Topological Terms of Nonlinear Sigma models

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Topological Terms of Nonlinear Sigma models

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Topological Terms of Nonlinear Sigma models

Introduction: WZW term and *Θ*-term

Example: 0+1 dimensional O(3) NLSM

$$S_0 = \int d\tau \ \frac{1}{g}(\partial_\tau \vec{n})^2$$

Hamiltonian: $H \sim g(\vec{L})^2$ $E \sim gl(l+1)$

Ground state: $|\mathrm{GS}\rangle = |l=0\rangle$

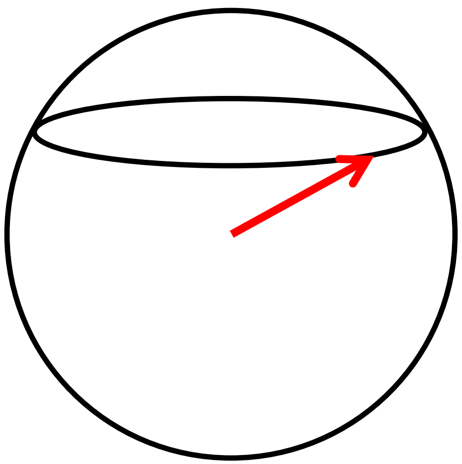
Nondegenerate ground state, gapped spectrum

Introduction: WZW term and *Θ*-term

Example: 0+1 dimensional O(3) NLSM, plus WZW term:

$$S = \int d\tau \ \frac{1}{g}(\partial_\tau \vec{n})^2 + i2\pi W[\vec{n}(\tau)]$$

W is proportional to the solid angle on the sphere enclosed by the closed loop.



$$W[\vec{n}(\tau)] = \frac{k}{4\pi}\int d\tau \ \left(1 - \cos\theta(\tau)\right)\partial_\tau\varphi(\tau)$$

$$W[\vec{n}(\tau)]' = \frac{k}{4\pi}\int d\tau \ \left(-1 - \cos\theta(\tau)\right)\partial_\tau\varphi(\tau)$$

k has to be an integer.

Introduction: WZW term and *Θ*-term

Example: 0+1 dimensional O(3) NLSM, plus WZW term:

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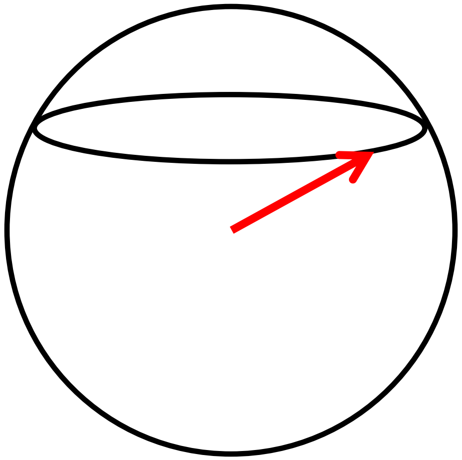


$$W[\vec{n}(\tau)] = \frac{ik}{8\pi}\int du d\tau \ \epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b \partial_\nu n^c$$

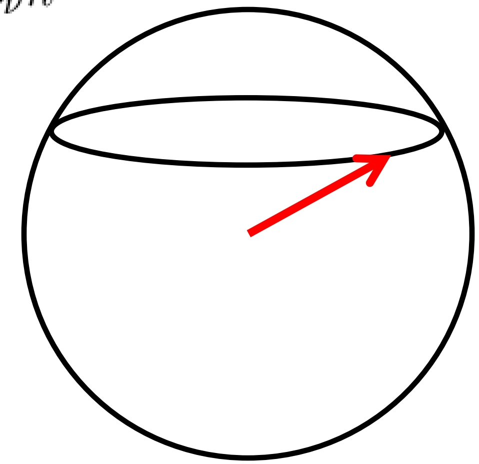
$$\vec{n}(\tau, u), \quad \vec{n}(\tau, 1) = \vec{n}(\tau) \quad \vec{n}(\tau, 0) = \hat{z}$$

k has to be an integer.

Introduction: WZW term and *Θ*-term

Example: 0+1 dimensional O(3) NLSM, plus WZW term:

$$S = \int d\tau \ \frac{1}{g}(\partial_\tau \vec{n})^2 + 2\pi \int du d\tau \ \frac{ik}{8\pi} \epsilon_{abc} \epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c$$



This is equivalent to a point particle moving on a sphere with *2kπ* flux through the sphere.

The Landau level degeneracy: *k+1*. Notice the Difference from the flat space.

This is the model describing a single spin, with *S* = *k/2*.

Topological Terms of Nonlinear Sigma models

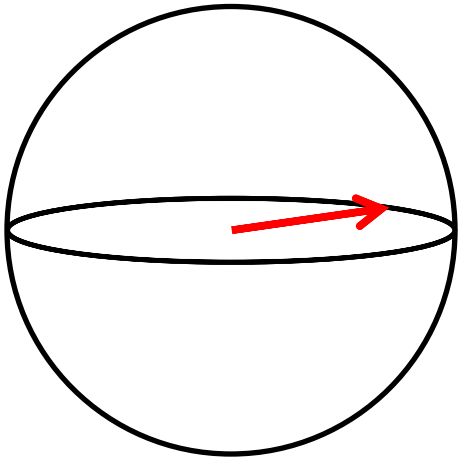
Introduction: WZW term and Θ-term

Breaking O(3) symmetry down to inplane O(2) symmetry

$$L = \int d\tau \, \frac{1}{g}(\partial_\tau \varphi)^2 + \frac{i\Theta}{2\pi} \, \partial_\tau \varphi$$

The WZW term reduces to the Θ-term
$\Theta = k\pi$

Hamiltonian: $$H \sim g \left(\hat{n} - \frac{\Theta}{2\pi} \right)^2$$



When *k* is odd, the ground state is doublet degenerate, otherwise no degenerate. Topological term increases the degeneracy.

Topological Terms of Nonlinear Sigma models

Introduction: WZW term and Θ-term

Let us consider 1+1d. Take 1d AF spin chain, define $\vec{n} \sim (-1)^j \vec{S}_j$

$$\sum_j (-1)^j WZW[\vec{n}(\tau)]_j \sim \int d\tau\, dx\; \frac{i\Theta}{8\pi} \epsilon_{abc} \epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c$$

Spin chain is described by the 1+1d O(3) NLSM with $\Theta = 2\pi S$ (Haldane).

$$L = \frac{1}{g} (\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi} \epsilon_{abc} \epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c$$

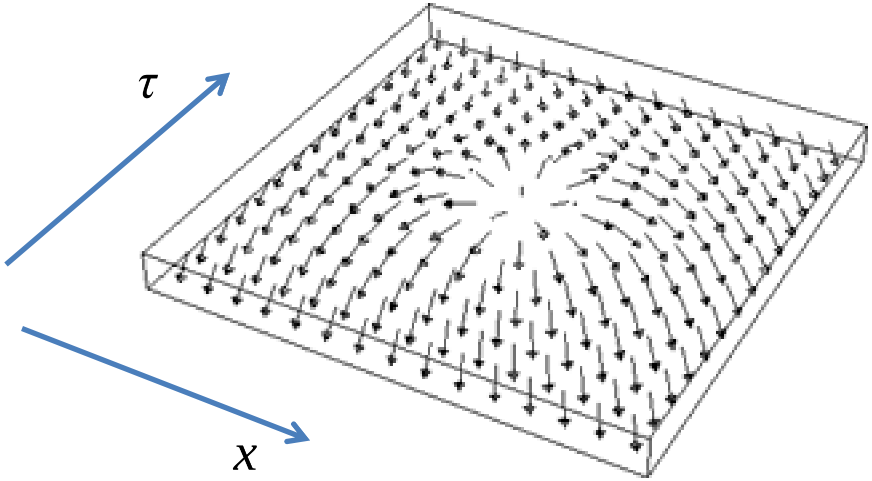
Without Θ-term, NLSM is gapped, disordered, and nondegenerate. According to LSM theorem, spin-1/2 chain is either gapless or two fold degenerate i.e. when Θ=π, the NLSM is either gapless or two fold degenerate.

Topological Terms of Nonlinear Sigma models

$$L = \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c$$

1+1d

Since $\pi_2(S^2) = Z$, there are "instantons" in the space-time.



The Θ-term gives phase factor exp(iΘ) to every instanton.

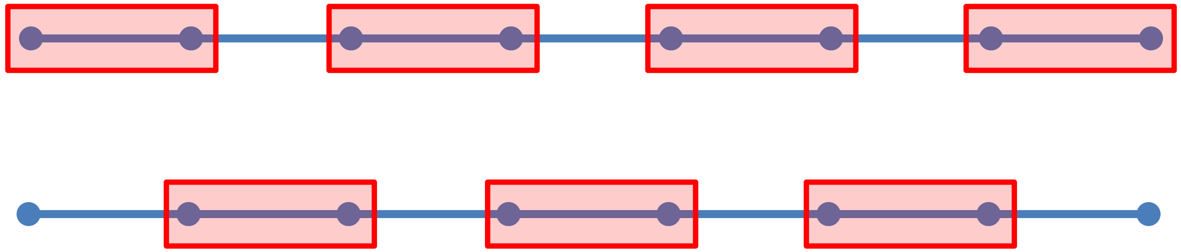
Topological Terms of Nonlinear Sigma models

$$L = \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c$$ 1+1d

$\Theta = \pi$

Gapless CFT, NN AF spin-1/2 chain

Gapped, 2-fold deg, J_1-J_2 spin chain



Topological Terms of Nonlinear Sigma models

$$L = \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c$$ 1+1d

$\Theta = \pi$

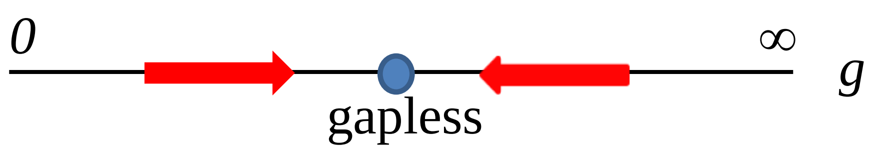
Gapless CFT, NN AF spin-1/2 chain

Gapped, 2-fold deg, J_1-J_2 spin chain

For NN spin-1/2 chain, the Neel order parameter and VBS order parameter have the same scaling dimension ½. Thus…

$$\phi^a = (\vec{n},\ Q)$$

$$S = \int d\tau dx\ \frac{1}{g}(\partial_\mu \phi^a)^2 + 2\pi \int du d\tau dx\ \frac{ik}{12\pi^2}\epsilon_{abcd}\epsilon_{\mu\nu\rho}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c\partial_\rho\phi^d$$



Topological Terms of Nonlinear Sigma models

Goal:

The goal of this work, is to understand the effect of the Θ term on the 2+1 dimensional principal chiral model defined on compact simple Lie groups, such as SU(N), SO(N), Sp(N), with $\pi_3 = Z$.

$$S = \int d\tau d^2x \; \frac{1}{g} \mathrm{tr}[\partial_\mu U^\dagger \partial_\mu U] + \frac{i\Theta}{G} \int d\tau d^2x \; \epsilon_{\mu\nu\rho} \mathrm{tr}[(U^\dagger \partial_\mu U)(U^\dagger \partial_\nu U)(U^\dagger \partial_\rho U)]$$

Let us take *U* in SU(2) group. $U = \phi^0 I_{2\times 2} + i\vec{\phi} \cdot \vec{\sigma}$

$$S = \int d\tau d^2x \; \frac{1}{g} (\partial_\mu \phi^a)^2 + \frac{i\Theta}{12\pi^2} \int d\tau d^2x \; \epsilon_{abcd} \epsilon_{\mu\nu\rho} \phi^a \partial_\mu \phi^b \partial_\nu \phi^c \partial_\rho \phi^d$$

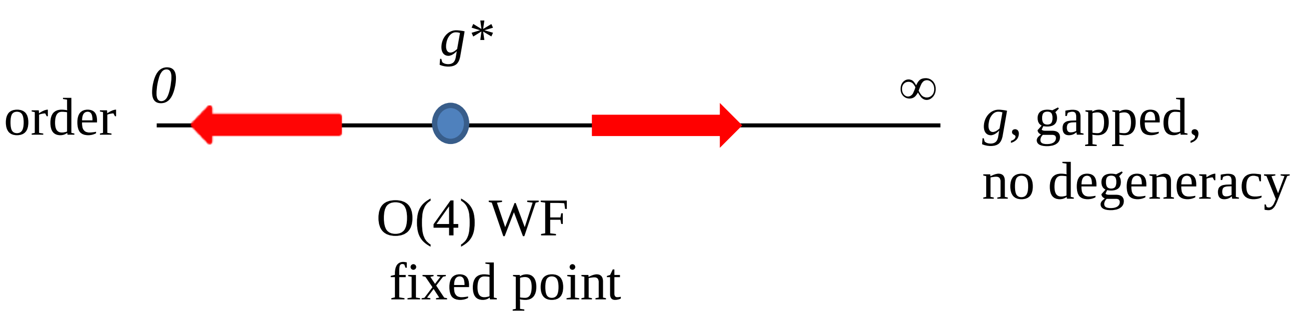
Topological Terms of Nonlinear Sigma models

$$S = \int d\tau d^2x \ \frac{1}{g}(\partial_\mu \phi^a)^2 + \frac{i\Theta}{12\pi^2}\int d\tau d^2x \ \epsilon_{abcd}\epsilon_{\mu\nu\rho}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c\partial_\rho\phi^d$$

Under time-reversal transformation, Θ becomes – Θ. The bulk physics (correlation, spectrum) is identical for Θ and Θ+2k π.

We focus on the time-reversal invariant case, where Θ = kπ.

Θ = 2kπ, bulk equivalent to Θ = 0.

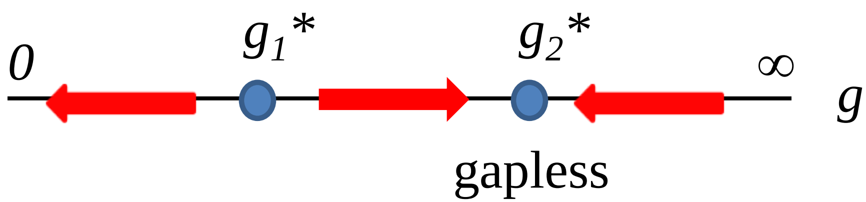


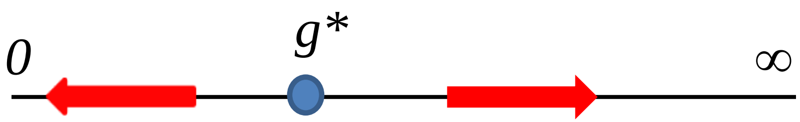
Topological Terms of Nonlinear Sigma models

$$S = \int d\tau d^2x \; \frac{1}{g}(\partial_\mu \phi^a)^2 + \frac{i\Theta}{12\pi^2}\int d\tau d^2x \; \epsilon_{abcd}\epsilon_{\mu\nu\rho}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c \partial_\rho \phi^d$$

$\Theta = \pi$, instantons will matter

Conclusion: two generic possibilities:

1: order 

2: order 

g, gapped,
2-fold degenerate

Topological Terms of Nonlinear Sigma models

Let us come back to investigate 1+1d O(3) NLSM:

$$L = \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu} n^a \partial_\mu n^b \partial_\nu n^c$$

Based on our knowledge of spin chains, this model is either gapless or two-fold degenerate when $\Theta = \pi$.

We will investigate this model without using spin-chain, and then apply the same argument to 2+1d NLSMs.

Topological Terms of Nonlinear Sigma models

$$L = \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c$$

1+1d

$\Theta = \pi$

$\Theta = 0$

Bulk gapped
No edge states

$\Theta = 2\pi$

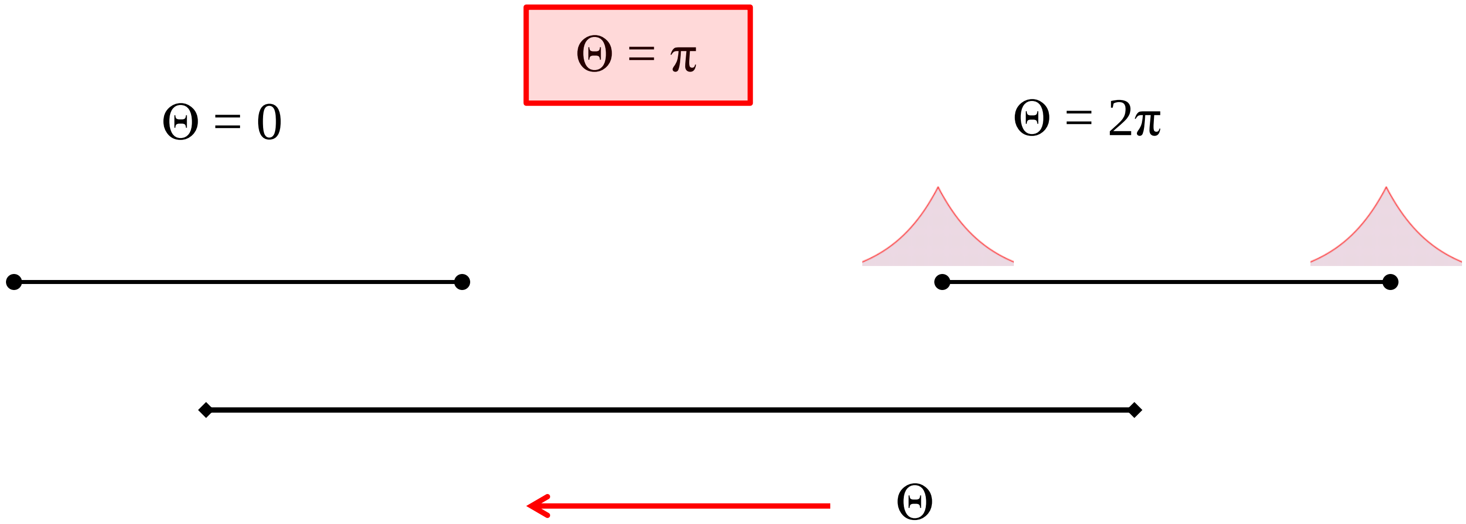
Bulk gapped
Edge NLSM +
WZW with k = 1.

Spin-1/2 state
Localized at the edge

Topological Terms of Nonlinear Sigma models

$$L = \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c$$

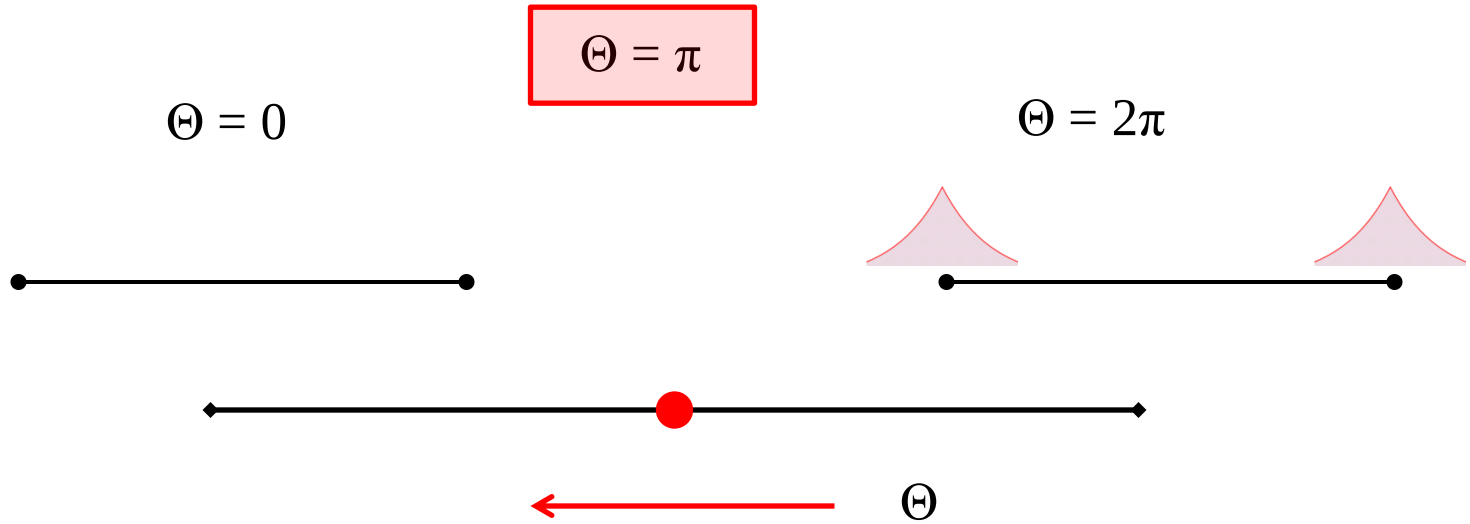
1+1d



Now tune Θ from 2π to 0, the spin-1/2 edge states have to disappear at some Θ, and the edge states can only be destroyed through a bulk transition, because single spin-1/2 is stable against discrete symmetry breaking, if O(3) symmetry is preserved.

Topological Terms of Nonlinear Sigma models

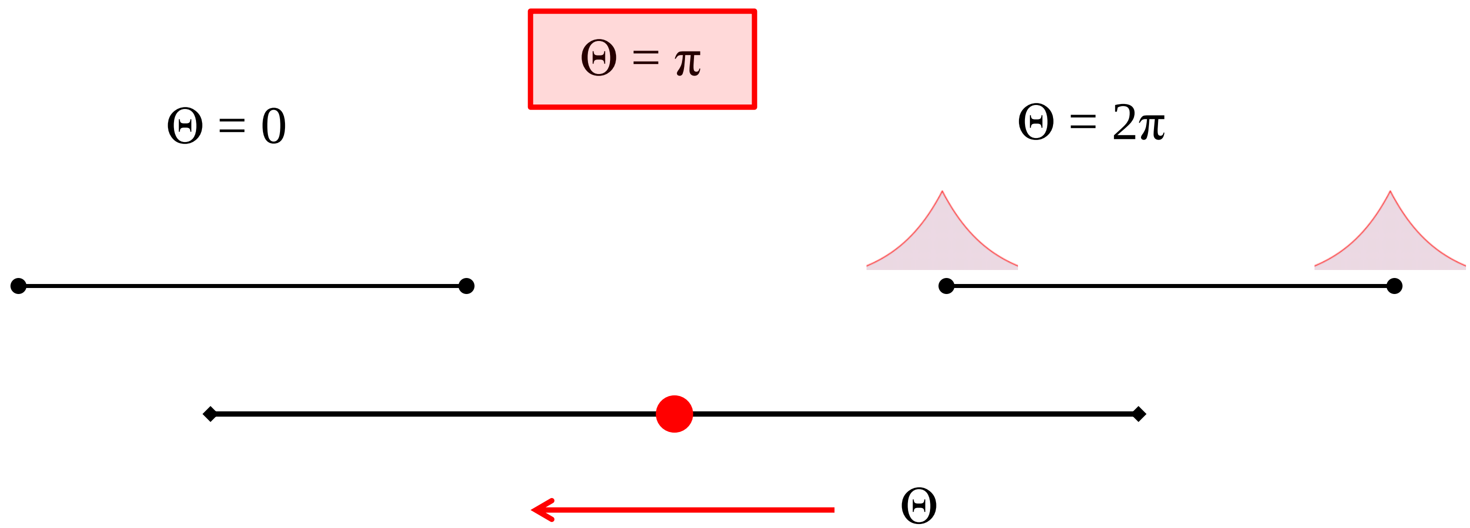
$$L = \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c \qquad \text{1+1d}$$



Possibility 1: a second order bulk transition, bulk gap closes at $\Theta = \pi$. While approaching this transition, the edge state becomes more and more delocalized, eventually continuously absorbed by the gapless bulk state at $\Theta = \pi$. System is gapless when $\Theta = \pi$.

Topological Terms of Nonlinear Sigma models

$$L = \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c$$ 1+1d



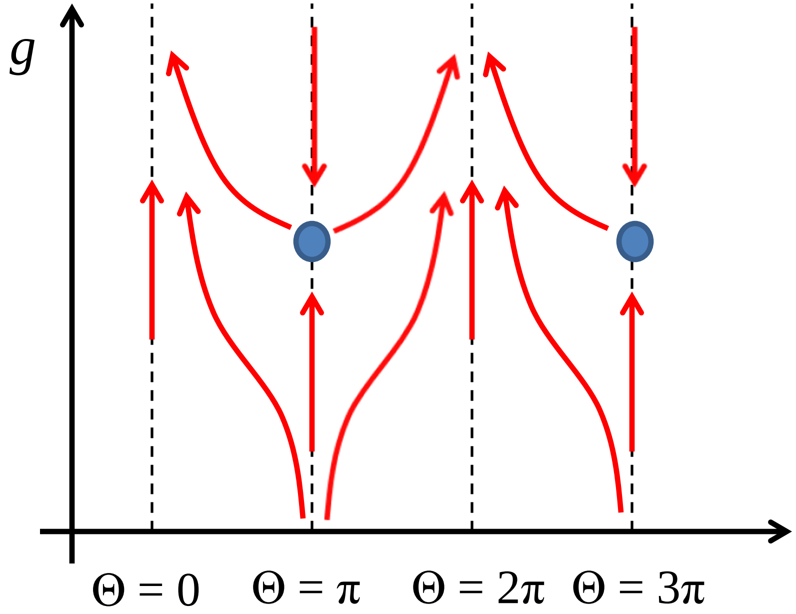
Possibility 2: a first order bulk transition, bulk level crossing Θ = π. The edge state disappears suddenly, and the system is two fold degenerate at Θ = π.

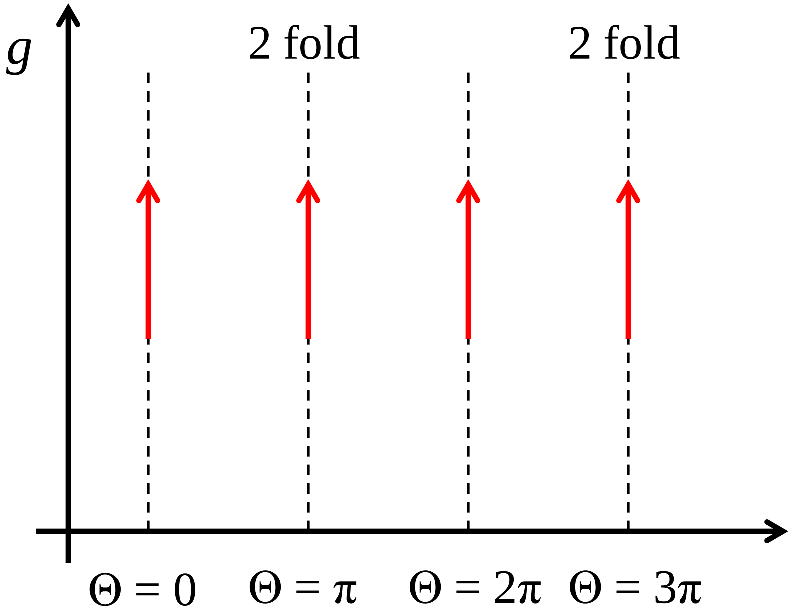
Two possibilities precisely consistent with LSM theorem.

Topological Terms of Nonlinear Sigma models

$$L = \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c$$

1+1d

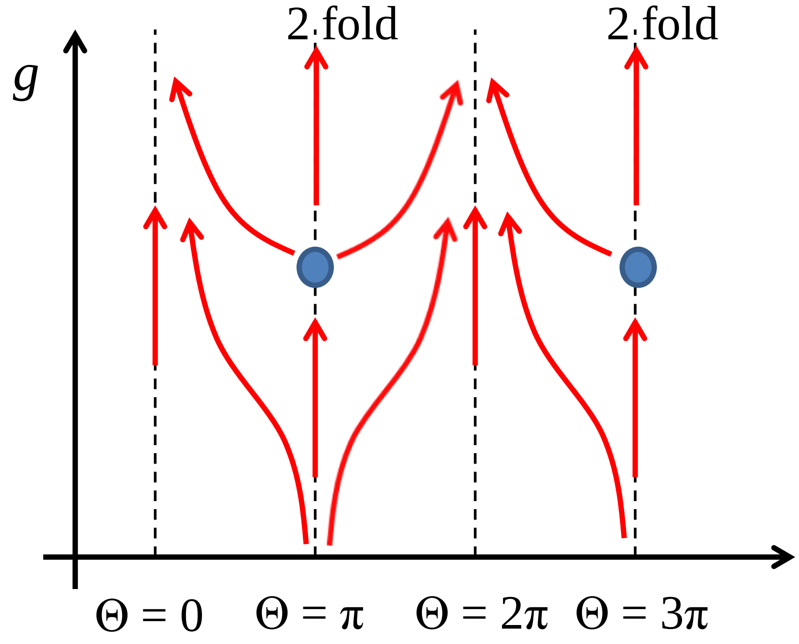




Topological Terms of Nonlinear Sigma models

$$L = \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c$$

1+1d



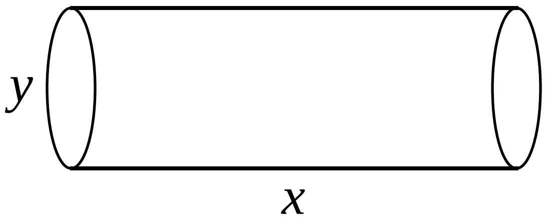
Topological Terms of Nonlinear Sigma models

$$S = \int d\tau d^2x \; \frac{1}{g}(\partial_\mu \phi^a)^2 + \frac{i\Theta}{12\pi^2}\int d\tau d^2x \; \epsilon_{abcd}\epsilon_{\mu\nu\rho}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c\partial_\rho\phi^d \qquad \text{2+1d}$$

$\Theta = \pi$

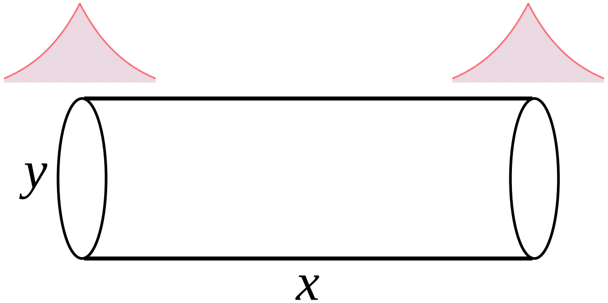
$\Theta = 0$

Bulk gapped
No edge states



$\Theta = 2\pi$

Bulk gapped
Edge O(4) NLSM +
WZW with k = 1.

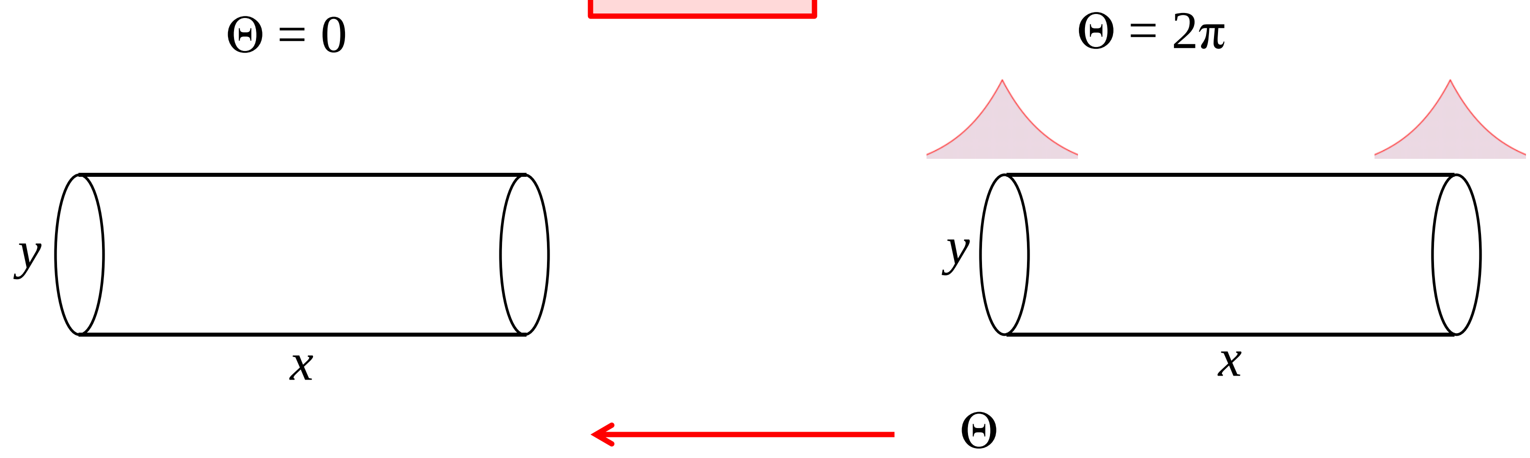


$SU(2)_1$ CFT at the edge

Topological Terms of Nonlinear Sigma models

$$S = \int d\tau d^2x \ \frac{1}{g}(\partial_\mu \phi^a)^2 + \frac{i\Theta}{12\pi^2}\int d\tau d^2x \ \epsilon_{abcd}\epsilon_{\mu\nu\rho}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c\partial_\rho\phi^d$$ 2+1d

$\Theta = \pi$

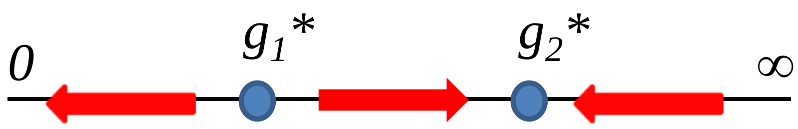


Now tune Θ from 2π to 0, the gapless edge state has to disappear. Again, the edge state cannot disappear without going through a bulk transition, because gapping a 1+1d $SU(2)_1$ CFT, one has to break O(4) ~ SU(2) x SU(2) down to SU(2).

Topological Terms of Nonlinear Sigma models

$$S = \int d\tau d^2x \ \frac{1}{g}(\partial_\mu \phi^a)^2 + \frac{i\Theta}{12\pi^2} \int d\tau d^2x \ \epsilon_{abcd}\epsilon_{\mu\nu\rho}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c \partial_\rho \phi^d$$ 2+1d

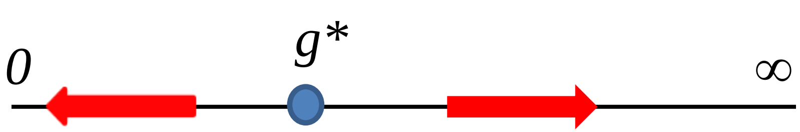
Possibility 1: a second order bulk transition, bulk gap closes at Θ = π. While approaching this transition, the edge state becomes more and more delocalized, eventually continuously absorbed by the gapless bulk state at Θ = π.

1: order 

g

gapless

Possibility 2: a first order bulk transition, bulk level crossing Θ = π. The edge state disappears suddenly, and the system is two fold degenerate at Θ = π.

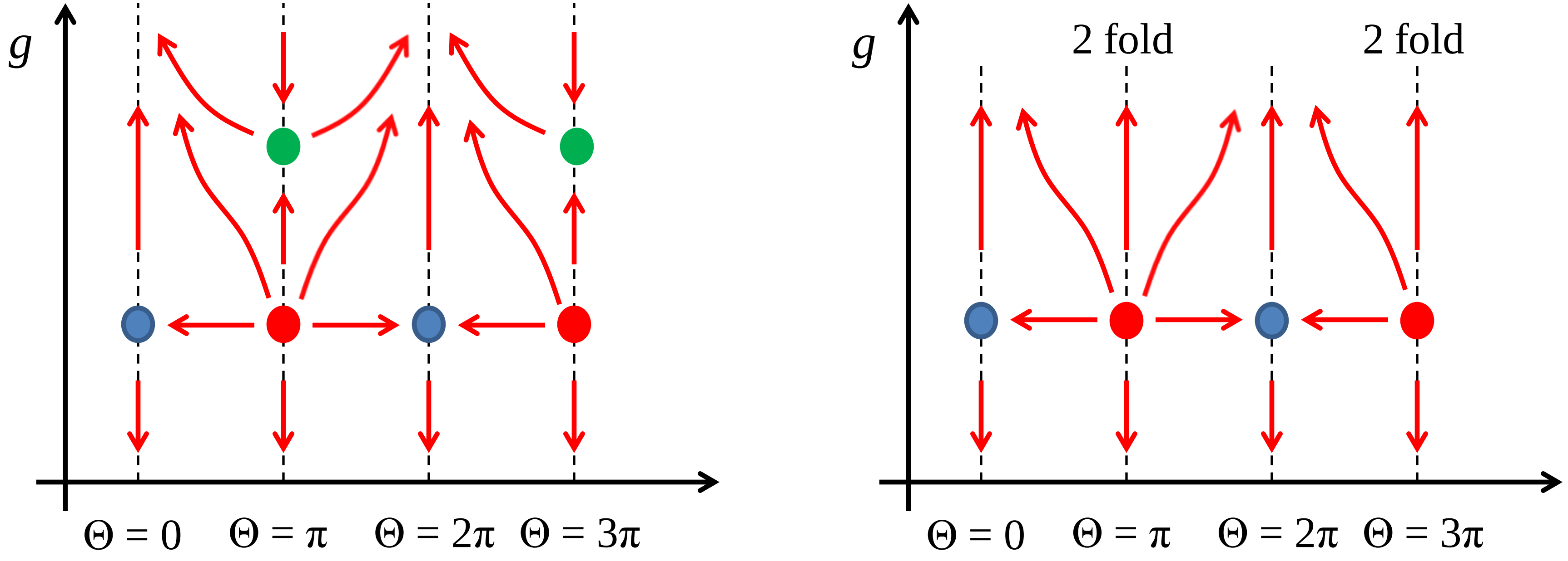
2: order 

g, gapped,
2-fold degenerate

Topological Terms of Nonlinear Sigma models

$$S = \int d\tau d^2x \ \frac{1}{g}(\partial_\mu \phi^a)^2 + \frac{i\Theta}{12\pi^2}\int d\tau d^2x \ \epsilon_{abcd}\epsilon_{\mu\nu\rho}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c\partial_\rho\phi^d$$

2+1d

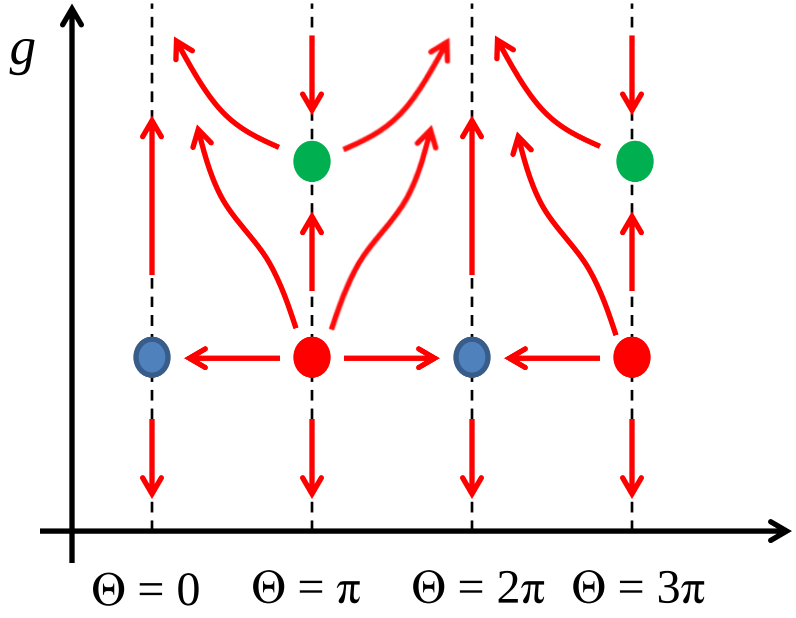


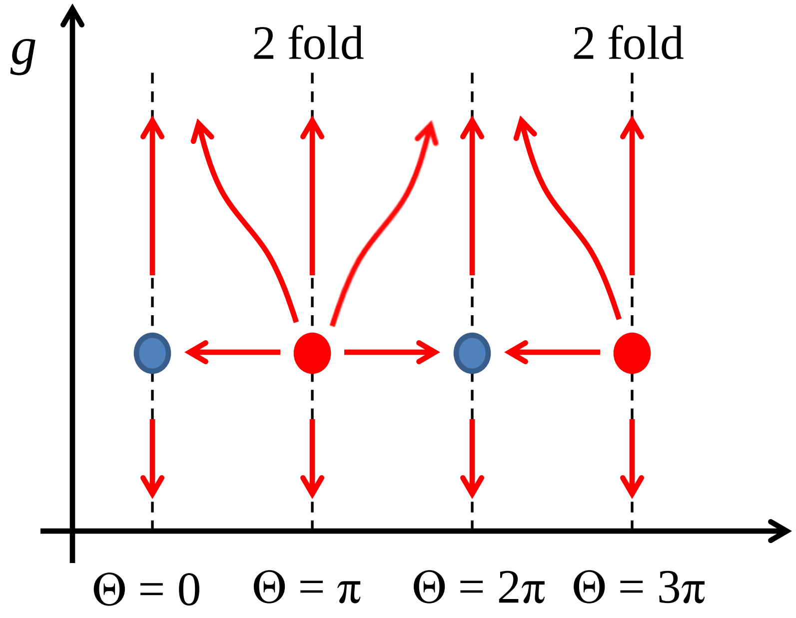
Xu and Ludwig, to appear soon

Topological Terms of Nonlinear Sigma models

$$S = \int d\tau d^2x \ \frac{1}{g}\mathrm{tr}[\partial_\mu U^\dagger \partial_\mu U] + \frac{i\Theta}{G}\int d\tau d^2x \ \epsilon_{\mu\nu\rho}\mathrm{tr}[(U^\dagger\partial_\mu U)(U^\dagger\partial_\nu U)(U^\dagger\partial_\rho U)]$$

Generalization to 2+1d principal chiral model





Xu and Ludwig, to appear soon

Topological Terms of Nonlinear Sigma models

How to generate the 2+1d O(4) NLSM with Θ-term?

$$L = \sum_{\alpha=1}^{N} \bar{\psi}_\alpha \gamma_\mu \partial_\mu \psi_\alpha + m\phi^a \bar{\psi}_\alpha \Gamma^a \psi_\alpha, \qquad a = 1 \cdots 4$$

Integrate out the fermions, obtain O(4) NLSM with Θ = Nπ.
Abanov, Wiegmann, 2000.

A true condensed matter physicist should not assume O(4) symmetry at the beginning, we should make it emerge by itself.

$$L = \bar{\psi}\gamma_\mu(\partial_\mu - iA_\mu)\psi + m\ \bar{\psi}\vec{\sigma}\psi \cdot \vec{n} + \frac{1}{e^2}(F_{\mu\nu})^2$$

Integrate out fermion, obtain O(4) NLSM with Θ = π, O(4) symmetry emerge. Senthil, Fisher, 2005.

Topological Terms of Nonlinear Sigma models

$$L = \bar{\psi}\gamma_\mu(\partial_\mu - iA_\mu)\psi + m\ \bar{\psi}\vec{\sigma}\psi \cdot \vec{n} + \frac{1}{e^2}(F_{\mu\nu})^2$$

Step 1: Integrate out fermion:

$$L = \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{1}{e^2}(F_{\mu\nu})^2 + i\pi \mathrm{Hopf}[\vec{n}] + iA_\mu \frac{1}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu\rho} n^a \partial_\nu n^b \partial_\rho n^c$$

Step 2: take CP(1) representation of O(3) vector: $\vec{n} = z^\dagger \vec{\sigma} z$

$$L = \frac{1}{g}|(\partial_\mu - ia_\mu)z_\alpha|^2 + \frac{1}{e^2}(F_{\mu\nu})^2 + \frac{i}{2\pi}\epsilon_{\mu\nu\rho} a_\mu \partial_\nu A_\rho + \frac{i\pi}{12\pi^2}\epsilon_{abcd}\epsilon_{\mu\nu\rho}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c \partial_\rho \phi^d$$

Step 3: Integrate out gauge field: $\vec{\phi} = (\mathrm{Re}[z_1], \mathrm{Im}[z_1], \mathrm{Re}[z_2], \mathrm{Im}[z_2])$

$$S = \int d\tau d^2x\ \frac{1}{g}(\partial_\mu \phi^a)^2 + \frac{i\Theta}{12\pi^2}\int d\tau d^2x\ \epsilon_{abcd}\epsilon_{\mu\nu\rho}\phi^a \partial_\mu \phi^b \partial_\nu \phi^c \partial_\rho \phi^d$$

Topological Terms of Nonlinear Sigma models

$$L = \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c \qquad \text{1+1d}$$

$\Theta = \pi$

g, gapped, 2-fold

The two degenerate ground states correspond to the states with $\Theta = 0$ and $\Theta = 2\pi$. We can view Θ as a dynamical field, and these two states are two different condensates of Θ.

$$S = \int d\tau dx \; i\frac{\Theta(x)}{2\pi}\partial_\tau \Phi(x)$$

$$\Phi(x) = \frac{1}{2}\int du \;\epsilon_{abc}n^a\partial_x n^b\partial_u n^c$$

Θ and Φ are conjugate variables. What is Φ?

Topological Terms of Nonlinear Sigma models

$$L = \frac{1}{g}(\partial_\mu \vec{n})^2 + \frac{i\Theta}{8\pi}\epsilon_{abc}\epsilon_{\mu\nu}n^a\partial_\mu n^b\partial_\nu n^c$$ 1+1d

$\Theta = \pi$ 

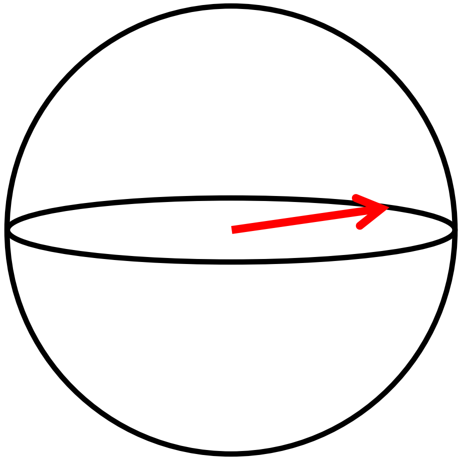
g, gapped, 2-fold

The two degenerate ground states correspond to the states with $\Theta = 0$ and $\Theta = 2\pi$. We can view Θ as a dynamical field, and these two states are two different condensates of Θ.

What is Φ? $n^x \sim \cos\varphi \quad n^y \sim \sin\varphi$

Breaking O(3) down to O(2):

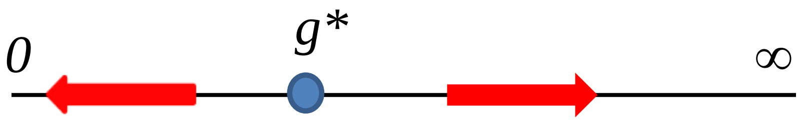
$$\int dx \; \Phi(x) \sim \int dx \; \partial_x \varphi$$



The two degenerate states are two coherent states of kink, total O(4) symmetry can emerge.

Topological Terms of Nonlinear Sigma models

$$S = \int d\tau d^2x \ \frac{1}{g}(\partial_\mu \phi^a)^2 + \frac{i\Theta}{12\pi^2}\int d\tau d^2x \ \epsilon_{abcd}\epsilon_{\mu\nu\rho}\phi^a\partial_\mu\phi^b\partial_\nu\phi^c\partial_\rho\phi^d \qquad \text{2+1d}$$



The same idea can be generalized to the 2+1d O(4) NLSM.

The two degenerate ground states, correspond to two different coherent states of the following quantity:

$$\Phi(x,y) \sim \int du \ \epsilon_{abcd}\phi^a\partial_x\phi^b\partial_y\phi^c\partial_u\phi^d \sim \epsilon_{abc}n^a\partial_x n^b\partial_y n^c$$

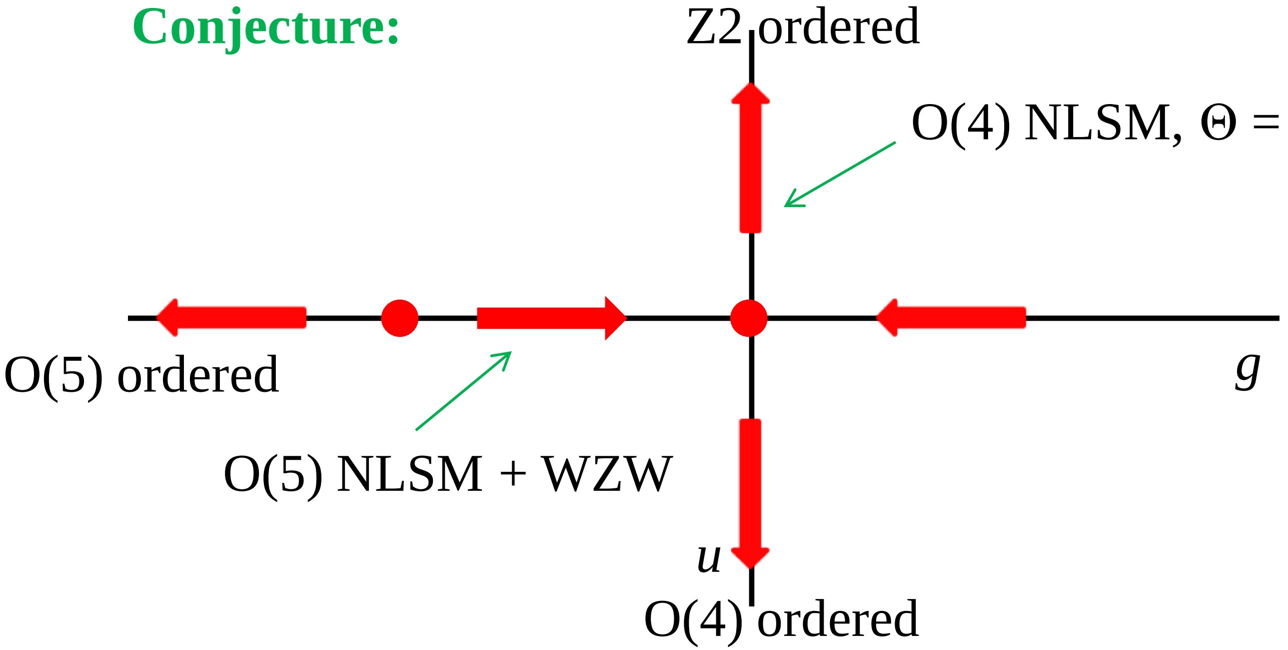
Topological Terms of Nonlinear Sigma models

$$S = \int d^3x \frac{1}{g}(\partial_\mu \vec{V})^2 + \frac{i3k}{4\pi}\int du\,dx\,dy\,d\tau\, \epsilon_{abcde}\epsilon_{\mu\nu\rho\sigma} V^a \partial_\mu V^b \partial_\nu V^c \partial_\rho V^d \partial_\sigma V^e$$

$$+u\left(\sum_{a=1}^{4}(V^a)^2 - (V^5)^2\right)$$

u breaks O(5) to O(4) x Z_2

Conjecture:



Topological Terms of Nonlinear Sigma models

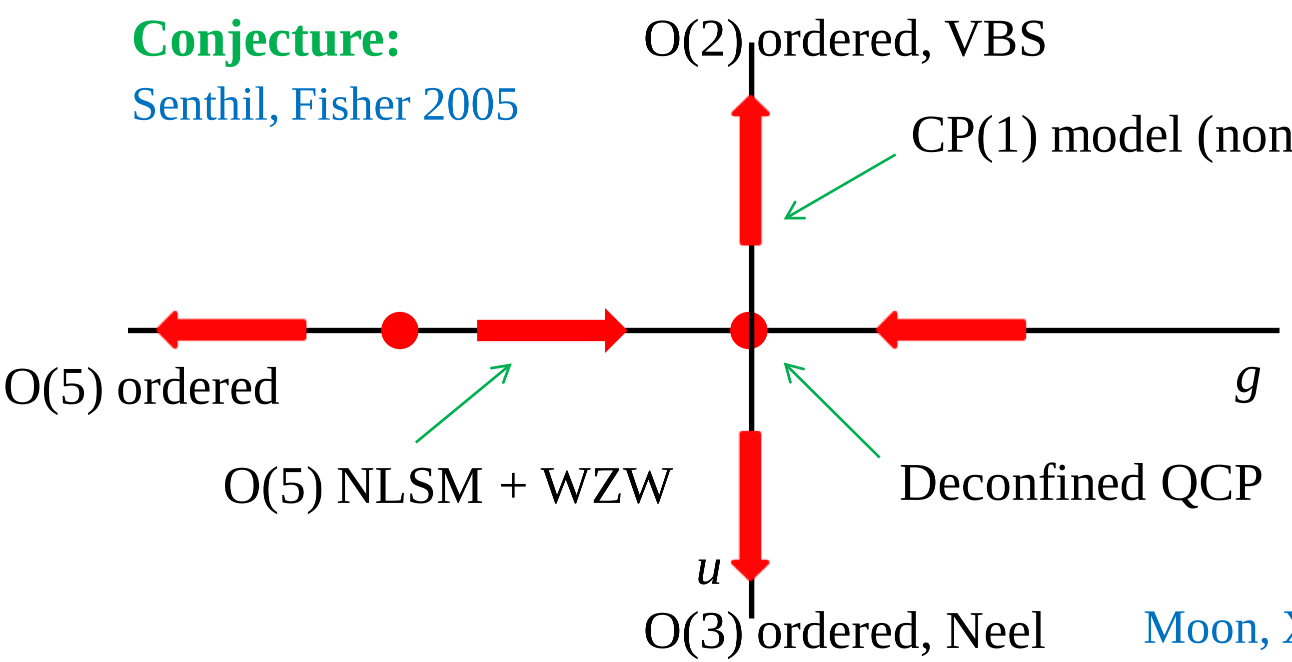
$$S = \int d^3x \frac{1}{g}(\partial_\mu \vec{V})^2 + \frac{i3k}{4\pi}\int du\,dx\,dy\,d\tau\,\epsilon_{abcde}\epsilon_{\mu\nu\rho\sigma}V^a\partial_\mu V^b\partial_\nu V^c\partial_\rho V^d\partial_\sigma V^e$$

$$+u\left(\sum_{a=1}^{3}(V^a)^2 - (V^4)^2 - (V^5)^2\right)$$

u breaks O(5) to O(3) x O(2)

Conjecture:

Senthil, Fisher 2005

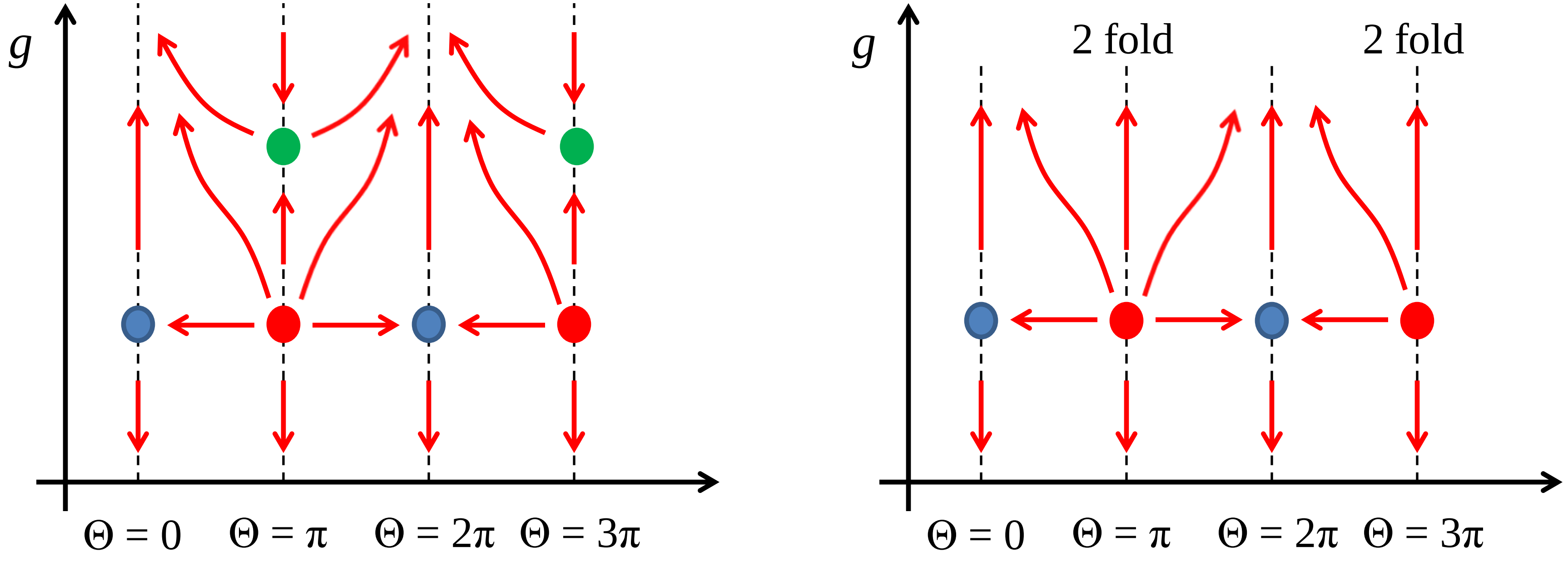


Moon, Xu, to appear

Topological Terms of Nonlinear Sigma models

Conclusion:

$$S = \int d\tau d^2x \ \frac{1}{g}\mathrm{tr}[\partial_\mu U^\dagger \partial_\mu U] + \frac{i\Theta}{G}\int d\tau d^2x \ \epsilon_{\mu\nu\rho}\mathrm{tr}[(U^\dagger\partial_\mu U)(U^\dagger\partial_\nu U)(U^\dagger\partial_\rho U)]$$



Xu and Ludwig, to appear soon