Fock vacuum: $a_k|0\rangle = 0$ for all modes k.

3. Algebra of the Bell operator and the time ordering T

We are now ready to face the issue of the construction of the Bell-CHSH operator in quantum field theory. We follow here the setup outlined in [8–12] and introduce the notion of "eligibility." A set of four field operators (A,A') and (B,B') are called eligible for the Bell-CHSH inequality if

(i) they are all Hermitian

$$A = A^\dagger, \qquad A' = A'^\dagger, \qquad B = B^\dagger, \qquad B' = B'^\dagger, \tag{33}$$

(ii) they obey the condition

$$(A,A') \quad \text{and} \quad (B,B') \text{ are bounded operators,}\\ \text{taking values in the interval } [-1,1], \tag{34}$$

(iii) Alice's operators (A,A') commute with Bob's operators (B,B'), namely,

$$[A,B] = 0, \qquad [A,B'] = 0, \qquad [A',B] = 0,\\ [A',B'] = 0. \tag{35}$$

Let us focus now on Eq. (35). Its fulfillment requires a well-specified localization property of both Alice and Bob operators in space-time. More precisely, relying on the relativistic causality (26), one is led to demand that the supports of Alice's test functions (f,f') belong to a space-time region Ω_A, which is spacelike with respect to the region Ω_B containing the supports of Bob's test functions (g,g'), i.e.,

$$(\text{supp}_{(f,f')}) \quad \text{spacelike with respect to} \quad (\text{supp}_{(g,g')}), \tag{36}$$

see Fig. 1.

These considerations make clear the key role of the relativistic causality when analyzing the Bell-CHSH inequality in quantum field theory. The use of smeared fields and of the test functions acquires a clear physical meaning: (f,f') and (g,g') act as space-time localizers for Alice's and Bob's operators, implementing in a practical way the fundamental principle of relativistic causality.

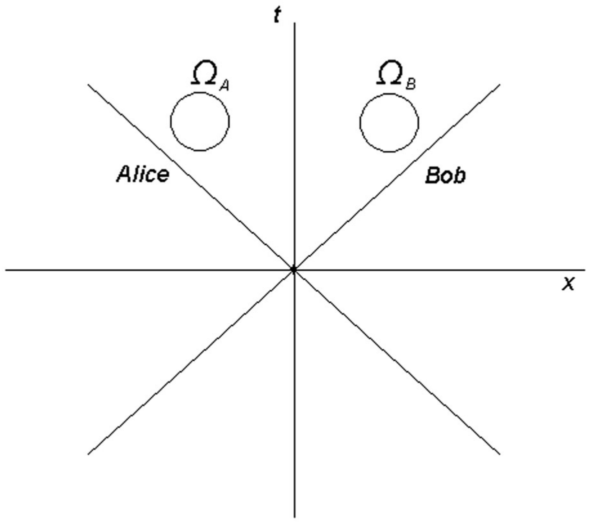

FIG. 1. Location of the labs of Alice and Bob in a two-dimensional space-time diagram.

Furthermore, the demand of spacelike separation between Alice and Bob has a relevant consequence for the time ordering T. In fact, if $O_1(x)$ and $O_2(y)$ are two field operators and $(x-y)^2 < 0$, it follows that

$$[O_1(x), O_2(y)] = 0, \qquad (x-y)^2 < 0, \tag{37}$$

so that the T product reduces to the identity, i.e.,

$$\begin{aligned} T(O_1(x)O_2(y)) &= \theta(x^0 - y^0)O_1(x)O_2(y) \\ &\quad + \theta(y^0 - x^0)O_2(y)O_1(x) \\ &= O_1(x)O_2(y). \end{aligned} \tag{38}$$

An immediate consequence of all this is that the Bell-CHSH combination is, by construction, left invariant by the time ordering T, namely,

$$T((A + A')B + (A - A')B')) = (A + A')B + (A - A')B'. \tag{39}$$

Although Eq. (39) looks a simple consequence of the requirements (33)–(35), it seems fair to state that it expresses a deep property of the Bell-CHSH particular combination. It paves the route for the Feynman path integral formulation.

C. Example of violation of the Bell-CHSH inequality in free quantum field theory

In order to provide an explicit example of the violation of the Bell-CHSH inequality in the vacuum state, we shall consider a model consisting of a pair of free massive real scalar fields (φ_A, φ_B),

$$\mathcal{L} = \frac{1}{2}(\partial^\mu\varphi_A\partial_\mu\varphi_A - m_A^2\varphi_A\varphi_A) + \frac{1}{2}(\partial^\mu\varphi_B\partial_\mu\varphi_B - m_B^2\varphi_B\varphi_B). \tag{40}$$

From the canonical quantization, we have

$$\begin{aligned} \varphi_A(t,\vec{x}) &= \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m_A)}(e^{-ikx}a_k + e^{ikx}a_k^\dagger), \\ k^0 &= \omega(k,m_A), \\ \varphi_B(t,\vec{x}) &= \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{2\omega(k,m_B)}(e^{-ikx}b_k + e^{ikx}b_k^\dagger), \\ k^0 &= \omega(k,m_B), \end{aligned} \tag{41}$$

where the only nonvanishing commutators among the annihilation and creation operators are

$$\begin{aligned} [a_k, a_q^\dagger] &= (2\pi)^3 2\omega(k,m_A)\delta^3(\vec{k} - \vec{q}), \\ [b_k, b_q^\dagger] &= (2\pi)^3 2\omega(k,m_B)\delta^3(\vec{k} - \vec{q}). \end{aligned} \tag{42}$$

To have well-defined operators in the Fock-Hilbert space, these fields are smeared with test functions, resulting in $(\varphi_A(h), \varphi_B(h))$. It is thus straightforward to evaluate the following commutation relations for the smeared fields:

$$\begin{aligned} [\varphi_A(h), \varphi_A(h')] &= i\Delta_{\mathrm{PJ}}^{m_A}(h,h'), \\ [\varphi_B(\tilde{h}), \varphi_B(\tilde{h}')] &= i\Delta_{\mathrm{PJ}}^{m_B}(\tilde{h},\tilde{h}'), \\ [\varphi_A(h), \varphi_B(\tilde{h})] &= 0, \end{aligned} \tag{43}$$

valid for any pair of test functions (h,h'), $(\tilde{h},\tilde{h}')$. The presence of the Pauli-Jordan function Δ_{PJ} in expressions (43) implements the relativistic causality in the model. In fact, if $supp_h$ and $supp_{h'}$ are spacelike, as well as those of $(\tilde{h},\tilde{h}')$, then the commutator of the corresponding smeared fields vanishes. The Fock vacuum of the model is defined as being the state $|0\rangle$ such that

$$a_k|0\rangle = 0, \qquad b_k|0\rangle = 0, \tag{44}$$

for any mode k.

Let us turn now to the Bell-CHSH inequality. We start with a general consideration of the algebraic relations fulfilled by the four operators (A, A'), (B, B').

Let us introduce the Hermitian Bell-CHSH field operator

$$\mathcal{C}_{\mathrm{CHSH}} = (A + A')B + (A - A')B'. \tag{45}$$

In agreement with [8–12], we shall say that the Bell-CHSH inequality is violated at the quantum level in the vacuum if

$$|\langle 0|\mathcal{C}_{\mathrm{CHSH}}|0\rangle| > 2. \tag{46}$$

We proceed by rewriting the vacuum expectation value of the Bell-CHSH operator as

$$\langle 0|\mathcal{C}_{\rm CHSH}|0\rangle >= \langle 0|\mathcal{U}^\dagger\mathcal{U}((A+A')\mathcal{U}^\dagger\mathcal{U}B+(A-A')\mathcal{U}^\dagger\mathcal{U}B') \times\mathcal{U}^\dagger\mathcal{U}|0\rangle, \tag{47}$$

where $\mathcal{U}$ stands for a unitary operator,

$$\mathcal{U}^\dagger\mathcal{U} = 1. \tag{48}$$

Introducing now the unitary equivalent operators

$$\hat{A} = \mathcal{U}^\dagger A\mathcal{U}, \qquad \hat{A}' = \mathcal{U}^\dagger A'\mathcal{U}, \qquad \hat{B} = \mathcal{U}^\dagger B\mathcal{U}, \qquad \hat{B}' = \mathcal{U}^\dagger B\mathcal{U}, \tag{49}$$

it is easily checked that $(\hat{A},\hat{A}')$ and $(\hat{B},\hat{B}')$ fulfill the same algebraic relations of (A,A') and (B,B'), namely,

$$\hat{A}^2 = 1, \qquad \hat{A}'^2 = 1, \qquad \hat{B}^2 = 1, \qquad \hat{B}'^2 = 1,$$
$$[\hat{A},\hat{B}] = 0, \qquad [\hat{A},\hat{B}'] = 0, \qquad [\hat{A}',\hat{B}] = 0, \qquad [\hat{A}',\hat{B}'] = 0. \tag{50}$$

Suppose now that the unitary operator $\mathcal{U}$ is such that its action on the vacuum $|0\rangle$ creates a two-mode entangled state $|\eta\rangle$, namely,

$$\mathcal{U}|0\rangle = |\eta\rangle. \tag{51}$$

This is the case, for instance, of the two-mode squeezed state. In such a case, the unitary operator is nothing but the squeezed operator[2]

$$\mathcal{U} = e^{\frac{r}{2}(a_f^\dagger b_g^\dagger - a_f b_g)}, \qquad |\eta\rangle = (1-\eta^2)^{\frac{1}{2}}\sum_{n=0}\eta^n|n_f n_g\rangle, \qquad \eta = {\rm tgh}(r), \tag{53}$$

and

$$|n_f n_g\rangle = \frac{1}{n}(a_f^\dagger)^n(b_g^\dagger)^n|0\rangle. \tag{54}$$

Therefore, we get the equality

$$\langle 0|(A+A')B+(A-A')B'|0\rangle = \langle\eta|(\hat{A}+\hat{A}')\hat{B} + (\hat{A}-\hat{A}')\hat{B}'|\eta\rangle. \tag{55}$$

This equation states that the vacuum expectation value of the Bell-CHSH operator can be obtained by evaluating the expectation value of the unitary equivalent combination $(\hat{A}+\hat{A}')\hat{B}+(\hat{A}-\hat{A}')\hat{B}'$ in the squeezed state $|\eta\rangle$. Equation (55) is very helpful, both from theoretical and computational points of view. In fact, following [9], the squeezed state $|\eta\rangle$ can be rewritten as the sum of even and odd modes, i.e.,

$$|\eta\rangle = (1-\eta^2)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty}\eta^{2n}|2n_f 2n_g\rangle + \sum_{n=0}^{\infty}\eta^{2n+1}|(2n_f+1) \times(2n_g+1)\rangle\right). \tag{56}$$

One defines the operators $\hat{A}_i = (\hat{A},\hat{A}')$ and $\hat{B}_k = (\hat{B},\hat{B}')$ as [9]

$$\hat{A}_i|2n_f\cdot\rangle = e^{i\alpha_i}|(2n_f+1)\cdot\rangle, \qquad \hat{A}_i|(2n_f+1)\cdot\rangle = e^{-i\alpha_i}|2n_f\cdot\rangle, \tag{57}$$

and

$$\hat{B}_k|\cdot 2n_g\rangle = e^{i\beta_k}|\cdot(2n_g+1)\rangle, \qquad \hat{B}_k|\cdot(2n_g+1)\rangle = e^{-i\beta_i}|\cdot 2n_g\rangle, \tag{58}$$

where (α_i,β_k) are arbitrary real quantities. The operators $\hat{A}_i$ act only on the first entry, while the operators B_k act only on the second.

From a quick computation, it turns out that

$$\langle\eta|(\hat{A}+\hat{A}')\hat{B}+(\hat{A}-\hat{A}')\hat{B}'|\eta\rangle = \frac{2\eta}{1+\eta^2}[\cos(\alpha_1+\beta_1)+\cos(\alpha_2+\beta_1) + \cos(\alpha_1+\beta_2)-\cos(\alpha_2+\beta_2)]. \tag{59}$$

Setting

$$\alpha_1 = 0, \qquad \alpha_2 = \frac{\pi}{2}, \qquad \beta_1 = -\frac{\pi}{4}, \qquad \beta_2 = \frac{\pi}{4}, \tag{60}$$

one gets

$$\langle 0|\mathcal{C}_{\rm CHSH}|0\rangle = \frac{2\eta}{1+\eta^2}2\sqrt{2}, \tag{61}$$

which attains Tsirelson's bound for $\eta\approx 1$,

[2]We point out that the test functions can be always normalized to 1, namely,

$$f\to\frac{f}{||f||}\Rightarrow ||f|| = 1. \tag{52}$$

As a consequence $[a_f,a_f^\dagger] = 1$. Similarly, $[b_g,b_g^\dagger] = 1$.

$$\langle 0|\mathcal{C}_{\rm CHSH}|0\rangle \approx 2\sqrt{2}. \tag{62}$$

This result is in full agreement with that of [9].

III. FEYNMAN PATH INTEGRAL FORMULATION OF THE BELL-CHSH INEQUALITY

Relying on the results and observations of the previous sections, let us discuss now the Feynman path integral formulation of the Bell-CHSH inequality. To that end, let us start by introducing the generating functional $\mathcal{Z}(j)$ of the time ordered correlation functions,

$$\mathcal{Z}(j) = \frac{\int [D\varphi] e^{i(S(\varphi)+\int d^4x j\varphi)}}{\int [D\varphi] e^{i(S(\varphi))}} = e^{-\frac{i}{2}\int d^4x d^4y j(x)\Delta_F(x-y)j(y)}, \tag{63}$$

where $\Delta_F(x-y)$ is the Feynman propagator, i.e.,

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)}}{p^2-m^2+i\varepsilon}. \tag{64}$$

It is useful to remind the reader here of the expression of $\Delta_F(x-y)$ in configuration space. Using the same notations of [25], $\Delta_F(x-y)$ can be written as

$$\Delta_F(x-y) = \frac{1}{2}(\theta(x^0-y^0)-\theta(y^0-x^0))\Delta_{\rm PJ}(x-y) - iH(x-y), \tag{65}$$

where $\Delta_{\rm PJ}$ is the Pauli-Jordan function (13), and H is the symmetric expression of Eq. (25). Explicitly,

$$\begin{aligned}\Delta_F(x-y) = &-\frac{1}{4\pi}\delta((x-y)^2) + \frac{m\theta((x-y)^2)}{8\pi\sqrt{(x-y)^2}} \\ &\times \left(J_1(m\sqrt{(x-y)^2}) - iN_1(m\sqrt{(x-y)^2})\right) \\ &- \frac{im\theta(-(x-y)^2)}{4\pi^2\sqrt{-(x-y)^2}} K_1\left(m\sqrt{-(x-y)^2}\right),\end{aligned} \tag{66}$$

where, from the second line, one observes the well-known noncausal behavior of the Feynman propagator. In the above expression, J_1 is the Bessel function, N_1 is the Neumann function, and K_1 is the modified Bessel function.

To achieve the equivalence between the path integral and the canonical formalism, we evaluate, for example, the correlation functions of two Weyl operators in both cases, the aim being that of showing that

$$\langle e^{i\varphi(h)}e^{i\varphi(h')}\rangle_{\rm Feyn} = \langle 0|e^{i\varphi(h)}e^{i\varphi(h')}|0\rangle_{\rm can}, \tag{67}$$

where the supports of the two test functions $(h,h') \in \mathcal{C}_0^\infty(\mathbb{R}^4)$ are taken as located in the positive half-plane $t>0$, as in Fig. 1, and are spacelike, i.e.,

$$(\mathrm{supp}_{(h)}) \quad \text{spacelike with respect to} \quad (\mathrm{supp}_{(h')}), \tag{68}$$

a feature that, as already underlined, reduces the chronological ordering T to unity.

The computation of the left-hand side of Eq. (67) is easily done with the help of (63), namely,

$$\mathcal{Z}(j) = \langle e^{i\varphi(j)}\rangle_{\rm Feyn}. \tag{69}$$

Therefore, setting

$$j = h + h', \tag{70}$$

it follows that

$$\begin{aligned}\langle e^{i\varphi(h)}e^{i\varphi(h')}\rangle_{\rm Feyn} &= e^{-\frac{i}{2}\Delta_F(h+h',h+h')} \\ &= e^{-\frac{i}{2}(\Delta_F(h,h)+2\Delta_F(h,h')+\Delta_F(h',h'))},\end{aligned} \tag{71}$$

where $(\Delta_F(h,h),\Delta_F(h,h'),\Delta_F(h',h'))$ denote the smeared expressions

$$\Delta_F(h,h') = \int d^4x d^4y h(x)\Delta_F(x-y)h'(y), \tag{72}$$

$$\begin{aligned}\Delta_F(h,h) &= \int d^4x d^4y h(x)\Delta_F(x-y)h(y), \\ \Delta_F(h',h') &= \int d^4x d^4y h'(x)\Delta_F(x-y)h'(y).\end{aligned} \tag{73}$$

One sees that Eq. (71) demands the evaluation of two kinds of smeared expressions involving the Feynman propagator. Let us first consider expression (72), where the smearing is done with respect to two different test functions: (h,h'). Reminding the reader that the supports of h and h' are spacelike, Eq. (68), one can rely directly on expression (65), from which one realizes that the Pauli-Jordan term $\Delta_{PJ}(x-y)$ does not contribute since it vanishes for spacelike separations. Thus,

$$\Delta_F(h,h') = -iH(h,h') = -i\int d^4x d^4y h(x)H(x-y)h'(y). \tag{74}$$

Though, owing to the general definition of the scalar product of test functions in terms of Wightman two-point function, Eqs. (22)–(24), it follows that

$$\Delta_F(h,h') = -i\langle h|h'\rangle. \tag{75}$$

Let us now focus on the expressions $\Delta_F(h,h)$ and $\Delta_F(h',h')$ in Eq. (73). These quantities require a different handling, as the Feynman propagator is smeared exactly over the same support. We proceed by moving to the Fourier space, i.e.,

$$\Delta_F(h,h) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\varepsilon} |h(p_0, \vec{p})|^2, \tag{76}$$

where $h(p_0, \vec{p})$ is the Fourier transform of $h(x)$,

$$h(p_0, \vec{p}) = \int d^x e^{ipx} h(x). \tag{77}$$

As is well known [16], being $h(x)$ a Schwartz-type test function, its Fourier transform displays an exponential decay at large $|\mathrm{p}|$. Moreover, since $h(x) \in \mathcal{C}_0^\infty(\mathbb{R}^4)$ and its support is located in the positive half-plane $t > 0$, Fig. 1, it follows that $h(p_0, \vec{p})$ can be analytically continued to an entire function in the complex p_0 plane, decaying very fast for large values of $\mathrm{Im}(p_0)$ in the positive complex p_0 half-plane.

Let us illustrate this relevant property with a simple one-dimensional example taken from Chap. II of [16]. Consider the function $f(x) \in \mathcal{C}_0^\infty(\mathbb{R})$ defined as

$$f(x) = \begin{cases} \mathcal{C} e^{-\frac{1}{(x-a)^2 (x-b)^2}} & \text{if } x \in [a,b], a, b > 0, \\ 0 & \text{if } x \notin [a,b], \end{cases} \tag{78}$$

where $\mathcal{C}$ is a normalization factor. The function $f(x)$ is a smooth function, infinitely differentiable, which is nonvanishing only in the interval $x \in [a,b]$. For the Fourier transform, we have

$$\hat{f}(p) = \int_{-\infty}^{\infty} dx e^{ipx} f(x). \tag{79}$$

Moreover, since $f(x)$ has compact support, the integral (79) becomes

$$\hat{f}(p) = \int_a^b dx e^{ipx} f(x), \tag{80}$$

and it can be analytically continued to an entire function in the complex plane

$$z = p + i\tau, \qquad \hat{f}(z) = \int_a^b dx e^{ipx} e^{-\tau x} f(x). \tag{81}$$

Notice that the analytic continuation of $\hat{f}(p)$ to an entire function is possible thanks to the fact that $f(x)$ has compact support, so that the integral (81) does exist for all values of τ. Of course, $\hat{f}(z)$ decays very fast to zero when $\mathrm{Im}(z) = \tau \to \infty$:

$$\lim_{\tau\to\infty} \hat{f}(p + i\tau) = 0. \tag{82}$$

Therefore, as a consequence of these properties, we can evaluate expression (76) by employing the residue Cauchy theorem in the complex p_0 plane, by closing the contour to infinity in the upper positive imaginary half-plane, getting nothing but

$$\Delta_F(h,h) = -i\|h\|^2. \tag{83}$$

Collecting everything, it turns out that

$$\langle e^{i\varphi(h)} e^{i\varphi(h')} \rangle_{Feyn} = e^{\frac{i}{2}\Delta_F(h+h')} = e^{-\frac{\|h+h'\|^2}{2}}, \tag{84}$$

which is exactly the result obtained from the canonical quantization (32). Finally, we get

$$\langle e^{i\varphi(h)} e^{i\varphi(h')} \rangle_{\mathrm{Feyn}} = \langle 0| e^{i\varphi(h)} e^{i\varphi(h')} |0\rangle_{\mathrm{can}}, \tag{85}$$

showing thus the equivalence between the Feynman path integral and the canonical quantization for the Bell-CHSH inequality.

IV. CONCLUSION

In this work, we have pursued the study of the Bell-CHSH inequality in relativistic quantum field theory by implementing its formulation within the Feynman path integral. Both canonical quantization and functional integral yield the same expression for the correlation function of Weyl operators.

This feature relies on the observation that, by construction, the Bell-CHSH combination is compatible with the fundamental principle of relativistic causality, as required by demanding that Alice and Bob be spacelike separated. Moreover, the localization of Alice and Bob in space-time can be given a precise mathematical formulation by employing a suitable set of smooth test functions with compact support, which act alike localizers for the bounded operators entering the Bell-CHSH inequality, which turns out to be compatible with the time ordering T, a key property for the path integral formulation.

We strengthen that the Feynman path integral formulation of the Bell-CHSH inequality opens the door to many applications, such as:

(i) Treatment of interacting field theories by employing the usual dictionary of Feynman diagrams;

(ii) Study of the Bell-CHSH inequality in Abelian and non-Abelian gauge theories in a manifest BRST invariant setting, through the use of the Faddeev-Popov BRST invariant action. In this regard, we refer to [15], where the BRST invariant formulation for the Weyl operators of Yang-Mills theories in presence of Higgs fields has been outlined.

(iii) Finally, the path integral formulation might enable us to estimate possible nonperturbative contributions to the Bell-CHSH inequality stemming from the existence of soliton sectors of the theory under investigation. From that point of view, non-Abelian gauge theories are particularly challenging. In addition to the existence of solitons, in this case, one is led to face the hard problem of the existence of the Gribov copies, intrinsically related to the Faddeev-Popov quantization procedure. For instance, in the case of pure Yang-Mills theories, Gribov copies lead to deep changes in the nonperturbative infrared region, exhibiting a strong connection with gluon confinement, see [26]. We hope to report soon on these matters.

ACKNOWLEDGMENTS

The authors would like to thank the Brazilian agencies CNPq and FAPERJ for financial support. This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior–Brasil (CAPES)–Finance Code 001. S. P. S. is supported by CNPq under Contract No. 301030/2019-7. G. P. is a FAPERJ postdoctoral fellow in the Pós-Doutorado Nota 10 program under the contracts E-26/205.924/2022 and E-26/205.925/2022.

[1] J. S. Bell, Phys. Phys. Fiz. **1**, 195 (1964).
[2] J. S. Bell, Rev. Mod. Phys. **38**, 447 (1966).
[3] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[4] S. J. Freedman and J. F. Clauser, Phys. Rev. Lett. **28**, 938 (1972).
[5] J. F. Clauser and M. A. Horne, Phys. Rev. D **10**, 526 (1974).
[6] J. F. Clauser and A. Shimony, Rept. Prog. Phys. **41**, 1881 (1978).
[7] R. Ashby-Pickering, A. J. Barr, and A. Wierzchucka, arXiv:2209.13990.
[8] S. J. Summers and R. Werner, J. Math. Phys. (N.Y.) **28**, 2440 (1987).
[9] Stephen J. Summers and Reinhard Werner, J. Math. Phys. (N.Y.) **28**, 2448 (1987).
[10] S. J. Summers and R. Werner, Commun. Math. Phys. **110**, 247 (1987).
[11] S. J. Summers, Bell'S inequalities and quantum field theory, Report No. CPT-88/P-2183, 1988.
[12] S. J. Summers and R. F. Werner, Lett. Math. Phys. **33**, 321 (1995).
[13] R. Haag, *Local Quantum Physics: Fields, Particles, Algebras* (Springer-Verlag, Berlin, 1992).
[14] R. Verch and R. F. Werner, Rev. Math. Phys. **17**, 545 (2005).
[15] G. Peruzzo and S. P. Sorella, Phys. Rev. D **106**, 125020 (2022).
[16] I. M. Gel'Fand and G. E. Shilov, *Generalized Functions, Vol. I* (Academic Press, New York, 1964).
[17] M. Giustina *et al.*, Phys. Rev. Lett. **115**, 250401 (2015).
[18] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2010).
[19] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, 2002).
[20] B. Zwiebach, *Mastering Quantum Mechanics* (MIT Press, Cambridge, MA, 2022).
[21] J. S. Bell, J. Phys. (Paris), Colloq. **42**, 41 (1981).
[22] B. S. Cirelson, Lett. Math. Phys. **4**, 93 (1980).
[23] B. S. Tsirelson, J. Math. Sci. **36**, 557 (1987).
[24] G. Scharf, *Finite Quantum Electrodynamics: The Causal Approach* (Springer, New York, 1995).
[25] R. Greiner, *Field Quantization* (Springer, New York, 1996).
[26] N. Vandersickel and D. Zwanziger, Phys. Rep. **520**, 175 (2012).