量子力学から量子幾何をつくる

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Sendai N.C.T.

[Refs.] arXiv: 2308. ****, arXiv:2212.05277 and references therein.

Quantum mechanics and M(atrix) theory

On the relation between the quantum mechanics of Heisenberg, Born, and Jordan, and that of Schroedinger, *Schroedinger* (Annalen der Physik *1926*)

Matrix mechanics (1925) ⟷ Wave mechanics (1926)


(from wiki)

$$F^{kl} = \int \rho(x) u_k(x) [F, u_l(x)] dx,$$

$$F_{ij} = \langle \psi_i | F(x, p = -i\partial_x) | \psi_j \rangle$$


(from wiki)

『なぜ行列を使うとよいのか？』
超ひも理論定式化の大きな武器となった「行列模型」（マトリクス）は、じつは量子力学の世界では古くからなじみの計算手法なんです。

$$S = \frac{1}{4}\mathrm{tr}([A_\mu, A_\nu]^2) + \cdots$$

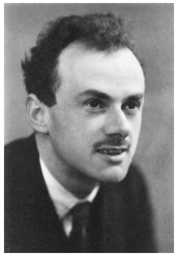
Conventional non-commutative scheme

Canonical quantization

Born, Jordan (1925)

(from wiki)

Dirac (1925)



(from wiki)

Poisson bracket $\{x, p\}$ ➡ Commutator $[\hat{x}, \hat{p}] = i\hbar\{x, p\}$

Conventional non-commutative scheme

$\{x, y\}$ ➡ $[\hat{x}, \hat{y}]$

Symplectic manifold

Quantization of symplectic manifold

This is the basic idea behind the conventional quantization methods:

Deformation quantization, geometric quantization, Berezin-Toeplitz quantization …

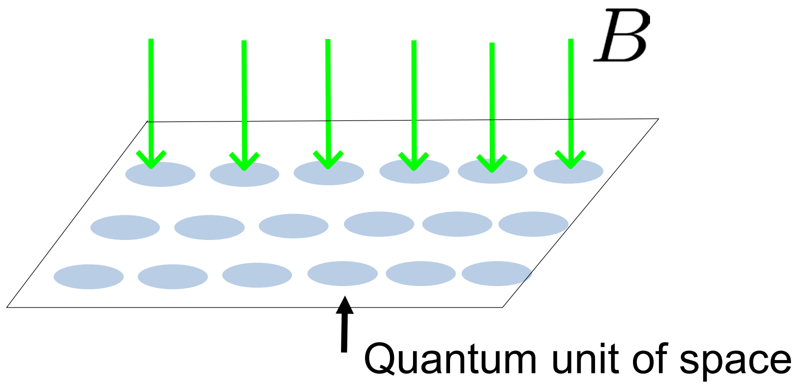
➡ Restricted to symplectic manifolds → General manifold ?
Restricted to the commutator formalism → General NCG such as Nambu bracket ?

NCG only in the lowest Landau level ?

QM on magnetic plane

$$X = x + i\frac{1}{B}D_y \qquad Y = y - i\frac{1}{B}D_x$$

$$\Rightarrow \quad [X, Y] = i\frac{1}{B}$$



``Noncommutative field theory'' *Douglas, Nekrasov* Rev.Mod.Phys. 73 *(2001)* 977

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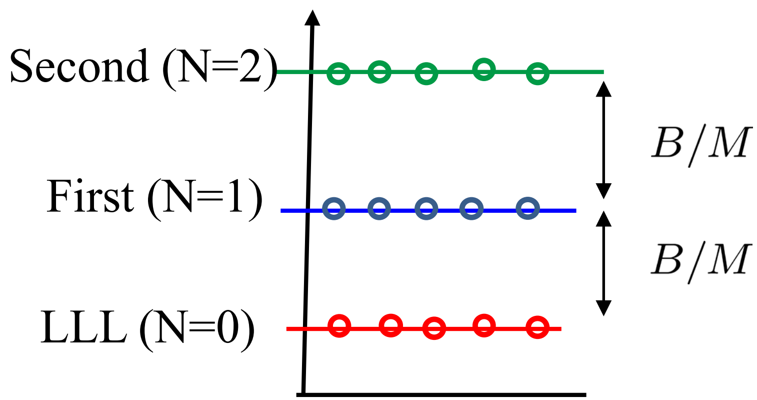
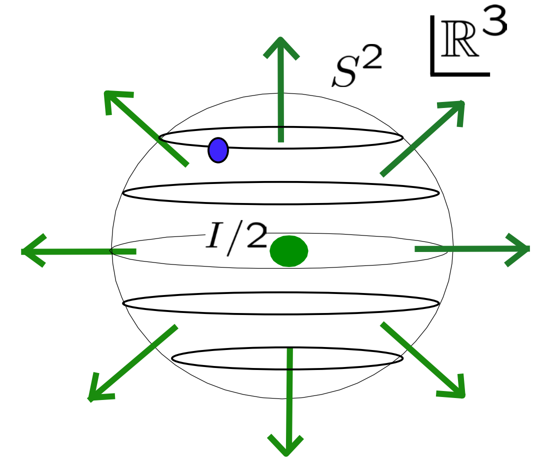


NCG geometry appears in the LLL, but
why lowest Landau level ???

Why magnetic field ???

The Landau model on two-sphere

Let's consider the simplest case. Wu, Yang ('76) Haldane ('83)



$$H = -\frac{1}{2M}\sum_{i=1}^{3} {D_i}^2\Big|_{r=1} = \frac{1}{2M}\sum_{i=1}^{3} {\Lambda_i}^2$$

$$(\Lambda_i = -i\epsilon_{ijk}x_j D_k)$$

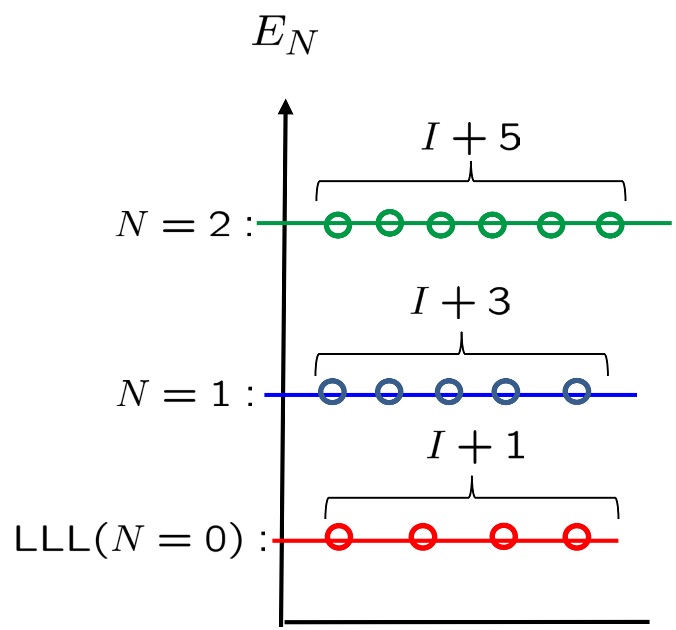
⇓

SU(2) Casimir index

$$\ell = N + \frac{I}{2} \qquad (N = 0, 1, 2, \cdots)$$

Landau levels

$$E_N = \frac{1}{2M}(I(N+\frac{1}{2}) + N(N+1))$$



Eigenstates= SU(2) irreps. : monopole harmonics

$$d(\ell) = 2\ell + 1$$

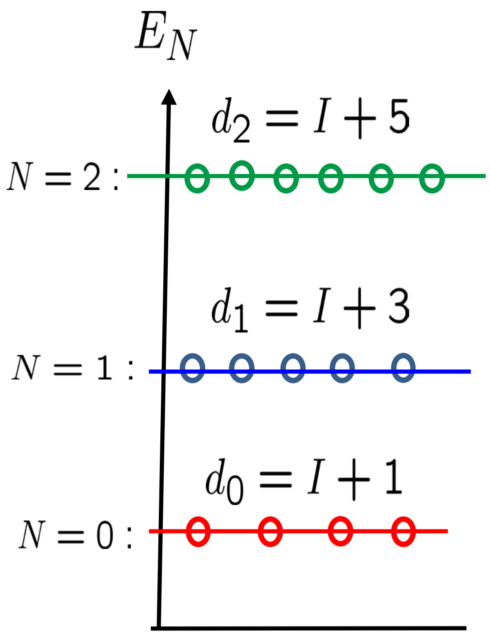
$$Y_m^{(N)}(\theta,\phi) \qquad (m = \ell, \ell-1, \cdots, -\ell)$$

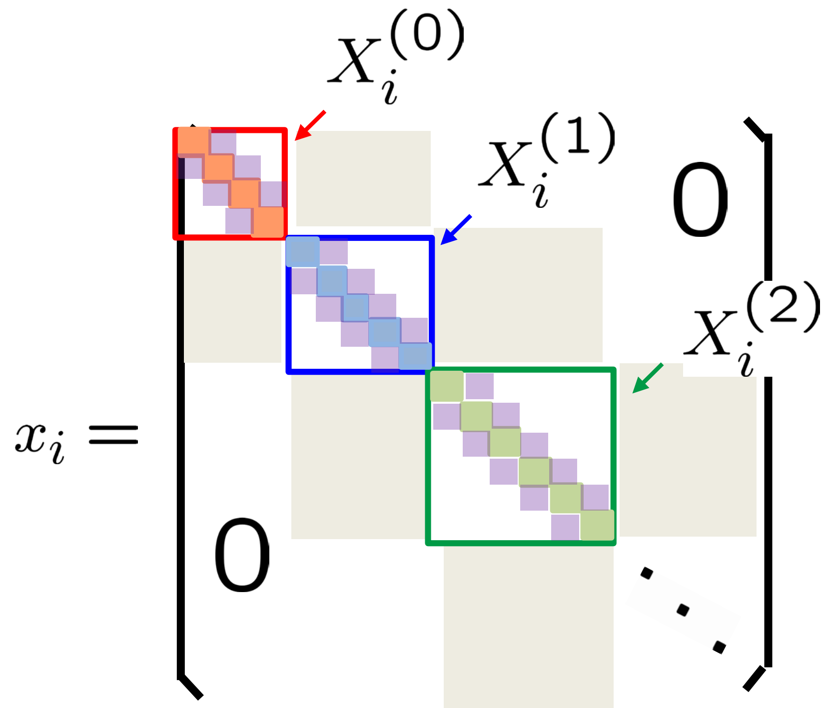
magnetic quantum #

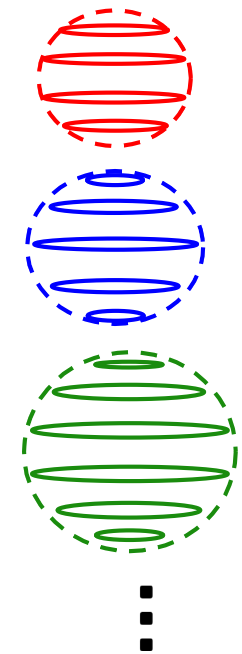
Fuzzy geometry in arbitrary Landau levels

KH ['16]

$$(x_i)_{\alpha\beta} = \langle Y_\alpha | x_i | Y_\beta \rangle \qquad [x_i, x_j] = 0$$







Fuzzy two-sphere

Matrix coordinates

$$X_i^{(N)} = \langle Y^{(N)} | x_i | Y^{(N)} \rangle = 2\alpha_N S_i^{(l=N+\frac{I}{2})}$$

$$\sum_{i=1}^{3} X_i^{(N)} X_i^{(N)} = 4{\alpha_N}^2 l(l+1) \cdot \mathbf{1}$$

$$[X_i^{(N)}, X_j^{(N)}] = 2i\alpha_N \epsilon_{ijk} X_k^{(N)}$$

The fuzzy sphere geometry also appears in the higher Landau levels!

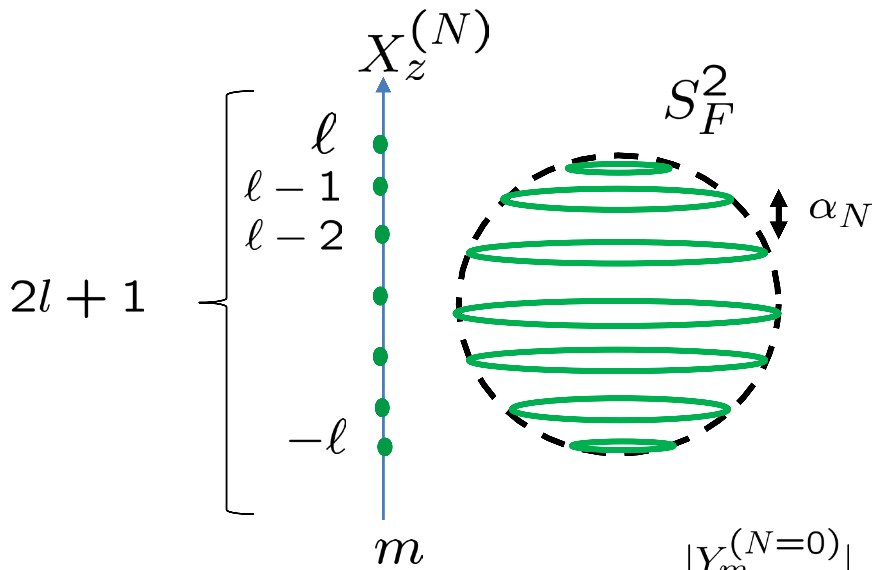
Nth Landau level matrix geometry

- Fuzzy geometry

$$X_i^{(N)} = \alpha_N S_i^{(l=N+\frac{I}{2})}$$

diagonalization

$$X_3^{(N)} = \alpha_N \begin{pmatrix} \ell & 0 & 0 & 0 & 0 \\ 0 & \ell-1 & 0 & 0 & 0 \\ 0 & 0 & \ell-2 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & -\ell \end{pmatrix}$$

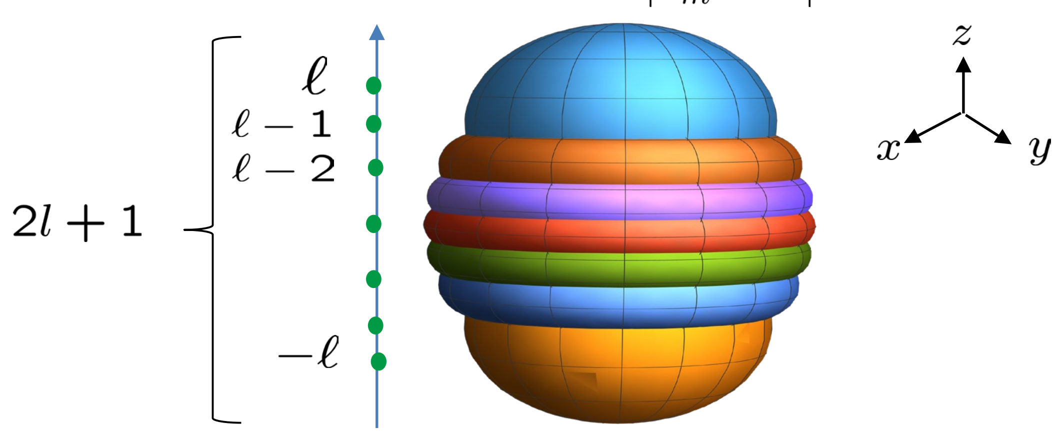


- Landau level Hilbert space

$$Y_m^{(N)}(\theta, \phi)$$

$(m = \ell, \ell-1, \cdots, -\ell)$

: $(2\ell+1)$ dim. basis states



``Points'' of fuzzy S2 = Landau level eigenstates = SU(2) irreps.
⇨ SU(2)-symmetric NC space = SU(2) irreps.

Irreps. : a set of the finite dof that returns to itself by group operation G
= Finite # of points of the fuzzy space

Behind the scene

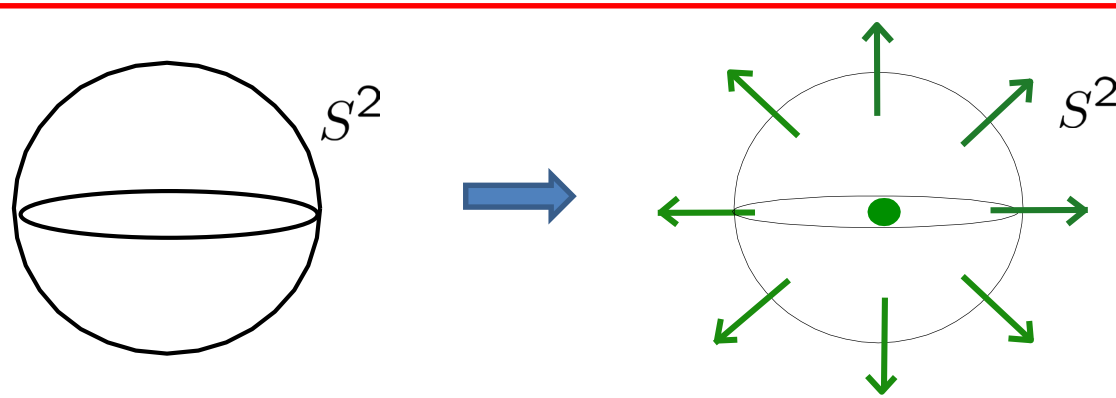
Reinterpretation S^2 ⇒ S_F^2

- Global sym. of manifold, SO(3)
- Points on the manifold (= infinite dof)
: set closed on the group action SO(3)
- Stabilizer group SO(2)
= sym. that does not change the points

- Global sym. group of the system, SU(2)
- Irreps. (= finite dof) of SU(2)
: set closed on the group action SU(2)
- Gauge group U(1)
= sym. that does not change physical states

(External symmetry → Internal symmetry)

$S^2 \simeq SO(3)/SO(2)$ ⇒ QM with SU(2) global sym. and U(1) gauge sym.



(Magnetic field is a consequence of the gauge symmetry.)

General prescription

$$\mathcal{M} \simeq G/H \;\Rightarrow\; \mathcal{M}_F$$

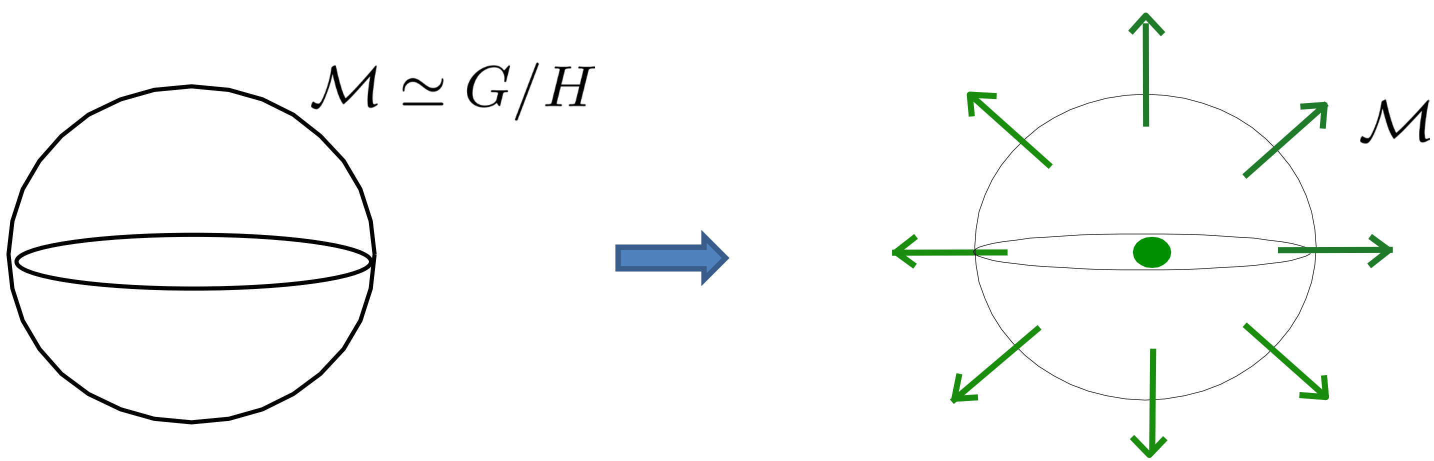
Just replace SU(2) → G, and U(1) → H.

$\mathcal{M} \simeq G/H$ ⇒ QM with gauge symmetry H on the manifold $\mathcal{M}$

$$H = -\frac{1}{2M}\sum_i {D_i}^2\Big|_{\mathcal{M}}$$

⇒ Solve the eigenvalue problem ⇒ Matrix coordinates $\mathcal{M}_F$

$$|\psi_\alpha^{(N)}\rangle \qquad (X_a^{(N)})_{\alpha\beta} = \langle\psi_\alpha^{(N)}|x_a|\psi_\beta^{(N)}\rangle$$



Important Points

- The quantum geometry is not postulated a priori,
  but naturally emerges in the context of physics.

- The original system is totally physical,
  and the background frame work is a consistent QM.

→ No need to worry about mathematical inconsistency!

- Following that prescription, we can automatically construct the fuzzy manifold corresponding to G/H
  (not restricted to symplectic manifolds. Odd D is also OK!)

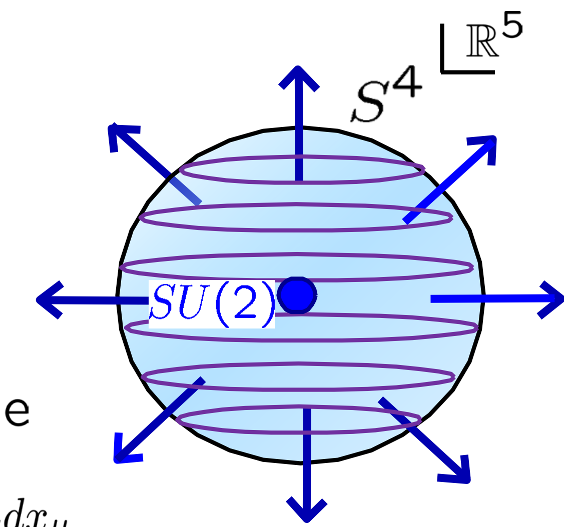
``Relativistic Landau models and generation of fuzzy spheres'' *K.H.* ('16)

as a consequence of level projection. In this work, we proactively utilize the level projection as an effective tool to generate fuzzy geometry. The level projection is specifically

Find a new quantum geometry from quantum mechanics !

Relativistic model [*'16*], Susy model [*'16*], three-sphere [*'17*], four-sphere [*'20, '21. '23*]…

From idea to a concrete model



Classical geometry $S^4 \simeq SO(5)/SO(4)$

Stabilizer group $SO(4) \sim SU(2) \otimes SU(2)$

Gauge symmetry $SU(2)$ ⇨ Yang's $SU(2)$ monopole

$$A = -\frac{1}{r(r+x_5)} \eta^i_{\mu\nu} S_i^{(I/2)} x_\nu dx_\mu$$

Quantum mechanics

$$H = -\frac{1}{2M} \sum_{a=1}^{5} {D_a}^2 \Big|_{r=1}$$

Yang ('78) Zhang, Hu ('01)

SO(5) Casimir index (p,q) $q = N, \quad p = N + I$

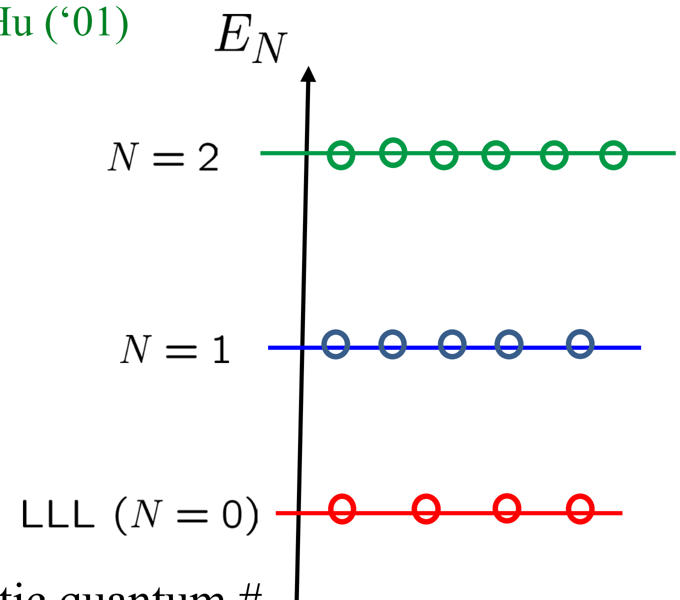
$N = 0, 1, 2, \cdots$

Landau levels:

$$E_N = \frac{1}{2M}(N(N+3) + I(N+1))$$

Eigenstates = SO(5) irreps. : SU(2) monopole harmonics

$Y^{(N)}_{j,m_j;\ k,m_k}(\xi, \chi, \theta, \phi)$ $SU(2) \bigotimes SU(2) = SO(4)$ magnetic quantum #



Matrix coordinates

$$(X_a^{(N)})_{\alpha\beta} = \langle Y_\alpha^{(N)} | x_a | Y_\beta^{(N)} \rangle$$

KH ['23, '21]

⇨ Simplest case $(I/2 = 1/2)$ $X_a^{(N=0)} = \frac{1}{5}\gamma_a$: SO(5) gamma matrices

⇨ In LLL $X_a^{(N=0)} = \frac{1}{I+4}(\gamma_a \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \gamma_a \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes \gamma_a)_{\mathrm{sym}}$

reproduces Berezin-Toeplitz result by *Ishiki, Matsumoto, Muraki['18]*

⇨ In higher LLs

$$X_5^{(N)} = -\frac{2n+I+2}{(2N+I+2)(2N+I+4)} \cdot 2s \cdot \delta_{j,j'}\delta_{k,k'}\delta_{m_j,m'_j}\delta_{m_k,m'_k}$$

$$X_{\mu=1,2,3,4}^{(N)} = \sum_{\sigma,\tau=+,-} \langle \sin\xi \rangle \ \langle y_\mu \rangle \ \delta_{j',j+\frac{\sigma}{2}} \ \delta_{k',k+\frac{\tau}{2}} \qquad (s = \frac{I}{2}, \ \frac{I}{2}-1, \ \frac{I}{2}-2, \ \cdots, \ -\frac{I}{2})$$

$$\langle \sin\xi \rangle = -\frac{4s}{(2N+I+2)(2N+I+4)}\sqrt{(N-n+\frac{1-\sigma}{2})(N+n+I+2+\frac{1+\sigma}{2})} + \cdots \qquad \langle y_1 \rangle = \frac{\sqrt{(2j+1)(2k+1)}}{2}(-1)^{n+I+\frac{\tau}{2}} \begin{Bmatrix} j+\frac{\sigma}{2} & k+\frac{\tau}{2} & \frac{I}{2} \\ k & j & \frac{1}{2} \end{Bmatrix} \sum_{\kappa=+,-} (-1)^{\frac{\kappa-1}{2}} C^{j+\frac{\sigma}{2},m'_j}_{\frac{1}{2},\frac{\kappa}{2};j,m_j} C^{k+\frac{\tau}{2},m'_k}_{\frac{1}{2},\frac{\kappa}{2};k,m_k}$$

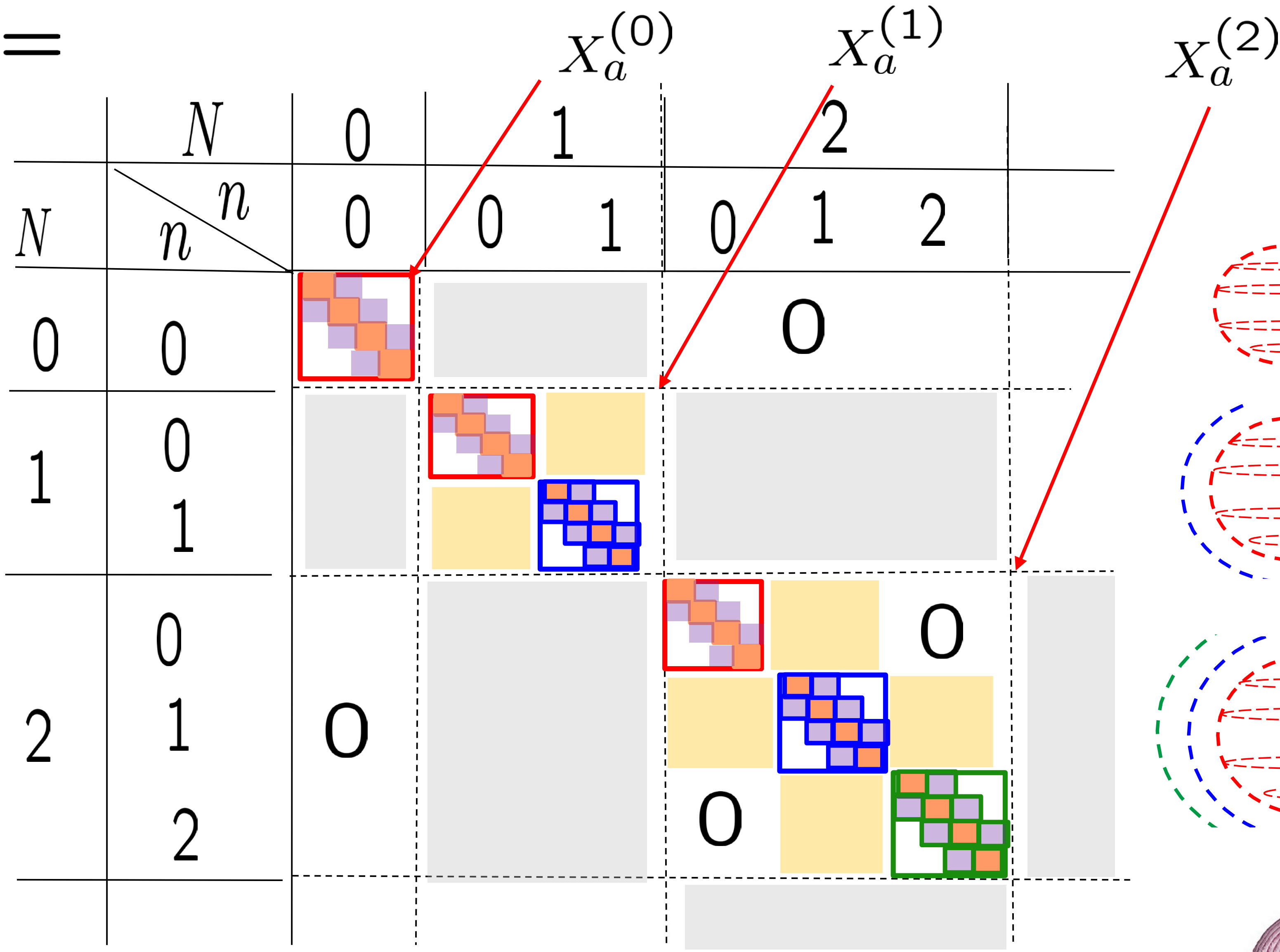
$$\sum_{a=1}^{5} X_a^{(N)} X_a^{(N)} = c_1(N,I) \ 1 \qquad [X_a^{(N)}, X_b^{(N)}, X_c^{(N)}, X_d^{(N)}] = c_3(N,I)\epsilon_{abcde}X_e$$

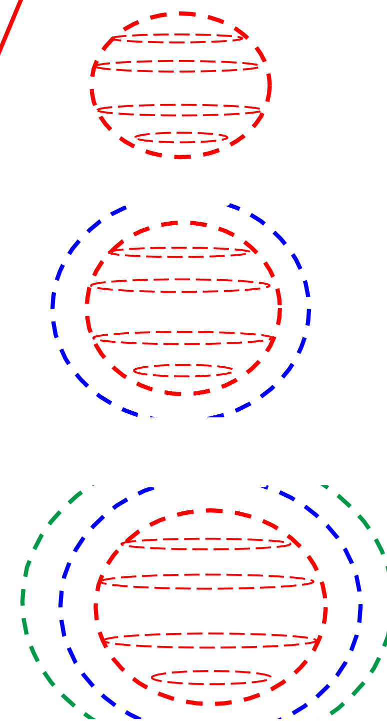
$$([X_1, X_2, X_3, X_4] \equiv \epsilon_{abcd5}X_aX_bX_cX_d)$$

Realization of the fuzzy four-sphere geometry !

Matrix geometry

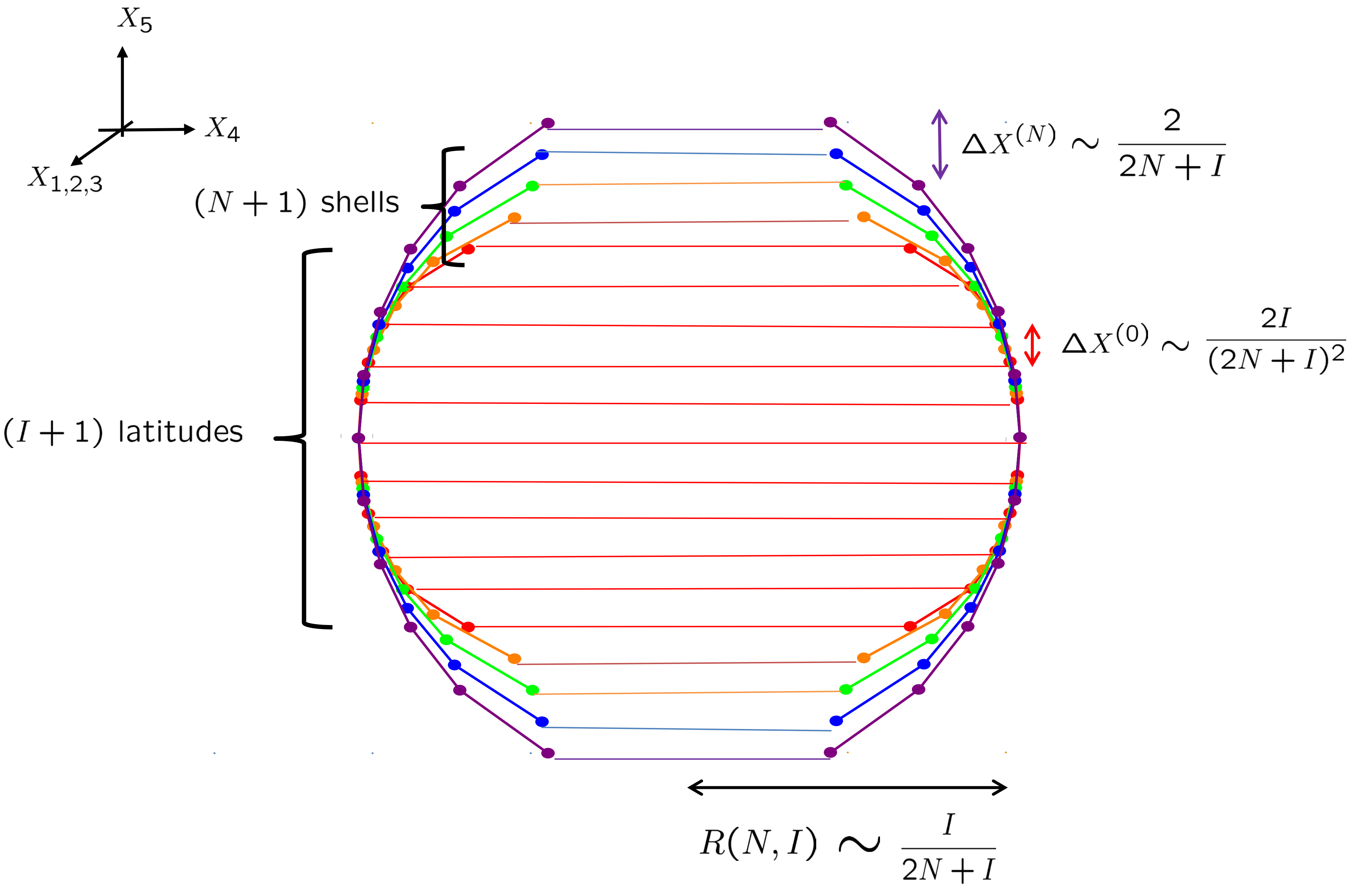
$x_a =$





Nth LL geometry becomes a fuzzy four-sphere-like object with N+1 shells.

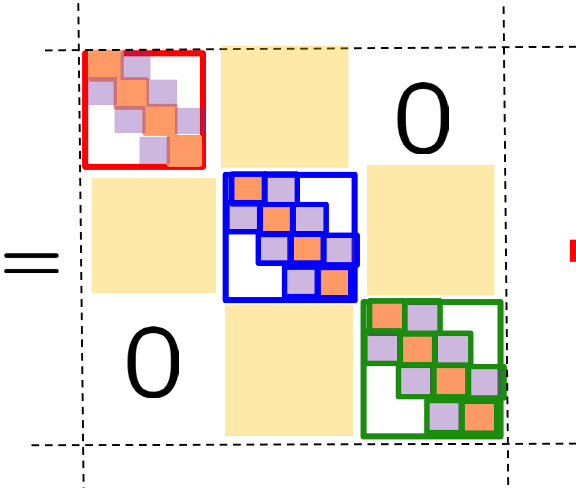
Nth Landau level matrix geometry

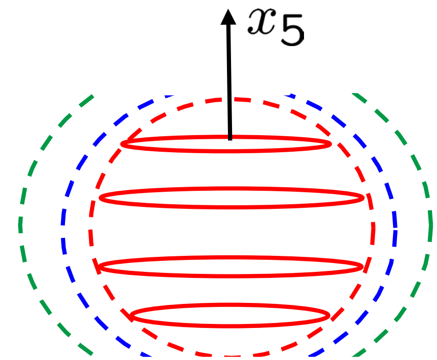


Pure quantum geometry

N(=2)th LL matrix coordinates

$$X_a^{(N=2)} =$$

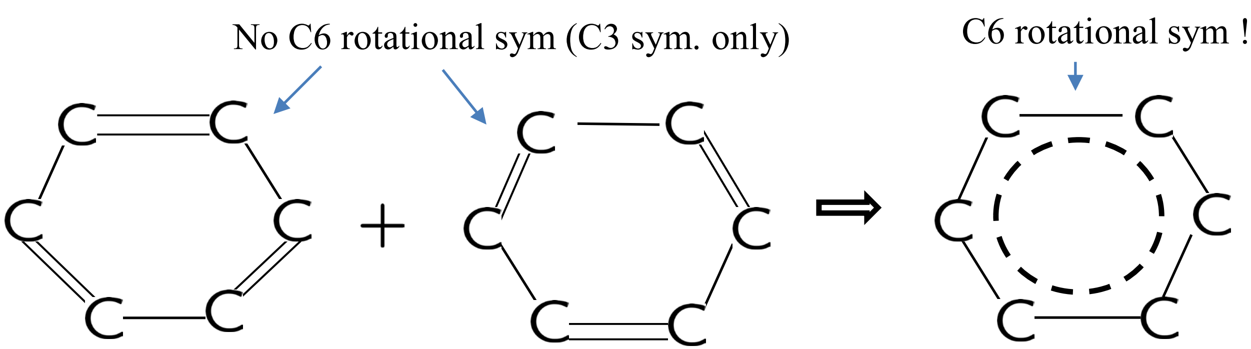




The nested matrix geometry does have the SO(5) rotational symmetry, but each of the shells does not !

→ Purely quantum mechanical geometry

* the covalent bond of benzene



purely quantum mechanical structure

Algebraic property

1. The lowest Landau level matrix geometry

➢ Lie algebraic structure

$$[X_a, X_b] = 4i\Sigma_{ab} \qquad [X_a, \Sigma_{bc}] = i\delta_{ab}X_c - i\delta_{ac}X_b \qquad [\Sigma_{ab}, \Sigma_{cd}] = i\delta_{ac}\Sigma_{bd} - \cdots$$

⇨ SU(4) algebra

➢ Quantum Nambu structure $[X_a, X_b, X_c, X_d] = 4!\epsilon_{abcde}X_e$

2. Higher Landau level matrix geometry

➢ No Lie algebraic structure

$$[X_a, X_b] \neq 4i\Sigma_{ab} \qquad [X_a, \Sigma_{bc}] = i\delta_{ab}X_c - i\delta_{ac}X_b \qquad [\Sigma_{ab}, \Sigma_{cd}] = i\delta_{ac}\Sigma_{bd} - \cdots$$

➢ Quantum Nambu structure

$$[X_a, X_b, X_c, X_d] = 4!\epsilon_{abcde}X_e$$

→ Pure quantum Nambu geometry !

New matrix model solution

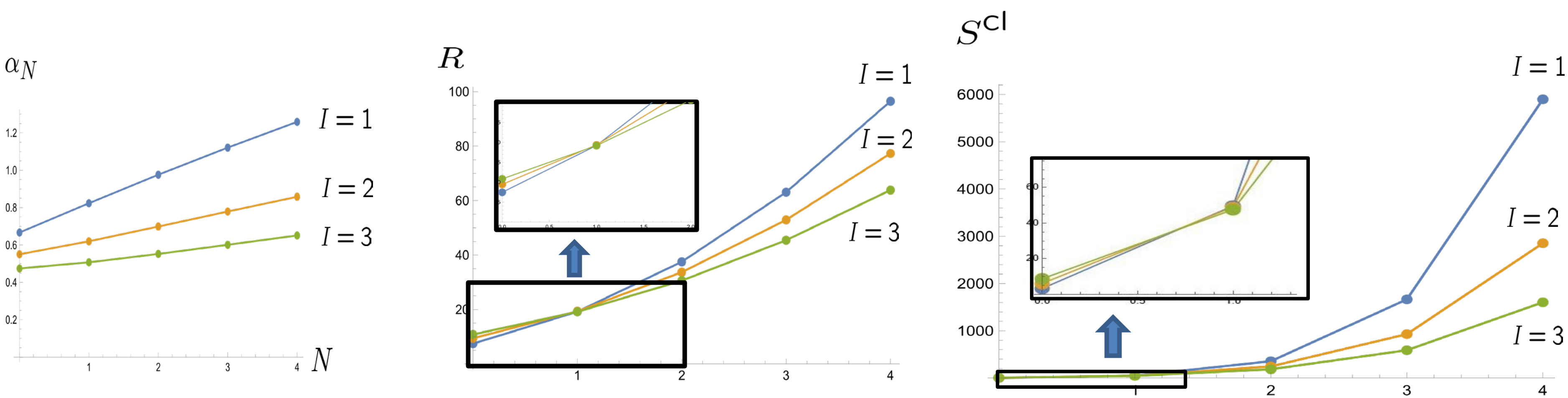
Yang-Mills matrix model : $S = \frac{1}{4}\mathrm{tr}([A_a, A_b]^2) + \frac{1}{5}\epsilon_{abcde}\mathrm{tr}(A_a A_d A_c A_d A_e)$ *Kimura (02)*

➡ EOM : $[[A_a, A_b], A_b] = \epsilon_{abcde} A_b A_c A_d A_e$

Known solution : $A_a \propto X_a^{(N=0)}$: LLL matrix geometry (The usual fuzzy four-sphere)

Castelino, Lee, Taylor (98)

Higher LL matrix coordinates → New solution $A_a = \alpha_N X_a^{(N)}$



Summary

Conventional non-commutative scheme: Quantization of classical manifolds

➡ Preset non-commutative scheme: Directly from quantum Hilbert space

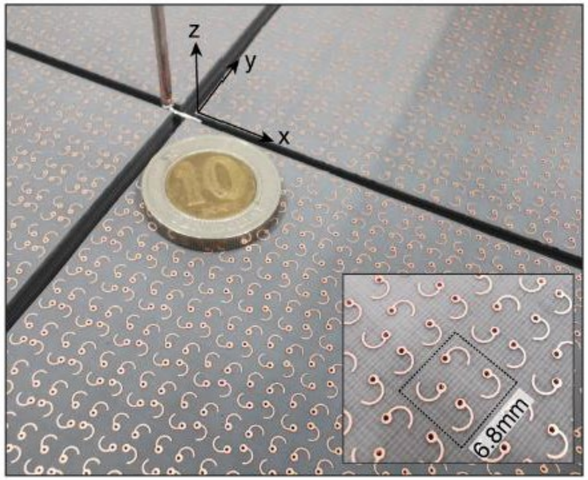
``Quantum native''

- ◆ A concrete prescription for generating the matrix geometry of $\mathcal{M} = G/H$
- ◆ Obtained quantum space is interesting by itself: pure quantum Nambu geometry
- ◆ A practical method to provide a new solution of the matrix model

■ Artificial dimension by metamaterial

``Gauge field induced chiral zero mode in five-dimensional Yang monopole metamaterials''

Ma, Jia et a. (2023) PRL



Amazingly, possible relation to the real world !