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The fundamental collective degree of freedom of fractional quantum Hall states is identified as a unimodular two-dimensional spatial metric that characterizes the local shape of the correlations of the incompressible fluid. Its quantum fluctuations are controlled by a topologically-quantized "guiding-center spin". Charge fluctuations are proportional to its Gaussian curvature.

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In this Letter, I point out the apparently previously-unnoticed geometric degree of freedom of the fractional quantum Hall effect (FQHE), that fundamentally distinguishes it from the integer effect, and will provide the basis for a new description of its collective properties as a fluctuating quantum geometry.

The simplest model Hamiltonian for N interacting electrons bound to a two-dimensional (2D) planar "Hall surface" traversed by a uniform magnetic flux density is

$$H = \sum_{i=1}^{N} \frac{1}{2m} g^{ab}\pi_{ia}\pi_{ib} + \frac{1}{A}\sum_{\boldsymbol{q}} V(\boldsymbol{q}) \sum_{i<j} e^{i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}. \quad (1)$$

Here $\boldsymbol{r}_i - \boldsymbol{r}_j = (r_i^a - r_j^a)\boldsymbol{e}_a$, $[r_i^a, r_j^b] = 0$, are the relative displacements of the particles on the 2D surface with orthonormal tangent vectors $\boldsymbol{e}_a$, $a = 1, 2$, and $\pi_{ia} = \boldsymbol{e}_a \cdot \boldsymbol{\pi}_i$ are the components of the gauge-invariant dynamical momenta, with commutation relations

$$[r_i^a, \pi_{jb}] = i\delta_{ij}\hbar\delta_b^a, \quad [\pi_{ia}, \pi_{jb}] = i\delta_{ij}\epsilon_{ab}\hbar^2/\ell_B^2. \quad (2)$$

I use Einstein summation convention: $q_a r^a = \boldsymbol{q}\cdot\boldsymbol{r}$ (index placement distinguishes real-space vectors r^a from dual (reciprocal) vectors q_a); δ_b^a is the Kronecker symbol, and $\epsilon_{ab} = \epsilon^{ab}$ is the 2D antisymmetric Levi-Civita symbol. A periodic boundary condition (pbc) can be imposed on a fundamental region of the plane with area $A = 2\pi\ell_B^2 N_\Phi$, which restricts wavevectors $\boldsymbol{q}$ to the reciprocal lattice; N_Φ is an integer, and $2\pi\ell_B^2$ is the area through which a London magnetic flux quantum h/e passes.

The parameters of the Hamiltonian are: (1) a Galileian effective mass tensor mg_{ab}, where g_{ab} is a positive-definite "Galileian metric" with $\det g = 1$ (*i.e.*, a *unimodular* metric) and inverse g^{ab}, and $m > 0$ is the effective mass that controls the cyclotron frequency $\omega_B = \hbar/m\ell_B^2$; (2) $V(\boldsymbol{q})$ which is the Fourier transform of an unretarded translationally-invariant two-body interaction potential.

In principle, the real function $V(\boldsymbol{q})$ is the Fourier transform of the long-ranged unscreened Coulomb potential, with the small-$\boldsymbol{q}$ behavior

$$\lim_{\lambda\to 0} \lambda V(\lambda\boldsymbol{q}) \to \frac{e^2}{2\varepsilon}(\tilde{g}^{ab}q_a q_b)^{-1/2}, \quad (3)$$

where $\tilde{g}^{ab}$ is the inverse of a unimodular *Coulomb metric* $\tilde{g}_{ab}$, controlled by the dielectric properties of the surrounding 3D insulating media, while the large-$\boldsymbol{q}$ behavior of $V(\boldsymbol{q})$ is controlled by the quantum well that binds electrons to the surface. The singularity of $V(0)$ does not affect incompressibility, and can be screened by a metallic plane placed parallel to the surface.

There is no fundamental reason for the Coulomb and Galileian metrics to coincide, unless there is an atomic-scale discrete $(n > 2)$-fold rotational symmetry of the surface, and no tangential magnetic flux. I will argue that the usual implicit assumption of rotational symmetry hides key geometric features of the FQHE.

In the presence of the magnetic field, the canonical degrees of freedom $\{\boldsymbol{r}_i, \boldsymbol{p}_i\}$ are reorganized into two independent sets, the dynamical momenta $\{\boldsymbol{\pi}_i\}$, which I will call "left-handed" degrees of freedom, and the "guiding centers" $\{\boldsymbol{R}_i\}$, the "right-handed" degrees of freedom,

$$R_i^a = r_i^a - \hbar^{-1}\epsilon^{ab}\pi_{ib}\ell_B^2, \quad [R_i^a, R_j^b] = -i\delta_{ij}\epsilon^{ab}\ell_B^2, \quad (4)$$

with $[R_i^a, \pi_{jb}] = 0$. The pbc further restricts the guiding-center variables to the set of unitary operators $\rho_{\boldsymbol{q},i} = \exp i\boldsymbol{q}\cdot\boldsymbol{R}_i$, which obey the Heisenberg algebra

$$\rho_{\boldsymbol{q},i}\rho_{\boldsymbol{q}',i} = e^{i\frac{1}{2}\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2}\rho_{\boldsymbol{q}+\boldsymbol{q}',i}, \quad \boldsymbol{q}\times\boldsymbol{q}' \equiv \epsilon^{ab}q_a q_b'; \quad (5)$$

reciprocal vectors $\boldsymbol{q}, \boldsymbol{q}'$ compatible with the pbc obey $(\exp i\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2)^{N_\Phi} = 1$. The pbc can be expressed as

$$(\rho_{\boldsymbol{q},i})^{N_\Phi}|\Psi\rangle = (\eta_{\boldsymbol{q}})^{N_\Phi}|\Psi\rangle \quad (6)$$

for all states in the Hilbert space, where $\eta_{\boldsymbol{q}} = 1$ if $\frac{1}{2}\boldsymbol{q}$ is an allowed reciprocal vector, and $\eta_{\boldsymbol{q}} = -1$ otherwise. This leads to the recurrence relation

$$\rho_{\boldsymbol{q}+N_\phi\boldsymbol{q}',i} = \left(\eta_{\boldsymbol{q}'}e^{i\frac{1}{2}\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2}\right)^{N_\Phi}\rho_{\boldsymbol{q},i} = \pm\rho_{\boldsymbol{q},i}. \quad (7)$$

For a given particle label i, the set of independent operators $\boldsymbol{\rho}_{\boldsymbol{q},i}$ can be reduced to a set of N_Φ^2 operators where $\boldsymbol{q} \in \mathrm{BZ}$ takes one of a set of N_Φ^2 distinct values that define an analog of a "Brillouin zone". Let

$$\delta^2_{\boldsymbol{q},\boldsymbol{q}'} \equiv \frac{1}{N_\Phi}{\sum_{\boldsymbol{q}''}}' e^{i\boldsymbol{q}''\times(\boldsymbol{q}-\boldsymbol{q}')}. \quad (8)$$

(Primed sums are over the BZ.) Then $\delta^2_{\boldsymbol{q},\boldsymbol{q}'} = 0$ if $\boldsymbol{q}$ and $\boldsymbol{q}'$ are distinct, and has the value N_Φ if they are equivalent; with this definition $\delta^2_{\boldsymbol{q},\boldsymbol{q}'}$ becomes $2\pi\delta^2(\boldsymbol{q}\ell_B - \boldsymbol{q}'\ell_B)$ in the limit $N_\Phi \to \infty$, where $\delta^2(\boldsymbol{x})$ is the 2D Dirac delta-function. It is convenient to choose the BZ so it has inversion symmetry: $\boldsymbol{q} \in$ BZ $\to -\boldsymbol{q} \in$ BZ, and $\rho_{\boldsymbol{q}=0,i}$ is the identity. The set of $N_\Phi^2 - 1$ operators $\{\rho_{\boldsymbol{q},i}, \boldsymbol{q} \in \text{BZ}, \boldsymbol{q} \neq 0\}$ are the generators of the Lie algebra $SU(N_\Phi)$. Both $\rho_{\boldsymbol{q},i}$ and also (as noted by Girvin, MacDonald and Platzman[1]) the "coproduct" $\rho_{\boldsymbol{q}} = \sum_i \rho_{\boldsymbol{q},i}$, obey

$$[\rho_{\boldsymbol{q}}, \rho_{\boldsymbol{q}'}] = 2i \sin(\tfrac{1}{2}\boldsymbol{q} \times \boldsymbol{q}' \ell_B^2)\rho_{\boldsymbol{q}+\boldsymbol{q}'}. \quad (9)$$

In this form of the Lie algebra, the quadratic Casimir is

$$C_2 = \frac{1}{2N_\Phi}{\sum_{\boldsymbol{q}\neq 0}}' \rho_{\boldsymbol{q}}\rho_{-\boldsymbol{q}} = \frac{N(N_\Phi^2 - N)}{2N_\Phi} + \sum_{i<j} P_{ij}, \quad (10)$$

where P_{ij} exchanges guiding centers of particles i and j. For $N = 1$, the $\rho_{\boldsymbol{q},i}$ form the N_Φ-dimensional fundamental (defining) $SU(N_\Phi)$ representation of one-particle states of a Landau level, with $C_2 = (N_\Phi^2 - 1)/2N_\Phi$.

The high-field condition is defined by

$$\hbar\omega_B \gg \frac{1}{A}\sum_{\boldsymbol{q}} V(\boldsymbol{q}) f(\boldsymbol{q})^2, \quad f(\boldsymbol{q}) = e^{-\frac{1}{4}q_g^2\ell_B^2}, \quad (11)$$

where $f(\boldsymbol{q})$ is the lowest-Landau-level form-factor, and $q_g^2 \equiv g^{ab}q_a q_b$. In this limit, the low-energy eigenstates of the model have all the particles in the lowest Landau level, and can be factorized into a simple *unentangled product* of states of right-handed and left-handed degrees of freedom:

$$|\Psi_{0,\alpha}\rangle = |\Psi_0^L(g)\rangle \otimes |\Psi_\alpha^R\rangle, \quad (12)$$

where $|\Psi_0^L(g)\rangle$ is a trivial harmonic-oscillator coherent state, fully symmetric under interchange of the dynamical momenta of any pair of particles, and parametrized only by the Galileian metric g_{ab}; its defining property is

$$a_i|\Psi_0^L(g)\rangle = 0, \quad a_i \propto \omega^a(g)\pi_{ia}, \quad i = 1, \ldots, N, \quad (13)$$

where the complex unit vector $\omega^a(g)$ is obtained by solution of the generalized Hermitian eigenvector problem

$$\omega_a(g) = g_{ab}\omega^b(g) = i\epsilon_{ab}\omega^b(g), \quad \omega_a(g)^*\omega^a(g) = 1. \quad (14)$$

In contrast, the non-trivial states $|\Psi_\alpha^R\rangle$ are the eigenstates of the "right-handed" (guiding-center) Hamiltonian

$$H_R = \frac{1}{2A}\sum_{\boldsymbol{q}} V(\boldsymbol{q}) f(\boldsymbol{q})^2 \rho_{\boldsymbol{q}}\rho_{-\boldsymbol{q}}. \quad (15)$$

The reduction of the problem by discarding "left-handed" degrees of freedom, "frozen out" by Landau quantization, makes numerical study of the problem by exact diagonalization of H_R for finite N, N_Φ tractable. This may also be characterized as a "quantum geometry" description: once the "left-handed" degrees of freedom are removed, the notion of *locality*, fundamental to both classical geometry and Schrödinger's formulation of quantum mechanics, is absent. The commutation relations (4) imply a fundamental uncertainty in the "position" of the particles, now only described by their guiding centers. A Schrödinger wavefunction can only be constructed after "gluing" $|\Psi_\alpha^R\rangle$ together with some $|\Psi^L\rangle$, after which the composite state can be projected onto simultaneous eigenstates of the commuting set $\{\boldsymbol{r}_i\}$: *e.g.*,

$$\Psi_\alpha(\{\boldsymbol{r}_i\}, g) = \langle\{\boldsymbol{r}_i\}|\Psi_0^L(g)\rangle \otimes |\Psi_\alpha^R\rangle. \quad (16)$$

Note that the construction (16) of a Schrödinger wavefunction involves an *extraneous quantity* (g_{ab}) *that is not directly determined by* $|\Psi_\alpha^R\rangle$ *itself*, and thus is a non-primitive construction that does not represent $|\Psi_\alpha^R\rangle$ in its purest form. This suggests a reconsideration of the meaning of the "Laughlin state", usually presented in the form of the "Laughlin wavefunction"[2], which is fundamental to current understanding of the FQHE.

The conventional presentation of FQHE states is as an N-particle Schrödinger wavefunction with the form

$$\Psi(\{\boldsymbol{r}_i\}) = F(\{z_i\})\prod_{i=1}^{N} e^{-\frac{1}{2}z_i^* z_i}, \quad (17)$$

where $z_i = \omega_a(g)r_i^a/\sqrt{2}\ell_B$. Such wavefunctions, formulated in the "symmetric gauge", obey (13) with $a_i \equiv \frac{1}{2}z_i + \partial/\partial z_i^*$. The original Laughlin wavefunction[2] was the polynomial

$$F(\{z_i\}) = F_L^q(\{z_i\}) \equiv \prod_{i>j}(z_i - z_j)^q; \quad (18)$$

it was subsequently adapted[3] to a pbc with the form

$$F_{L,\alpha}^q(\{z_i\}) = \prod_{i>j} w(z_i - z_j)^q \prod_{k=1}^{q} w((\textstyle\sum_i z_i) - a_{k,\alpha}), \quad (19)$$

where $w(z)$ is given in terms of an elliptic theta function: $w(z) = \theta_1(\pi z/L_1|L_2/L_1)\exp(z^2/2N_\Phi)$, with $L_1L_2^* - L_1^*L_2 = 2\pi i N_\Phi$ (the wavefunction is (quasi) periodic under $z_i \to z_i + mL_1 + nL_2$). The additional q parameters $a_{k,\alpha}$ of (19), with $\sum_k a_{k,\alpha} = 0$, characterize the q-fold topological degeneracy of the Laughlin state with a pbc.

The Laughlin wavefunction was originally presented as a "variational wavefunction", albeit one with no continuously-tunable parameter, since q is an integer fixed by statistics. Its initial success was that, as a "trial wavefunction", it had a lower Coulomb energy than obtained in Hartree-Fock approximations, and explained the existence of a strong FQHE state at $\nu \equiv N/N_\Phi = 1/3$, but not at $\nu = 1/2$. In the wavefunction language, its defining characteristic is that, as a function of any particle coordinate z_i, there is an order-q zero at the location of every other particle, which heuristically "keeps particles apart", and lowers the Coulomb energy.

Subsequent to its introduction, the Laughlin state's essential validity was further confirmed by this author's observation[4] that, at $\nu = 1/q$, it is also uniquely characterized as the zero-energy eigenstate of a two-body "pseudopotential Hamiltonian"

$$H_R = \sum_{m=0}^{q-1} V_m P_m(g), \quad V_m > 0, \tag{20}$$

where

$$P_m(g) = \frac{1}{N_\Phi} \sum_{\boldsymbol{q}} L_m(q_g^2 \ell_B^2) e^{-\frac{1}{2} q_g^2 \ell_B^2} \rho_{\boldsymbol{q}} \rho_{-\boldsymbol{q}}, \tag{21}$$

where $L_m(x)$ is a Laguerre polynomial. Numerical finite-size diagonalization[5] for $q = 3$ showed that this H_R had the gapped excitation spectrum of an incompressible FQHE state, and that this gap did not close along a path that adiabatically interpolated between it and the Hamiltonian of the Coulomb interaction with $\tilde{g}_{ab} = g_{ab}$.

This raises the question that does not seem to have been previously considered: what if the "Coulomb metric" $\tilde{g}_{ab}$ and the "Galileian metric" g_{ab} do *not* coincide? The "pseudopotential" definition of the Laughlin *state* (as opposed to the Laughlin *wavefunction*) defines a *continuously-parametrized family* of $\nu = 1/q$ Laughlin states $|\Psi^q_{L,\alpha}(\bar{g})\rangle$ by

$$P_m(\bar{g})|\Psi^q_{L,\alpha}(\bar{g})\rangle = 0, m < q. \tag{22}$$

The continuously-variable parameter here is a unimodular *guiding-center metric* $\bar{g}_{ab}$ that is in principle distinct from the Galileian metric g_{ab}, and is *not* fixed by the one-body physics of the Landau orbits. Physically, it characterizes the *shape* of the correlation functions of the Laughlin state. If the shape of Landau orbits is used as the definition of "circular", the correlation hole around the particles deforms to "elliptical" when $\bar{g}_{ab} \neq g_{ab}$.

If a wavefunction (13) is constructed by "gluing together" $|\Psi^L_0(g)\rangle$ with the "Laughlin *state*" $|\Psi^R\rangle = |\Psi^q_{L,\alpha}(\bar{g})\rangle$, it does *not* correspond to the Laughlin *wavefunction* (19) *unless* $\bar{g}_{ab} = g_{ab}$, as there is no longer a q'th order zero of the wavefunction when $z_i = z_j$. Despite this, I will not call $|\Psi^q_L(\bar{g})\rangle$ with $\bar{g}_{ab} \neq g_{ab}$ a "generalization" of the Laughlin state, but propose it as a definition of the *family* of Laughlin states that exposes the geometrical degree of freedom $\bar{g}_{ab}$ hidden by the wavefunction-based formalism. I argue that FQHE states should be described completely within the framework of the "quantum geometry" of the guiding-center degrees of freedom alone, and no "preferred status" should be accorded to the metric choice $\bar{g}_{ab} = g_{ab}$. If the states $|\Psi^q_{L,\alpha}(\bar{g})\rangle$ are used as variational approximations to the ground state of a generic H_R given by (15), $\bar{g}_{ab}$ must be chosen to minimize the correlation energy $E(\bar{g}) = \langle\Psi^q_{L\alpha}(\bar{g})|H_R|\Psi^q_{L\alpha}(\bar{g})\rangle$. If the Coulomb ($\tilde{g}_{ab}$) and Galileian ($g_{ab}$) metrics coincide, the energy will be minimized by the choice $\bar{g}_{ab} = \tilde{g}_{ab} = g_{ab}$; otherwise, $\bar{g}_{ab}$ will be a compromise intermediate between the two physical metrics.

A more profound consequence of the identification of the variable geometric parameter $\bar{g}_{ab}$ follows from the observation that the correlation energy will be a quadratic function of local deformations $\bar{g}_{ab}(\boldsymbol{r}, t)$ around the minimizing value, whether or not this is equal to g_{ab}. This unimodular metric, or "shape of the circle" defined by the correlation function of the FQHE state, may be identified as the natural *local collective degree of freedom* of a FQHE state (defined on lengthscales large compared to ℓ_B), and not merely a variational parameter.

In its finite-N polynomial form (18), the Laughlin state $|\Psi^q_L(g)\rangle$ is an eigenstate of $L_R(g, 0)$ where $L_R(g, \boldsymbol{r}) = g_{ab}\Lambda^{ab}(\boldsymbol{r})$ generates rotations of the guiding-centers about a point $\boldsymbol{r}$; here $\Lambda^{ab}(\boldsymbol{r}) = \Lambda^{ba}(\boldsymbol{r})$ are the three generators of area-preserving linear deformations[7] of the guiding-centers around $\boldsymbol{r}$:

$$\Lambda^{ab}(\boldsymbol{r}) = \frac{1}{4\ell_B^2} \sum_i \{\delta R^a_i(\boldsymbol{r}), \delta R^b_i(\boldsymbol{r})\}, \tag{23}$$

with $\delta R^a_i(\boldsymbol{r}) \equiv R^a_i - \boldsymbol{r}$. Leaving $\boldsymbol{r}$ implicit, these obey the non-compact Lie algebra[7]

$$[\Lambda^{ab}, \Lambda^{cd}] = -\frac{i}{2}\left(\epsilon^{ac}\Lambda^{bd} + \epsilon^{bd}\Lambda^{ac} + a \leftrightarrow b\right), \tag{24}$$

which is isomorphic to $SO(2,1)$, $SL(2,R)$, and $SU(1,1)$, with a Casimir $C_2 = -\frac{1}{2}\det\Lambda \equiv -\frac{1}{4}\epsilon_{ac}\epsilon_{bd}\Lambda^{ab}\Lambda^{cd}$.

FQHE states with $\nu = p/q$ can be simply understood as condensates of "composite bosons"[6] which are "elementary droplets" of the incompressible fluid consisting of p identical charge-e particles "bound to q London flux quanta" (*i.e.*, occupying q one-particle orbitals of the Landau level), which behave as a boson under interchange. This requires that the Berry phase cancels any bare statistical phase: $(-1)^{pq} = \xi^p$, where $\xi = -1$ ($+1$) for fermions (bosons). For a condensate of charge-pe objects, the elementary fractionally-charged vortex has charge $\pm e^* = \pm(\nu e^2/h) \times (h/pe) = \pm e/q$. This work aims to extend the description of the "composite boson" by giving it (2D orbital) "spin" and geometry.

Polynomial FQHE wavefunctions like (18) that describe $\bar{N} = N/p = N_\Phi/q$ elementary droplets are generically eigenstates of $L_R(g, 0)$ with eigenvalue $\frac{1}{2}pq\bar{N}^2 + \bar{s}\bar{N}$, where $\bar{s}$ is a variant of the so-called "shift" that I will identify as a fundamental FQHE parameter, the *guiding-center spin*, that characterizes the geometric degree of freedom of FQHE states. It can also be obtained as the limit $\bar{N} \to \infty$ of

$$\bar{s} = \frac{1}{\bar{N}} \sum_{m=0}^{q\bar{N}-1} (m + \tfrac{1}{2})(n_m(\bar{g}, \boldsymbol{r}) - \nu), \tag{25}$$

where $n_m(\bar{g}, \boldsymbol{r})$, $m \geq 0$ are the occupations of guiding-center orbitals defined as the eigenstates of $L_R(\bar{g}, \boldsymbol{r})$.

Note that the "superextensive" ($\propto \bar{N}^2$) contribution to the eigenvalue derives from the uniform background density contribution $\nu\delta^2_{\boldsymbol{q},0}$ to $\rho_{\boldsymbol{q}}$, and can be removed (regularized) by defining $\Lambda^{ab}(r)$ in the thermodynamic limit $N_\Phi = q\bar{N} \to \infty$ using the limit of the $\boldsymbol{q} \neq 0$ $SU(N_\phi)$ generators alone, which become continuous functions $\rho(\boldsymbol{q})$ of $\boldsymbol{q}$, with $\lim_{\lambda\to 0}\rho(\lambda\boldsymbol{q}) = 0$. Then

$$\Lambda^{ab}(\boldsymbol{r}) = \lim_{\lambda\to 0}\left(-\frac{1}{2}\frac{1}{(\lambda\ell_B)^2}\frac{\partial}{\partial q_a}\frac{\partial}{\partial q_b}\rho(\lambda\boldsymbol{q})e^{-i\lambda\boldsymbol{q}\cdot\boldsymbol{r}}\right). \quad (26)$$

The Laughlin state $|\Psi^q_L(\bar{g})\rangle$ is an eigenstate of $\bar{g}_{ab}\Lambda^{ab}(\boldsymbol{r})$ with $\bar{s} = \frac{1}{2}(1-q)$. Note that for fermionic particles ($\xi = -1$), $\bar{s}$ is odd under particle-hole transformations, and vanishes when the Landau-level is completely filled (here $q = 1$). A spin-statistics selection rule requires that

$$(-1)^{2\bar{s}}(-1)^{2s} = (-1)^{pq} = \xi^p, \quad (-1)^{2s} = (-1)^p, \quad (27)$$

where s is the (orbital) "Landau-orbit spin" of the elementary droplet ($s = -\frac{1}{2}, -\frac{3}{2}, \ldots$ for particles with Landau index $0, 1, \ldots$). The expression for $\bar{s}$ may now be inverted to define the (local) unimodular guiding-center metric $\bar{g}_{ab}(\boldsymbol{r})$ by the expectation value

$$\lim_{\bar{N}\to\infty}\frac{1}{\bar{N}}\langle\Psi^R|\Lambda^{ab}(\boldsymbol{r})|\Psi^R\rangle = \tfrac{1}{2}\bar{s}\bar{g}^{ab}(\boldsymbol{r}), \quad \det\bar{g} = 1, \quad (28)$$

so if $\bar{\rho}(\boldsymbol{r})$ is the local droplet density, $\frac{1}{2}\bar{s}\bar{\rho}(\boldsymbol{r})g^{ab}(\boldsymbol{r})$ is the local density of the deformation generator.

The quantization of $2\bar{s}$ as an integer is a topological property deriving from the incompressibility of FQHE states. A simple picture that is reminiscent of Jain's notion of "quasi-Landau-levels"[8] supports this: the "elementary droplet", with a shape fixed by $\bar{g}_{ab}(\boldsymbol{r})$, supports q single-particle orbitals with guiding-center spins $\frac{1}{2}, \frac{3}{2}, \ldots, \frac{q-1}{2}$. The way these are occupied by the p particles of the droplet, determines the guiding-center spin of the droplet as the actual total guiding center spin of the configuration, minus that ($\frac{1}{2}pq$) given by putting p/q particles in each orbital. The repulsive exchange and correlation fields of particles outside the droplet will give each of the internal levels a mean energy for orbiting around an effective potential minimum at its center. The droplet will be stable, with a quantized guiding center spin that is adiabatically conserved as the droplet changes shape, provided there is a finite positive energy gap between the highest occupied and lowest empty single-particle state in the droplet. Collapse of this gap implies that the system has become compressible with an unquantized or indeterminate value of $\bar{s}$.

The geometrical degree of freedom exposed here also suggests a new look at the problem of formulating a continuum description of incompressible FQHE states. Elsewhere, I will present a continuum description combining Chern-Simons fields with the geometry field $\bar{\omega}_a(\boldsymbol{r}, t)$, where $\bar{g}_{ab} = \bar{\omega}^*_a\bar{\omega}_b + \bar{\omega}^*_b\bar{\omega}_a$, but mention here some fundamental formulas that emerge. First, the electric charge density is given by $pe\bar{\rho}(\boldsymbol{r})$, where $\bar{\rho}(\boldsymbol{r})$ is the local elementary droplet (composite boson) density, and

$$\bar{\rho}(\boldsymbol{r}, t) = \frac{1}{2\pi pq}\left(\frac{pe}{\hbar}B(\boldsymbol{r}) + \bar{s}K(\boldsymbol{r}, t)\right), \quad (29)$$

Here $B(\boldsymbol{r})$ is the externally-imposed 2D (normal) magnetic flux density, (assumed to be time-independent, but not necessarily spatially uniform), and $K(\boldsymbol{r}, t)$ is the instantaneous Gaussian curvature of the unimodular guiding-center-metric field $\bar{g}_{ab}(\boldsymbol{r}, t)$, given by $K = \epsilon^{ab}\partial_a\Omega^{\bar{g}}_b$, $\Omega^{\bar{g}}_a = \epsilon^{bc}\bar{\omega}^*_b\nabla^{\bar{g}}_a\bar{\omega}_c$, where $\Omega^{\bar{g}}_a$ is the spin connection gauge-field and $\nabla^{\bar{g}}_a$ is the covariant derivative (Levi-Civita connection) of $\bar{g}_{ab}$. This formula could perhaps have been anticipated from the work of Wen and Zee[9], who considered coupling Chern-Simons fields to curvature, but the curvature they apparently had in mind was not the collective dynamical internal degree of freedom described here, but that due to placing the FQHE system on a curved 2D surface embedded in 3D Euclidean space, as in formal calculations of the FQHE on a sphere surrounding a monopole[3, 4]. The second formula is that the canonical conjugate of the geometry field $\bar{\omega}_a(\boldsymbol{r})$ is

$$\bar{\pi}^a_{\bar{\omega}}(\boldsymbol{r}) = \hbar\bar{s}\bar{\rho}(\boldsymbol{r})\epsilon^{ba}\bar{\omega}_b(\boldsymbol{r})^*, \quad (30)$$

so the momentum density (translation generator density) is $\bar{\pi}^b_{\bar{\omega}}\nabla^{\bar{g}}_a\bar{\omega}_b = \hbar s\bar{\rho}\Omega^{\bar{g}}_a$. These formulas parallel those of quantum Hall ferromagnets, with guiding-center spin and Gaussian curvature replacing true electron spin and Berry curvature. On large lengthscales, the elementary charge $e^* = \pm e/q$ quasiparticles appear as rational cone-singularities of the metric field $\bar{g}_{ab}(\boldsymbol{r}, t)$ with localized Gaussian curvature $K = \pm 4\pi/(2\bar{s})$.

In summary, the prevalent assumption of rotational invariance of FQHE fluids conceals a fundamental geometric degree of freedom, the shape of their correlations, described by a unimodular spatial metric field that exhibits quantum dynamics.

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