Symmetry and phases in cold atomic and condensed matter systems

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Novel correlation effects in cold atom condensates

- Accurate realization of models with short ranged interactions.
- Higher spin particles.
  - Spinor condensates, internal Josephson effect, large s approximations, quarteting, etc.
- Higher symmetry groups.
  - Symmetry principles provide non-perturbation information on strongly correlated systems. Organizational principle for different competing states.
- Geometric phases.
  - U(1) Berry's phase and the SU(2) holonomy of s=3/2 fermions.

Outline

- Generic SO(5) symmetry in spin 3/2 systems.
- Quantum phases of spin 3/2 Hubbard model, Quartet condensation in the spin 3/2 model.
- Non-Abelian holonomy of s=3/2 fermions.
- Emergent non-Abelian gauge structure from Fermi liquid theory: dynamic generation of the spin-orbit interaction.
- New QMC algorithm without the sign problem: reliable determination of quantum phases.

Lecture I

Progress on ultra-cold fermions

- Quantum degeneracy in ultra cold fermionic gases:

$^{40}K :| F, m\rangle = |\frac{9}{2}, \frac{9}{2}\rangle, |\frac{9}{2}, \frac{7}{2}\rangle$: JILA, Science 285, 1703 (1999);

^{6}Li : Rice group, Science 291, 2570 (2001).

- BCS v.s BEC Molecule condensation:

^{40}K JILA, PRL 92, 40403(2004);

^{6}Li MIT, PRL.92, 120403; Innsbruck, PRL 92, 120401(2004);

- Fermions on the optical lattice:

$^{40}K + ^{87}Rb$ G. Modugno et al, PRA 68,11601(2003).

Spin degrees of freedom

- Optical traps and lattices: playground for high spin physics.

- Spin one bosons $^{23}Na\,(F=1),\ ^{87}Rb\,(F=1)$

  T. L. Ho, PRL 81,742 (1998), W. Ketterle, cond-mat/0005001,

  Zhou, PRL 87, 80491 (2001), E. Demler and F. Zhou PRL 88, 163001(2002)

- High spin fermions: $^{22}Na\,(F=\frac{5}{2}),\ ^{40}K\,(F=\frac{9}{2}),$

  $^{86}Rb\,(F=\frac{5}{2}),\ ^{132}Cs\,(F=\frac{3}{2})$

  T. L. Ho and S. Yip, PRL 82, 247(1999). S. Yip and T. L. Ho, PRA 59, 4653(1999).

- The simplest high spin fermionic atoms: F=3/2.

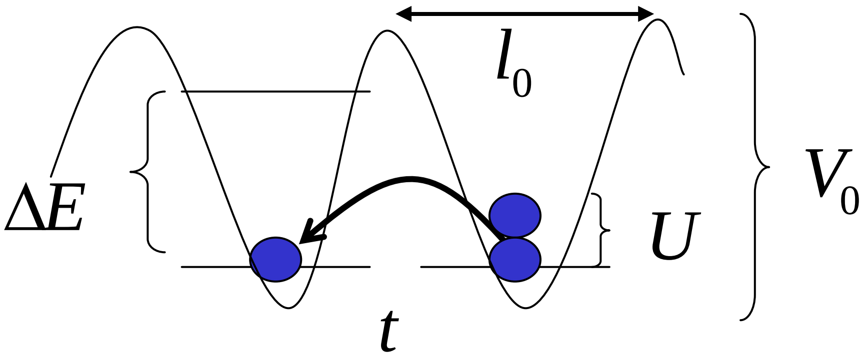
  metastable $^{3}He^{*},\ ^{132}Cs,\ ^{9}Be,\ ^{135}Ba,\ ^{137}Ba$

Cold atoms on a optical lattice: the Hubbard model

$$H = \sum_{\langle ij \rangle,\sigma} -t\{c^{+}_{i,\alpha} c_{j,\alpha} + h.c.\} - \mu \sum_{i} c^{+}_{i,\alpha} c_{i,\alpha}$$
$$+ U_0 \sum_{i} c^{+}_{i,\uparrow} c_{i,\uparrow} c^{+}_{i,\downarrow} c_{i,\downarrow}$$

- Validity of the single band Hubbard model in the optical lattice.

$a_{s:}$ scattering length, $E_{r:}$ recoil energy



$$\frac{U}{\Delta E} \approx \frac{\pi^2 a_s}{2 l_0} \left(\frac{V_0}{E_r}\right)^{1/4} < 0.1,$$

$$l_0 \sim 5000\,A,\ a_s \sim 100\, a_B$$

$$\left(\frac{V_0}{E_r}\right)^{1/4} \approx 1 \sim 2$$

The spin 3/2 Hubbard model

$$H = \sum_{\langle ij\rangle,\sigma} -t\{c^+_{i,\alpha} c_{j,\alpha} + h.c.\} - \mu \sum_i c^+_{i,\alpha} c_{i,\alpha}$$
$$+ U_0 \sum_i P^+_{00}(i) P_{00}(i) + U_2 \sum_{i,m=\pm 2,\pm 1,0} P^+_{2m}(\vec{r}) P_{2m}(\vec{r})$$

- The generic Hamiltonian with spin SU(2) symmetry.
- F=0 (singlet), 2(quintet); m=-F,-F+1,...F.

$$P^+_{Fm}(\vec{r}) = \sum_{\alpha\beta} \langle \tfrac{3}{2}\tfrac{3}{2}; Fm \mid \tfrac{3}{2}\tfrac{3}{2}; \alpha\beta \rangle \psi^+_\alpha(\vec{r}) \psi^+_\beta(\vec{r})$$

- *s*-wave singlet and quintet scattering lengths:

$$g_{0,2} = 4\pi\hbar^2 a_{0,2} / M \ ,$$

Energy levels of in spin 3/2 systems

Energy level degeneracy

$E_0 = 0$

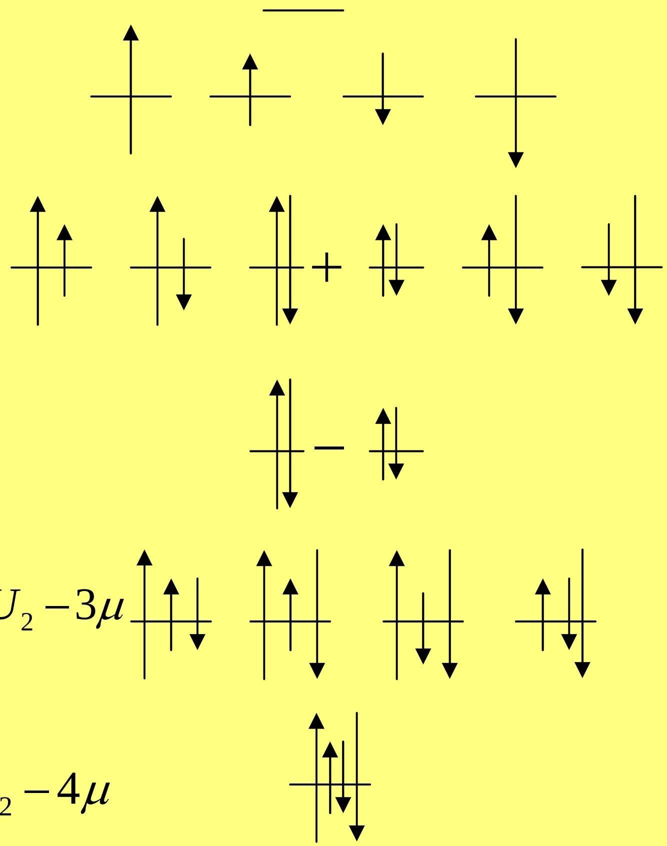
$E_1 = -\mu$

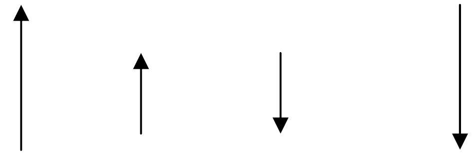
$E_2 = U_2 - 2\mu$

$E_3 = U_0 - 2\mu$

$E_4 = \frac{1}{2}U_0 + \frac{5}{2}U_2 - 3\mu$

$E_5 = U_0 + 5U_2 - 4\mu$





$S_z : \frac{3}{2}, \quad \frac{1}{2}, \quad -\frac{1}{2}, \quad -\frac{3}{2}$

SO(5) structure

- Level degeneracy: 1, 4, 5.
- E_1, E_4, E_6 states are spin SU(2) and SO(5) singlets.
- E_2, E_5 quartet states are the 4-d SO(5) spinor states.
- E_3 spin-2 quintet states are the 5-d SO(5) vector states.
- $U_0=U_2 \Longleftrightarrow E_2=E_3 \Longleftrightarrow$ SU(4)

Generic SO(5) symmetry

- The kinetic energy part is SU(4) symmetric, which is broken into SO(5) by interactions at $g_0 \neq g_2$.

$$SU(4) \cong SO(6) \quad \underset{g_0 = g_2}{\overset{g_0 \neq g_2}{\rightleftarrows}} \quad SO(5)$$

	Kinetic part	g_o term	g_2 term
Spin SU(2)	s=3/2 (d=4)	s=0 (d=1)	s=2 (d=5)
SO(5)	spinor (d=4)	singlet (d=1)	vector (d=5)

- Spin 3/2 system high Tc superconductivity

The **particle-hole** channel SO(5) vs. the **particle-particle** channel SO(5)

Wu, Hu and Zhang, PRL 91, 186402(2003); Demler, Hanke and Zhang, cond-mat/0405038

Spin SU(2) algebra: s=1/2, 3/2

- S=1/2 2-d spinor Representation and 2*2 Pauli matrices:

$$\psi(r)=(\psi_{1/2},\psi_{-1/2})^T \qquad \sigma_{1,}\ \sigma_{2,}\ \sigma_{3,} \quad \{\sigma_{a,}\ \sigma_b\}=2\delta_{ab}$$

- S=3/2 4-d spinor Representation.

$$\psi(r)=(\psi_{3/2},\psi_{1/2},\ \psi_{-1/2},\ \psi_{-3/2})^T$$

$$F_x=\begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix} \quad F_y=\begin{pmatrix} 0 & -\frac{\sqrt{3}}{2}i & 0 & 0 \\ \frac{\sqrt{3}}{2}i & 0 & -i & 0 \\ 0 & i & 0 & -\frac{\sqrt{3}}{2}i \\ 0 & 0 & \frac{\sqrt{3}}{2}i & 0 \end{pmatrix} \quad F_z=\begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix}$$

SO(5) Algebra (I)

- Spin 3/2 nematic matrices:

$$\frac{1}{\sqrt{3}}\{F_x,F_y\}=\Gamma^1,\quad \frac{1}{\sqrt{3}}\{F_z,F_x\}=\Gamma^2,$$
$$\frac{1}{\sqrt{3}}\{F_y,F_z\}=\Gamma^3,\quad F_z^2-\frac{5}{4}=\Gamma^4,$$
$$F_x^2-F_y^2=\Gamma^5$$

Five vector ☝ matrices:

$$\{\Gamma^a,\Gamma^b\}=2\delta_{ab},\quad (1\le a,b\le 5)$$
$$\Gamma^1=\begin{pmatrix}0, -iI\\ iI, 0\end{pmatrix},\quad \Gamma^{2,3,4}=\begin{pmatrix}\vec{\sigma},\ 0\\ 0, -\vec{\sigma}\end{pmatrix},$$
$$\Gamma^5=\begin{pmatrix}0,\ I\\ I, 0\end{pmatrix}$$

- 3 spin operators F_i and
\+ 7 cubic symmetric traceless combinations $\xi^m_{ijk}\,F_i\,F_j\,F_k$

Ten SO(5)generators

$$\Gamma^{ab}=\frac{i}{2}[\Gamma^a,\Gamma^b]\ (a<b)$$

- $\Gamma^a,\quad \Gamma^{ab}$ together form the 15 generators of the SU(4) group.

SO(5) Algebra (II)

- Particle-hole channel bilinears:

s=1/2

4=1 scalar +3 tensors

$$n(\vec{r}) = \psi^+(\vec{r})\psi(\vec{r}),$$
$$\vec{S}(\vec{r}) = \psi^+(\vec{r})\frac{\vec{\sigma}}{2}\psi(\vec{r})$$

⟺

s=3/2

16=1 scalar+5 vectors +10 tensors

$$n(\vec{r}) = \psi^+(\vec{r})\psi(\vec{r}),$$
$$n_a(\vec{r}) = \psi^+(\vec{r})\frac{\Gamma^a}{2}\psi(\vec{r}),$$
$$L_{ab}(\vec{r}) = \psi^+(\vec{r})\frac{\Gamma^{ab}}{2}\psi(\vec{r})$$

- Under the spin SU(2) rotation:

  $n(\vec{r})$ spin 0, $n_a(\vec{r})$ spin 2, $L_{ab}(\vec{r})$ spin 1+ 3

SO(5) Algebra (III)

- Time reversal transformations: ***T= R C*** (*C*: complex conjugate)

s=1/2

$$R = i\sigma_y$$
$$TnT^{-1} = n,$$
$$T\vec{S}T^{-1} = -\vec{S}.$$

s=3/2

$$R = \begin{pmatrix} 0, & -i\sigma_y \\ -i\sigma_y, & 0 \end{pmatrix}$$
$$TnT^{-1} = n, \quad Tn_aT^{-1} = n_a,$$
$$TL_{ab}T^{-1} = -L_{ab}$$

- Pairing operators:

$$\text{spin 0 and SO(5) scalar: } \eta^+(\vec{r}) = \frac{1}{2}\psi^+(\vec{r})R\psi^+(\vec{r})$$

$$\text{spin 2 and SO(5) vectors: } \chi_a^{\ +}(\vec{r}) = \frac{-i}{2}\psi^+(\vec{r})\Gamma^a R\psi^+(\vec{r})$$

The exact particle-hole channel SO(5) symmetry

- Explicit SO(5) invariant Hamiltonian:

$$H=\int dr\,\psi^{+}(r)\left(\frac{-\hbar^{2}}{2m}\hat{\nabla}^{2}-\mu\right)\psi(r)+g_{0}\eta^{+}(\vec{r})\eta(\vec{r})+g_{2}\sum_{a=1\sim 5}\chi_{a}^{+}(\vec{r})\chi_{a}(\vec{r})$$

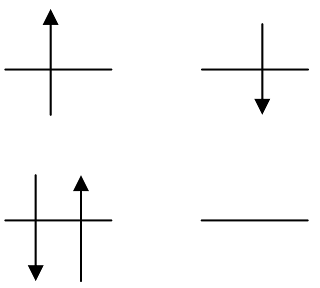
$$\text{axial form}: P_{2m}^{+}(r) \quad \Leftrightarrow \quad \text{polar form}: \chi_{a}^{+}(r)(a=1\sim 5)$$

$$\text{analogy}: Y_{2m}(\theta,\phi) \quad \Leftrightarrow \quad d_{xy},\ d_{zx},\ d_{yz},\ d_{r^2-3z^2},\ d_{x^2-y^2}$$

- The above proof is also valid for both the lattice and the continuum model.
- Particle-hole SO(5) symmetry vs. particle-particle SO(5) symmetry in the high Tc context.

C. J. Wu, J. P. Hu and SCZ, PRL 91, 186402(2003).

SO(4) symmetry in spin ½ Hubbard model



- $\eta^+(i) = c^+_{i\uparrow} c^+_{i\downarrow},\ \eta(i) = c_{i\downarrow} c_{i\uparrow},\ N(i) = \frac{1}{2}(n_i - 1)$
- Pseudo-spin SU(2) algebra.

$$I_x = \sum_i (-)^i \,\mathrm{Re}\,\eta^+(i),\ I_y = \sum_i (-)^i \,\mathrm{Im}\,\eta^+(i),\ I_z = \sum_i N(i)$$

- Exact SO(4) symmetry at half-filling and bipartite lattices.

$$SU_{psp}(2) \otimes SU_{sp}(2) \cong SO(4)$$

- Degeneracy of CDW and SC at negative U.

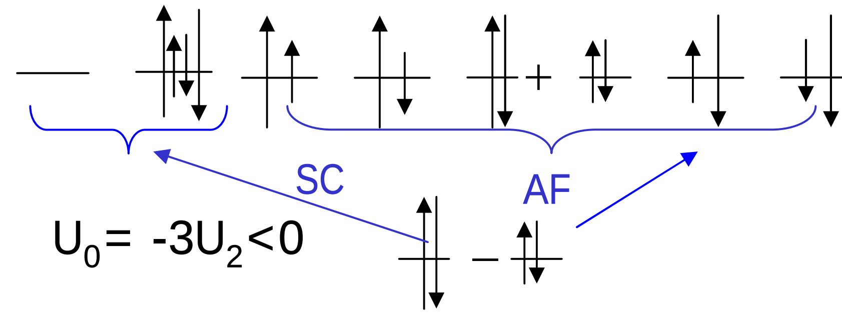
$$O_{cdw} = \sum_i (-)^i N(i), \quad \mathrm{Re}\, O_{sc} = \sum_i \mathrm{Re}\,\eta^+(i), \quad \mathrm{Im}\, O_{sc} = \sum_i \mathrm{Im}\,\eta^+(i),$$

- doublet modes I_x, I_y at (:
Rotating CDW to SC.

C.N. Yang and SCZ, Mod. Phys. Letter B 4, 759(1991).

Higher symmetries at half-filling and bipartite lattice

- SO(7) symmetry



$U_0 = -3U_2 < 0$

a) 5-vec AF spin nematics $(-)^i n_a(i)$
and singlet SC $\eta(i)$.

5-♍ models: rotate SC←→AF.

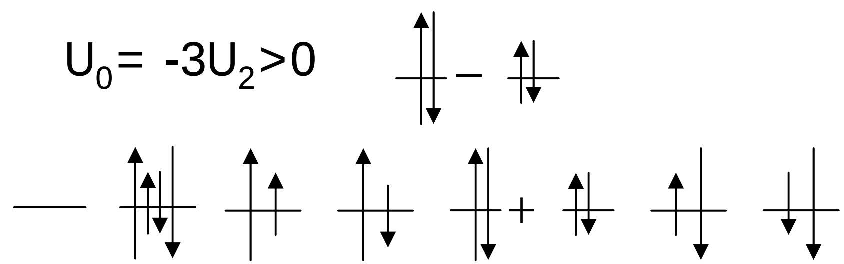
Analogy to the □ modes in high Tc.

b) 10-tensor AF $(-)^i L_{ab}(i)$,

CDW $(-)^i N(i)$, quintet SC $\chi^+(i)$

21d adjoint Rep. of SO(7)

$U_0 = -3U_2 > 0$

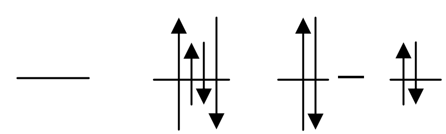


- SO(5)*SU(2) symmetry:

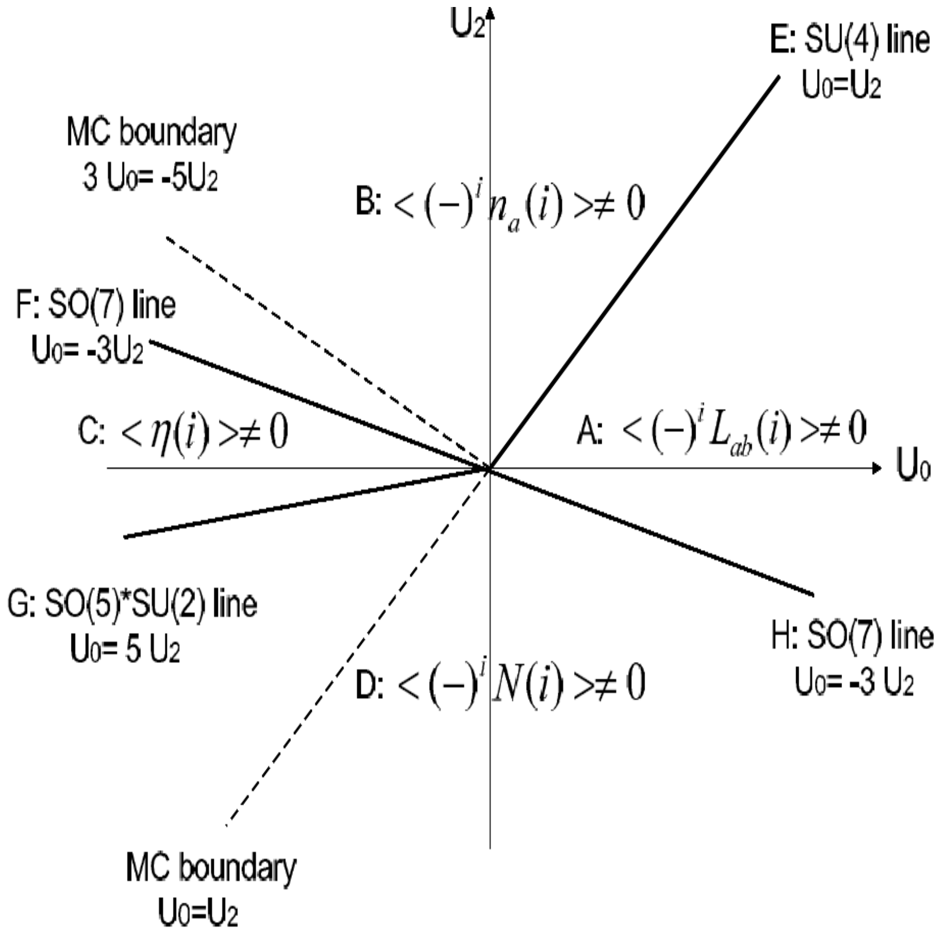
CDW $(-)^i N(i)$, singlet SC $\eta(i)$.

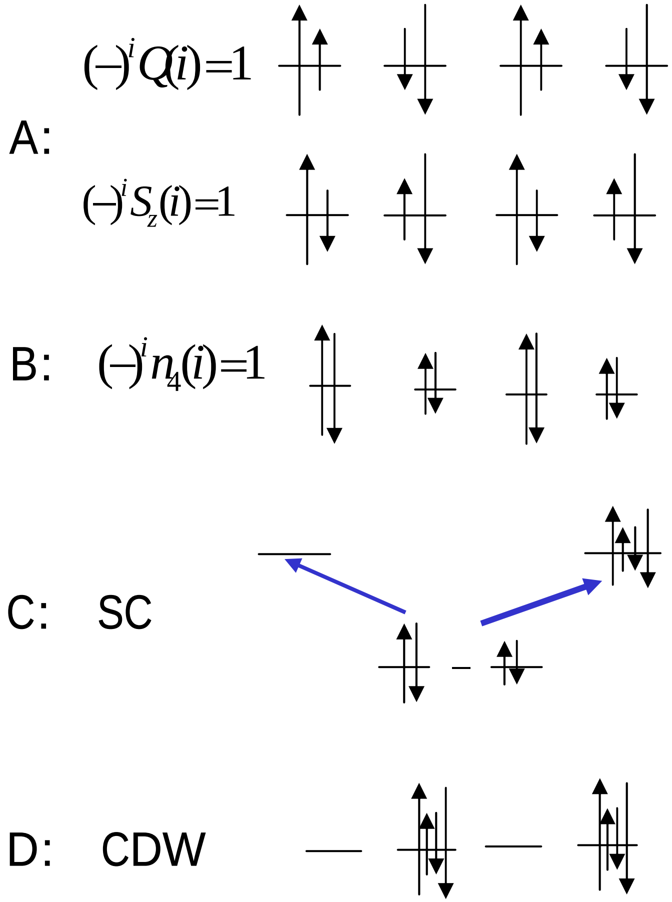
♒-modes: rotate CDW←→SC

$U_0 = 5U_2 < 0$



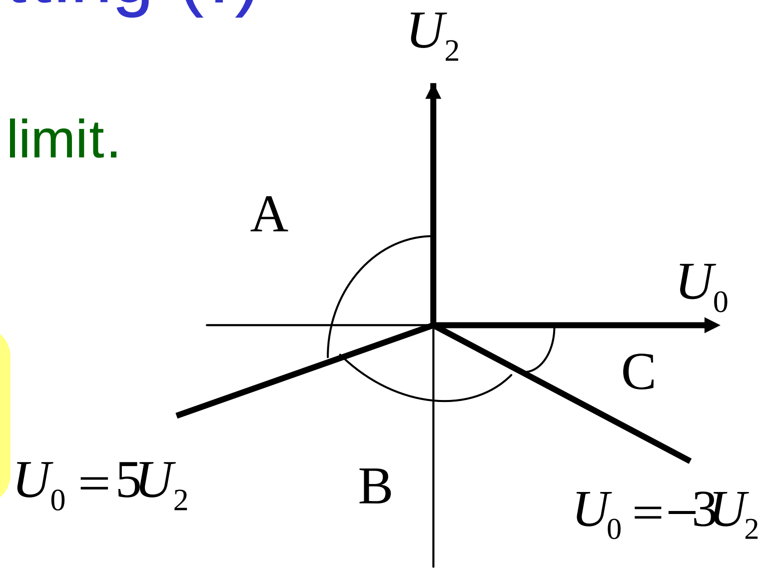
Phase diagram at half-filling, phase boundaries are exact!





Pairing vs. quartetting (I)

- The low density and strong coupling limit.



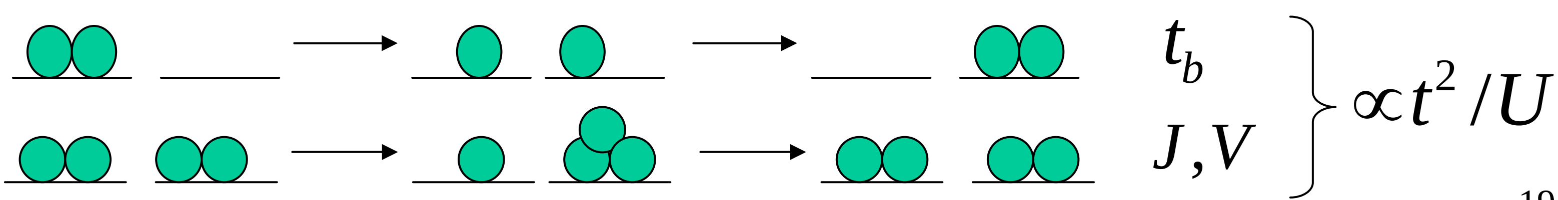
- Region A: composite singlet boson:

$$H_{eff} = -2t_b \sum_{\langle ij \rangle} \{ b_i^+ b_j + h.c. \} + V \sum_{\langle ij \rangle} b_i^+ b_i b_j^+ b_j$$

- Region C: composite quintet boson:

$$H_{eff} = -2t_b \sum_{\langle ij \rangle, a} \{ b_{ia}^+ b_{ja} + h.c. \} + V \sum_{\langle ij \rangle} b_{ia}^+ b_{ia} b_{jb}^+ b_{jb} + J \sum_{\langle ij \rangle} L_{ab}(i) L_{ab}(j)$$

- V=0 on the phase boundaries.

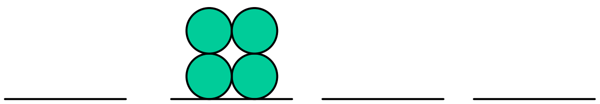


Pairing vs. quartetting (II)

- Region B: quartetting bosons:

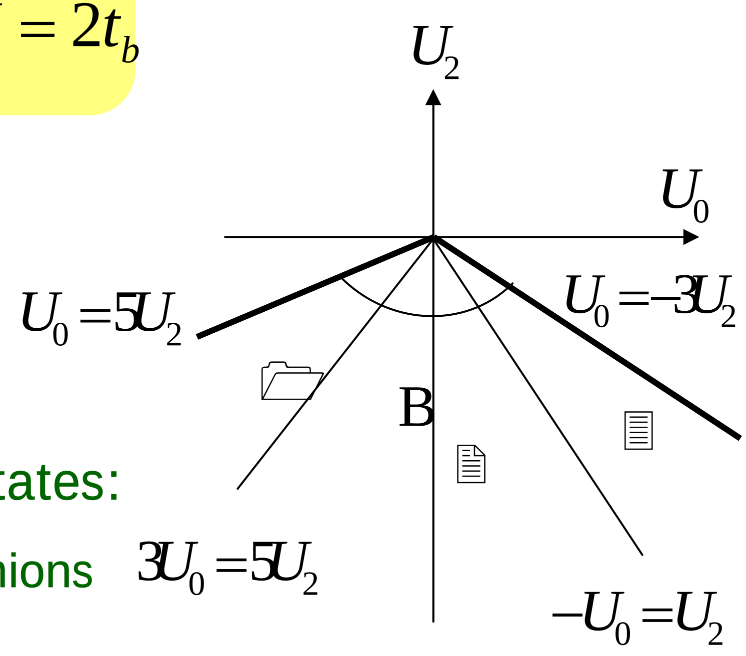
$$H_{eff} = -2t_b \sum_{\langle ij \rangle} \{b_i^+ b_j + h.c.\} + V \sum_{\langle ij \rangle} b_i^+ b_i b_j^+ b_j$$

$$t_b = \left(\frac{2t}{-U_0 + 5U_2}\right)^2 \left(\frac{t^2}{U_0 - 5U_2} + \frac{t^2}{-U_0 - 3U_2}\right), \quad V = 2t_b$$



- Deuteron condensation vs alpha-like quartetting in the nuclear matter.

  G. Ropke *et al,* PRL 80, 3177(1998).



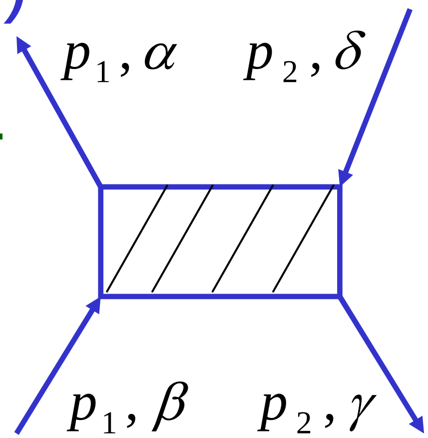
- Lowest energy quartetting breaking states:

  1: two singlet pairs; 2: single + three fermions

  3: two quintet pairs.

- vortices with $\Omega a^2 = \hbar / 4M$

Spin 3/2 Fermi liquid theory (I)



- 4 sets of Landau functions without SO(5) symmetry.

$$\frac{3}{2}(p) + \frac{3}{2}(h) \rightarrow 0 + 1 + 2 + 3$$

S. Yip and T. L. Ho, PRA 59, 4653(1999)

- Only three sets of Landau functions by the SO(5) symmetry.

Particle-hole spin triplet and septet → SO(5) adjoint Rep (tensor).

$F_{x,y,z}$ ⟷ (degenerate / Not accidental) $\xi^m_{ijk} F_i F_j F_k$, $m = 1 \sim 7$

$$3+7=10$$

- SO(5) symmetric Landau functions:

$$f_{\alpha\beta,\gamma\delta}(\hat{p}_1,\hat{p}_2) = f^S(\hat{p}_1,\hat{p}_2) + f^V(\hat{p}_1,\hat{p}_2)\frac{\Gamma^a_{\alpha\beta}}{2}\frac{\Gamma^a_{\gamma\delta}}{2} + f^T(\hat{p}_1,\hat{p}_2)\frac{\Gamma^{ab}_{\alpha\beta}}{2}\frac{\Gamma^{ab}_{\gamma\delta}}{2}$$

Spin 3/2 Fermi liquid theory (II)

- S-wave scattering approximation:

$$f^{S} = (g_0 + 5g_2)/16: \quad n \qquad \text{density channel (spin 0)}$$

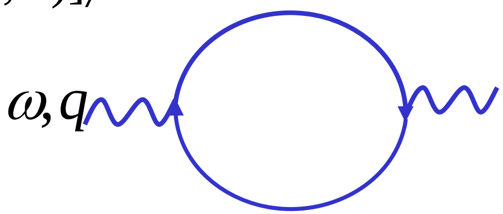
$$f^{V} = (g_0 - 3g_2)/4: \quad n_a \qquad \text{spin nematics channel (spin 2)}$$

$$f^{T} = -(g_0 + g_2)/4: \quad L_{ab} \qquad \text{SO(5) tensor channel (spin 1+3)}$$

- Collective modes: $\hat{O}\,(rt) = n(rt), n_a(rt), L_{ab}(rt)$

$$\chi_{s,v,t}(r-r', t-t') = -i\theta(t-t')\langle[\hat{O}_{s,v,t}(r,t), \hat{O}_{s,v,t}(r',t')]\rangle$$

- RPA approximation:



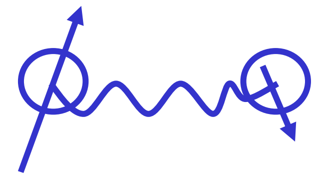
$$F_0^{S,V,T} = N_F f^{S,V,T}, \quad 1 = F_0 \int \frac{d\cos\theta}{2} \frac{\cos\theta}{\omega / v_f q - \cos\theta + i\eta}$$

$F_0 > 0$: propagating modes

$0 > F_0 > -1$: damped modes

$F_0 < -1$: Landau-Pomeranchuk instabilities

Quintet superfluid (I)



- $g_2 < 0$ favors quintet Cooper pairing. T. L. Ho and S. Yip, PRL 82, 247(1999).

  order parameters as symmetric and traceless tensor:

$$A_{ij} = |g_2| \langle | \psi^{+} [\{F_i, F_j\} - \delta_{ij} \tfrac{F(F+1)}{3}] R \psi^{+} | \rangle$$

- Without SO(5) symmetry, the Ginzburg-Landau free energy:

$$F = -\alpha' Tr(AA^{+}) + \beta_1' |TrA^2|^2 + \beta_2' |TrA^{*}A|^2 + \beta_3' Tr(A^{*2}A^2)$$

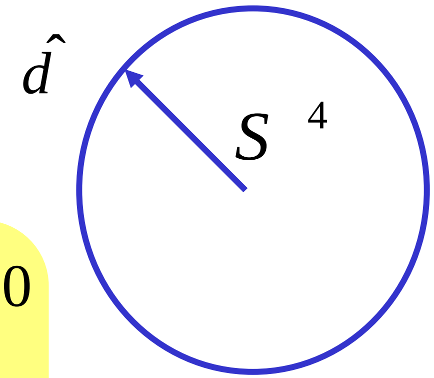
- Simplified G-L free energy with SO(5) symmetry.

$$F = \gamma \nabla \Delta_a^{*} \nabla \Delta_a + \alpha(T) |\Delta|^2 + \frac{\beta_1}{2} |\Delta|^4 + \frac{\beta_2}{2} L_{ab}^2$$

5-d complex order parameters: $\Delta_a = |g_2| \langle | \chi_a^{+}(\vec{r}) | \rangle$

SO(5) spin of Cooper pairs: $L_{ab} = \frac{1}{\sqrt{2}} (\Delta_a^{*} \Delta_b - \Delta_b^{*} \Delta_a)$

Quintet superfluid (II)



- Polar states vs. Axial states:

Unitary states: $\Delta_{pl,a} = |\Delta| e^{i\phi} \hat{d}_a, \Rightarrow \langle | L_{ab} | \rangle = 0$

Non-unitary states: $\Delta_{ax,a} = \frac{1}{\sqrt{2}} \Delta e^{i\phi} (\hat{d}_{1,a} + i\hat{d}_{2,a}),$

$\hat{d}_{1,a} \perp \hat{d}_{2,a} \quad \Rightarrow \langle | L_{ab} | \rangle \neq 0$

$\beta_2 > 0 \Rightarrow$ Polar states, $\beta_2 < 0 \Rightarrow$ axial states

- Polar state: Pairing on all fermi surface:

$$\chi_4^+(\vec{r}) = i(\psi_{\frac{3}{2}}^+ \psi_{-\frac{3}{2}}^+ + \psi_{\frac{1}{2}}^+ \psi_{-\frac{1}{2}}^+)$$

Axial state: Pairing on two fermi surface:

$$\chi_1^+ + i\chi_5^+ = 2\psi_{\frac{3}{2}}^+ \psi_{-\frac{1}{2}}^+$$

Polar state is more stable.

Quintet superfluid (III)

- Gross-Pitaevskii equation at T=0:

$$H_{GP} = \int d^D r \, \frac{\hbar^2}{4M} \nabla \Psi_a^* \nabla \Psi_a + \frac{c_1}{2} (\Psi_a^* \Psi_a - \rho_0)^2 + \frac{c_2}{2} \sum_{1 \le a < b \le 5} (\Psi_c^* L_{cd}^{ab} \Psi_d)^2$$

$$c_2 > 0 \Rightarrow \Psi_a = \sqrt{\rho} e^{i\phi} \hat{d}_a$$

- Goldstone modes in the polar state:

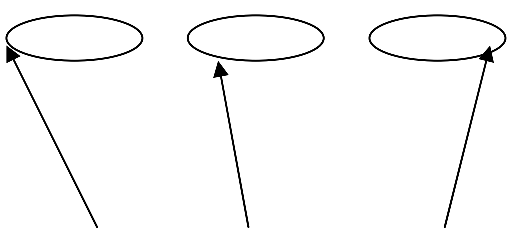
one phonon mode:

$$\partial_t^2 \phi = \frac{\rho c_0}{2M} \nabla^2 \phi, \qquad v_c = \sqrt{\rho_0 c_0 / 2M}$$

four spin nematics waves:

$$\partial_t L_{ab} = \frac{\rho}{2M} (\hat{d}_a \nabla^2 \hat{d}_b - \hat{d}_b \nabla^2 \hat{d}_a),$$

$$\partial_t \hat{d}_a = -c_2 L_{ab} \hat{d}_b \qquad v_s = \sqrt{\rho_0 c_2 / 2M}$$



The SO(5) version of the Leggett equation.

Lecture II

Non-abelian holonomy of SO(5) spinor states

- Wigner-Von Neumann classes for level crossing:

	Co-dimension	symmetry	systems
orthogonal	2	Time-reversal invariant, no Krammer degeneracy.	bosons
unitary	3	Time-reversal breaking	SO(3) spinor
symplectic	5	Time-reversal invariant, with Krammer degeneracy.	SO(5) spinor

- Example of SO(5) spinors.

fermions in high T_c: E. Demler and SCZ, Annal of Physics 🖹🖮🗁 ☎🗁🕽🖳 🖰🗏 ☎🗁🙵🙵🙵🕽🖎

electrons in p-doped semiconductor (Luttinger model)

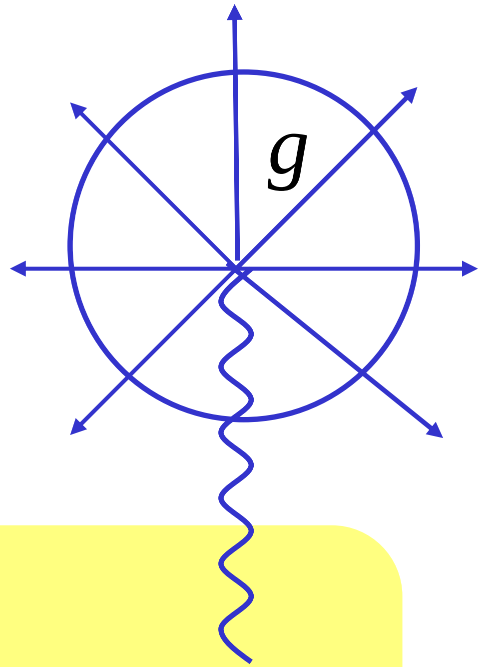
S. Murakami, N. Nagaosa, SCZ, science 301,1348 (2003🕽🖎

fermions in spin 3/2 spin nematics AF phase.

quasi-particles in the quintet pairing states.

C. H. Chern, H. D. Chen, C. J. Wu, J. P. Hu and SCZ, PRB 69 214512(2004).

SO(3) spinor and Abelian holonomy



- SO(3) spinor: $H_{\alpha\beta} = n_a(t)\sigma^a_{\alpha\beta},$

  the eigenstate for nondegeneration eigenvalue 1:

$$|\psi^1_+\rangle = \frac{1}{N}\frac{1+\hat{n}_a\sigma^a}{2}\begin{pmatrix}1\\0\end{pmatrix}, \quad |\psi^2_+\rangle = \frac{1}{N}\frac{1+\hat{n}_a\sigma^a}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

- Abelian Berry's phase and Dirac Monopole:

$$\gamma^1_+ = -i\int dn_a \left\langle \psi^1_+(n) \,|\, \frac{d}{dn} \,|\, \psi^1_+(n)\right\rangle = \int dn_a \; A^a_+(n)$$

$$A^a dn_a = \frac{1}{2(1+\hat{n}_z)}(\hat{n}_1 d\hat{n}_2 - \hat{n}_2 d\hat{n}_1) \qquad \tilde{A}^a dn_a = \frac{-1}{2(1-\hat{n}_z)}(\hat{n}_1 d\hat{n}_2 - \hat{n}_2 d\hat{n}_1)$$

$$B^a = \tfrac{1}{2}\varepsilon_{abc}\partial_b A^c = \frac{n^a}{n^2}$$

- First Chern number: $\oint B^a dS_a = g = 1$

SO(5) spinor and Non-Abelian holonomy

- SO(5) spinor: $H_{\alpha\beta} = n_a(t)\Gamma^a_{\alpha\beta},$

  eigenstates for double degenerate eigenvalue 1:

$$|\psi^1_+\rangle = \frac{1}{N}\frac{1+\hat{n}_a\Gamma^a}{2}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |\psi^4_+\rangle = \frac{1}{N}\frac{1+\hat{n}_a\Gamma^a}{2}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

- non-Abelian Berry's phase and Yang Monopole:

$$\gamma^1_+ = -i\int dn_a \begin{bmatrix} \left\langle \psi^1_+ \left| \frac{d}{dn} \right| \psi^1_+ \right\rangle, \left\langle \psi^1_+ \left| \frac{d}{dn} \right| \psi^4_+ \right\rangle \\ \left\langle \psi^4_+ \left| \frac{d}{dn} \right| \psi^1_+ \right\rangle, \left\langle \psi^4_+ \left| \frac{d}{dn} \right| \psi^4_+ \right\rangle \end{bmatrix} = \int dn_a \; A^a_+(n)$$

$$F^{ab} = \partial_a A^b - \partial_b A^a + i[A^a, A^b]$$

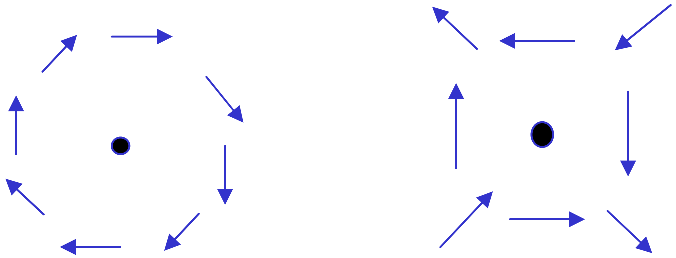
- Second Chern number:

$$8\pi^2 c_2 = Tr \oint_{S^4} d\sigma_{abcd} \; F^{ab} F^{cd} = 1$$

Real space topological defects

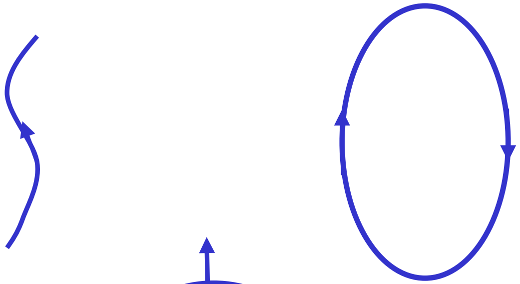
- Vortex and antivortex:

  2D superfluid



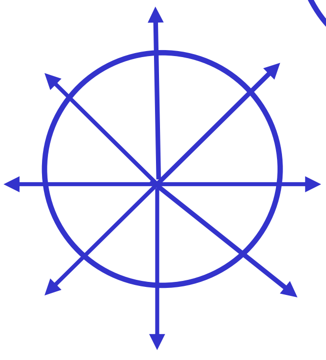
- Vortex string and loop:

  3D superconductor



- 3 D monopoles:

  cosmic cooling



New physics: Generation of topological singularities in the momentum space at quantum phase transitions.

Dynamic generation of spin-orbit coupling

- Spin-orbit coupling originates from the Dirac equation.

  Relativistic effect, single body effect.

  Rashba, Dresselhaus, Luttinger Hamiltonian

  Application in spintronics to control spin through electric fields

  S. Murakami, N. Nagaosa, SCZ, science 301,1348 (2003)

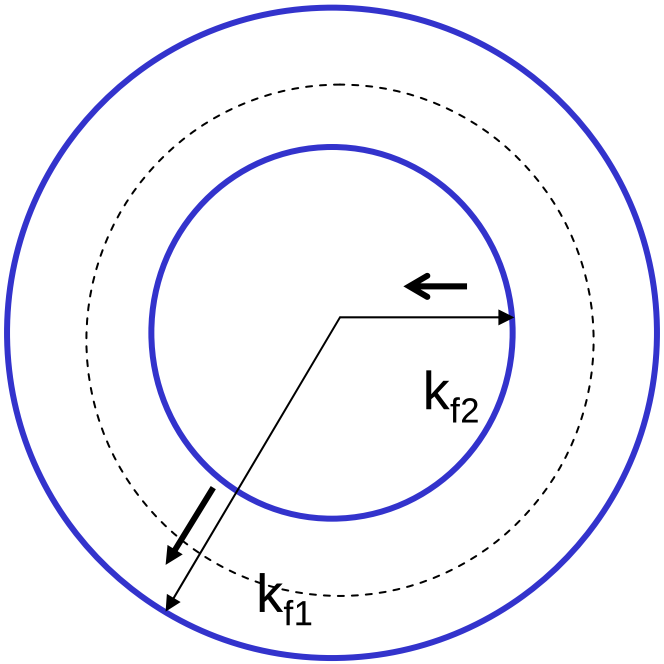
- Q: Can spin-orbit coupling as an order parameter be generated dynamically in a non-relativistic, strongly interacting system through phase transitions, like ferromagnetism?

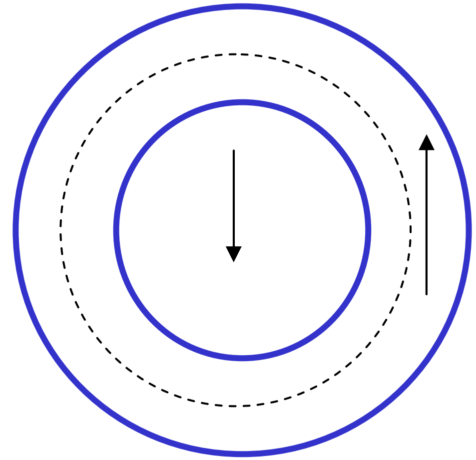
- A: Yes, through Landau-Pomeranchuk Fermi surface instability in the high angular momentum channel.

C. J. Wu and SCZ, Phys. Rev. Lett. **93**, 36403 (2004).

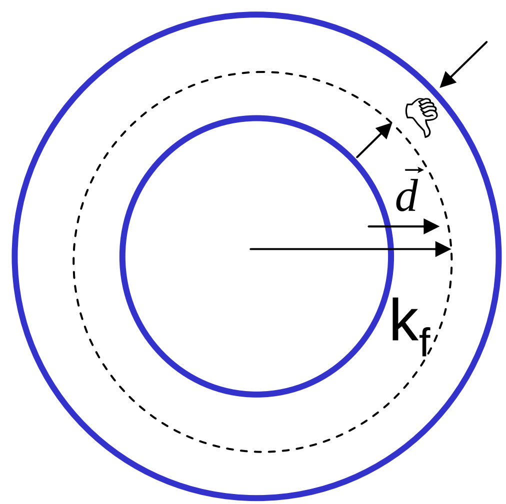
Dynamic generation of spin-orbit coupling

- ♌📭phase with spin-orbit coupling dynamically generated.





Ferromagnetic state



He-3 B phase

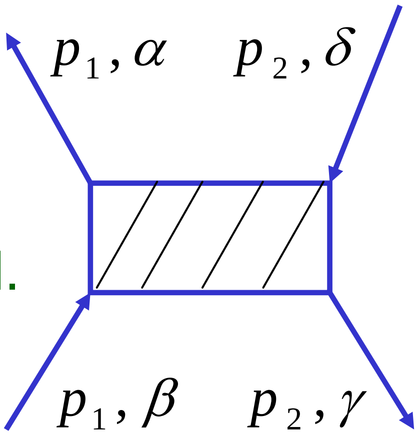
- Landau-Pomeranchuk instability in the channel F_1^a .
- Broken relative spin-orbit symmetry.

Landau-Pomeranchuk (L-P) instability (spin ½ Fermi liquid)

I. I. Pomeranchuk, Sov. Phys. JETP 8, 361(1959)

- Landau interaction functions:

$$f_{\alpha\beta,\gamma\delta}(\hat{p}_1,\hat{p}_2) = f^s(\hat{p}_1,\hat{p}_2) + f^a(\hat{p}_1,\hat{p}_2)\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\gamma\delta}$$



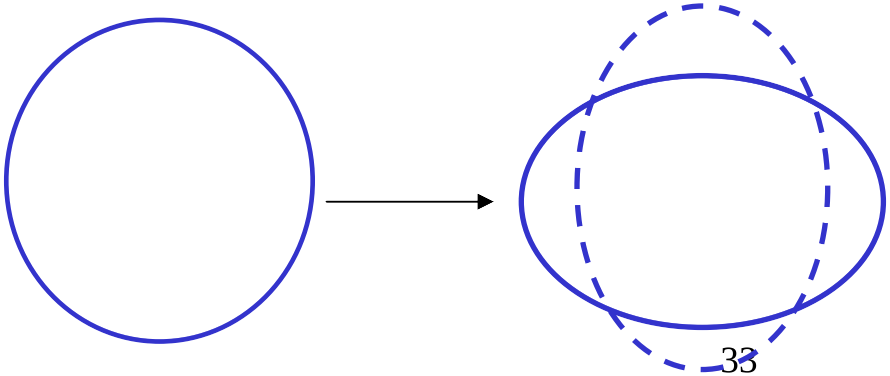
- Landau parameters and L-P instability in channel l.

$$F_l^{s,a} = N_f f_l^{s,a}, \quad F_l^{s,a} < -(2l+1)$$

- $l = 0$: phase separation: $F_0^s < -1$, ferromagnetism: $F_0^a < -1$

- $l = 2$: nematic Fermi liquid $F_2^s < -2(2D)$.

Fermi surface anisotropic distortions.



V. Oganesyan, S. Kivelson and E.Fradkin, PRB 64,195109(2001)

Model Hamiltonian and the mean field decoupling

- L-P instability in the F_1^a channel

$$H_{\text{int}} = \frac{1}{2}\int d^3\vec{r}\int d^3\vec{r}\,' f_1^a(\vec{r}-\vec{r}\,')\hat{Q}^{\mu a}(\vec{r})\hat{Q}^{\mu a}(\vec{r}\,')$$

$$\hat{Q}^{\mu\alpha}(\vec{r}) = \psi_\alpha^+(\vec{r})\sigma_{\alpha\beta}^\mu(-i\hat{\nabla}^a)\psi_\beta(\vec{r}), \quad \hat{\nabla}^a = \nabla^a / |\nabla|$$

- p-wave Cooper pairs in He-3: $\Delta^{\mu a}(\vec{r}) = \psi^+(\vec{r})(i\sigma_2\sigma^\mu)(-i\hat{\nabla}^a)\psi^+(\vec{r})$

- Order parameters form 3*3 real matrices:

$$n^{\mu a}(\vec{r}) = -\int d^3\vec{r}\,' f_1^a(\vec{r}-\vec{r}\,')\langle Q^{\mu a}(\vec{r})\rangle$$

- Mean field decoupling :

$$H_{MF} = \int d^3\vec{r}\,\psi_\alpha^+(\vec{r})(\varepsilon(\vec{\nabla}) - n^{\mu a}\sigma^\mu(-i\hat{\nabla}^a) - \mu)\psi(\vec{r})$$

Ginzburg-Landau free energy

- Symmetry constraints from $SO_L(3)\otimes SO_s(3)$, *P* and *T*

$$n^{\mu a} \rightarrow R_{s,\mu\nu}\, n^{\nu b} R^{-1}_{L,ba} \quad P\, n^{\mu a}\, P^{-1} = -n^{\mu a} \quad T\, n^{\mu a}\, T^{-1} = n^{\mu a}$$

$$F(n) = F(0) + A\, tr[n^T n] + B_1 (tr[n^T n])^2 + B_2 tr[(n^T n)^2]$$

- Phase structures at $A = \dfrac{1 + F_1^a/3}{2N_f\,|F_1^a|} < 0$

$$n^{\mu a} = \bar{n} \begin{Bmatrix} \hat{d}_\mu \hat{e}_a, & \alpha - phase \quad at \;\; B_2 < 0 \\ D_{\mu a}, & \beta - phase \quad at \;\; B_2 > 0 \end{Bmatrix}$$

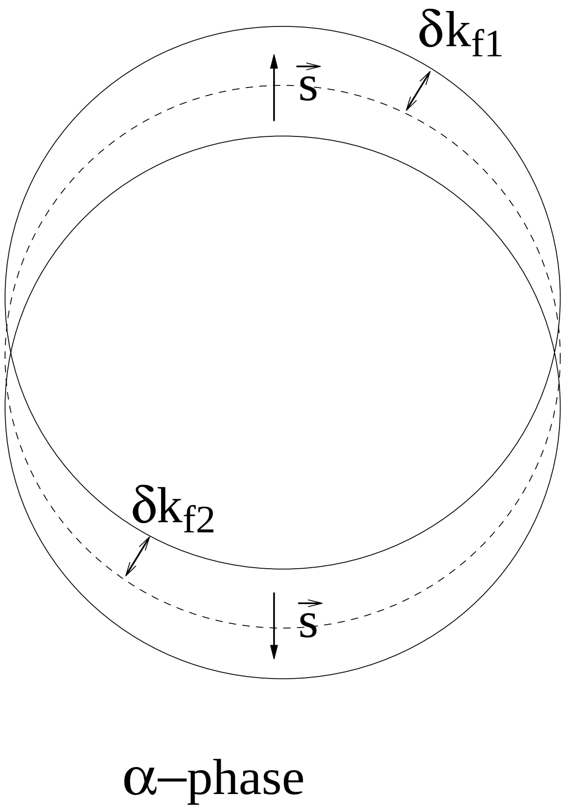
$D_{\mu a}$ is any SO(3) matrix, $\hat{d}_\mu$ and $\hat{e}_a$ are two unit vectors in the spin and orbit spaces, respectively.

Anisotropic ♋-phase (nematic Fermi liquid)

- Anisotropic Fermi surface distortions with

$$H_{MF} = \sum \psi^{+}(k)(\varepsilon(k) - \mu - \bar{n}\sigma_z \cdot \hat{k}_z)\psi(k)$$

$$n^{\mu a} = \bar{n}\,\delta_{\mu z}\,\delta_{az}$$



- Dispersion relation:

$$\xi^{\alpha}(k)_{1,2} = \varepsilon(k) - \mu \mp \bar{n}\cos\theta$$

- Remaining symmetry $SO(2)_L \otimes SO(2)_S$

spin-split state by J. E. Hirsch, PRB 41, 6820 (1988)

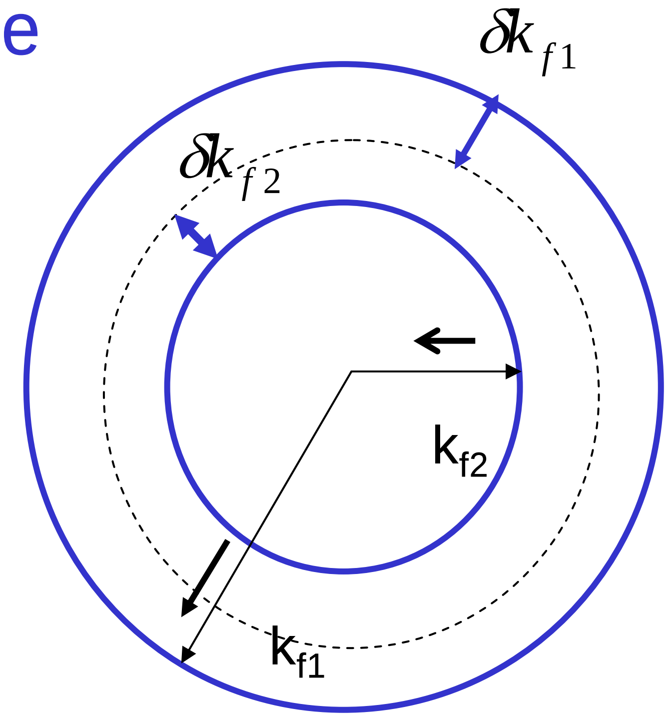
Isotropic β-phase

- Ansatz $n^{\mu a} = \bar{n}\,\delta_{\mu a},\ \bar{n} = \langle \psi^{+}\vec{\sigma}.(-i\vec{\nabla})\psi \rangle$

$$H_{MF} = \sum \psi^{+}(k)(\varepsilon(k) - \mu - \bar{n}\,\vec{\sigma}\cdot\hat{k})\psi(k)$$

$$\xi^{\beta}(k)_{1,2} = \varepsilon(k) - \mu \pm \bar{n}$$

- Helicity bands structure $\vec{\sigma}\cdot\hat{k}$:



- Total angular momentum remains conserved,

$$SO_{L+S}(3), \qquad \vec{J} = \vec{L} + \vec{S}$$

$\beta-$phase

- Generally, $n_{\mu a} = \bar{n}\, D_{\mu a},\qquad J_a = L_a + S_\mu D_{\mu a}$

- Broken relative spin-orbit symmetry.

F_2^V channel L-P instability in spin 3/2 systems

- L-P instability in the F_2^V channel in spin 3/2 system at $F_2^V < -5$

$$H_{\text{int}} = \frac{1}{2}\int d^3\vec{r}\int d^3\vec{r}' f_2^V(\vec{r}-\vec{r}')\hat{Q}^{\mu a}(\vec{r})\hat{Q}^{\mu a}(\vec{r}')$$

$$\hat{Q}^{\mu\alpha}(\vec{r}) = \psi_\alpha^+(\vec{r})\Gamma_{\alpha\beta}^\mu(\hat{d}^a(\vec{\nabla}))\psi_\beta(\vec{r})$$

$$d^1 = -\sqrt{3}\hat{\nabla}_x\hat{\nabla}_y, d^2 = -\sqrt{3}\hat{\nabla}_z\hat{\nabla}_x, d^3 = -\sqrt{3}\hat{\nabla}_x\hat{\nabla}_y,$$

$$d^4 = -\tfrac{\sqrt{3}}{2}(\hat{\nabla}_x^2 - \hat{\nabla}_y^2), d^5 = -\tfrac{1}{2}(2\hat{\nabla}_z^2 - \hat{\nabla}_x^2 - \hat{\nabla}_y^2)$$

- Order parameter matrices:

$$n^{\mu a}(\vec{r}) = -\int d^3\vec{r}' f_2^V(\vec{r}-\vec{r}')\langle Q^{\mu a}(\vec{r})\rangle$$

- Mean field decoupling:

$$H_{MF} = \int d^3\vec{r}\,\psi_\alpha^+(\vec{r})(\varepsilon(\vec{\nabla}) - n^{\mu a}\Gamma^\mu d(\hat{\nabla}^a) - \mu)\psi(\vec{r})$$

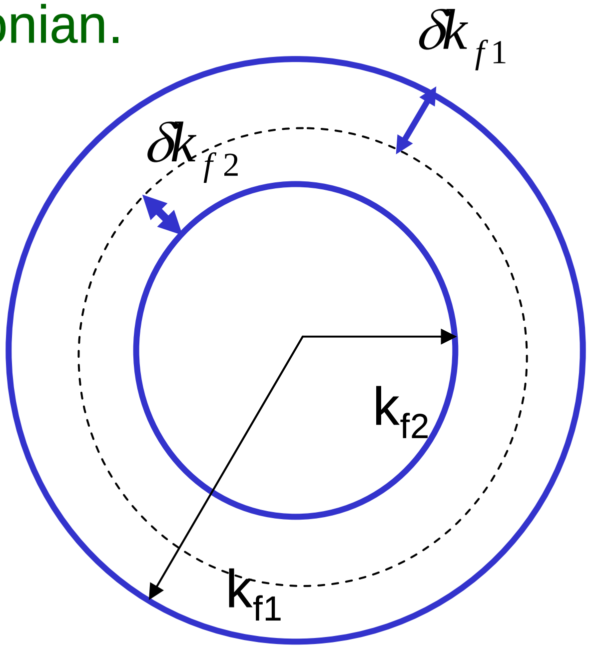
Ω-phase in spin 3/2 systems

- Dynamic generation of Luttinger-like Hamiltonian.

$$n^{\mu a} = \bar{n}\,\delta_{\mu a},\ \bar{n} = \langle | \psi^{+} \Gamma^{a} d^{a} (-i\vec{\nabla}) \psi \,| \rangle$$

$$H_{MF} = \sum \psi^{+}(k)(\varepsilon(k) - \mu - \bar{n}\Gamma^{a} \cdot \hat{d}^{a}(k))\psi(k)$$
$$= \sum \psi^{+}(k)\{\varepsilon(k) - \mu - \bar{n}(\vec{F} \cdot \vec{k})^{2} + \mathrm{const}\}\psi(k)$$

Hamiltonian for the valance band semiconductor.



- Helicity bands structure. $\Gamma^{a} \cdot d^{a}(\hat{k})$

spin nematics direction along $d^{a}(\hat{k})$

doubly degenerate (Kramers) $\xi^{\beta}(k) = \varepsilon(k) - \mu \pm \bar{n}$

- Only relative spin-orbit symmetry is broken.

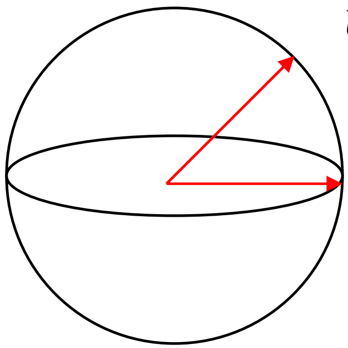
P,T and total angular momentum are conserved.

Emergent non-abelian gauge structure (I)

$$H_{MF} = \sum \psi^+(k)\{\varepsilon(k) - \mu - \bar{n}(\vec{F}\cdot\vec{k})^2\}\psi(k)$$

Diagonalize the first term with a *local* unitary transformation

$$U(k)\vec{k}\cdot\vec{F}U^+(k) = kF_z, U(\vec{k}) = e^{i\theta F_y}e^{i\varphi F_z}$$



Helicity basis $\lambda = \hat{k}\cdot\vec{F}$

$$H'(k) = U(\vec{k})H(k)U^+(\vec{k}) = \varepsilon(k) - \bar{n}(F_z^2 - \tfrac{5}{4}) + \text{const}$$

$$\varepsilon(k) + \begin{pmatrix} -\bar{n} & & & \\ & \bar{n} & & \\ & & \bar{n} & \\ & & & -\bar{n} \end{pmatrix} \quad \begin{matrix} \lambda = \frac{3}{2} : \text{large FS} \\ \lambda = \frac{1}{2} : \text{small FS} \\ \lambda = -\frac{1}{2} : \text{small FS} \\ \lambda = -\frac{3}{2} : \text{large FS} \end{matrix}$$

$$x_i = i\hbar\frac{\partial}{\partial k_i} - A_i$$

$$A_i = -iU(\vec{k})\frac{\partial}{\partial k_i}U^+(\vec{k}) : \text{gauge field in } k!$$

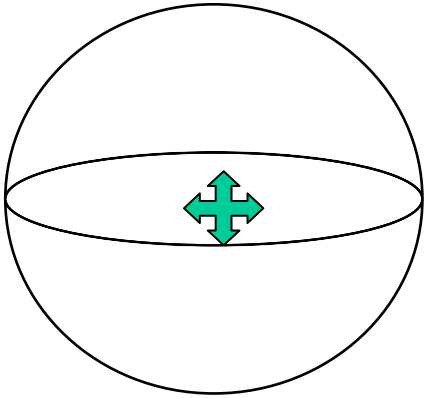
Emergent non-abelian gauge structure (II)

- Projection into large and small FS:
  only retain the intra-band matrix elements

$$A_i dk_i = \begin{pmatrix} -\frac{3}{2}\cos\theta d\varphi & \frac{\sqrt{3}}{2}(\sin\theta d\varphi + id\theta) & & \\ \frac{\sqrt{3}}{2}(\sin\theta d\varphi - id\theta) & -\frac{1}{2}\cos\theta d\varphi & \sin\theta d\varphi + id\theta & \\ & \sin\theta d\varphi - id\theta & \frac{1}{2}\cos\theta d\varphi & \frac{\sqrt{3}}{2}(\sin\theta d\varphi + id\theta) \\ & & \frac{\sqrt{3}}{2}(\sin\theta d\varphi - id\theta) & \frac{3}{2}\cos\theta d\varphi \end{pmatrix} \begin{matrix} \lambda = \frac{3}{2} \\ \lambda = \frac{1}{2} \\ \lambda = -\frac{1}{2} \\ \lambda = -\frac{3}{2} \end{matrix}$$

- Dirac monopole in the k-space.

S. Murakami, N. Nagaosa, SCZ, science 301,1348 (2003); cond-mat/0310005.



(Dirac monopole)

$$F_{ij} = \varepsilon_{ijk} \lambda \frac{k_k}{k^3}$$

Auxiliary Field Quantum Monte-Carlo method

Probability: positive number ⟷ Auxiliary field QMC ⟷ Fermions: Grassmann number

- Hubbard-Stratonovich (H-S) transformation.
- Integrate out fermions.
- Fermion functional determinants (FFD) as statistical weights.

  R. Blankenbecler et al, PRD 24, 2278(1981).

- Major difficulty: FFD is not positive definite generally.

→ **the sign problem.**

Factorization of the fermion functional determinant

- Negative U Hubbard model as an example.

$$H = -t\sum_{\langle ij\rangle}\{c^{+}_{i\sigma}c_{j\sigma} + h.c\} - \mu\sum_i n(i) + U\sum_i (n_\uparrow(i) - \frac{1}{2})(n_\downarrow(i) - \frac{1}{2})$$

$$Z = \int dn \exp\{ -\frac{|U|}{2}\int_0^\beta d\tau \sum_i (n(i,\tau)-1)^2 \det(I+B)\}$$

$$I + B = I + \Im \exp\{ -\int_0^\beta d\tau\; H_K + H_I(\tau)\}$$

$$H_I(\tau) = U\sum_{i\sigma} c^{+}_{i,\sigma}(\tau)c_{i,\sigma}(\tau)n(i,\tau)$$

- Factorize FFD into two identical parts.

$$\det(I + B) = \det(I + B_\uparrow)\det(I + B_\downarrow) \geq 0$$

- Other examples:

positive U Hubbard mode at half-filling;

anisotropic SU(4) mode: F. F. Assaad et al (2003).

Time reversal invariant decomposition

- Theorem: If there exists a anti-unitary transformation T

$$T^2 = -1, \qquad T(I+B)T^{-1} = (I+B)$$

for any H-S field configuration, then $\det(I+B) \geq 0$

Proof:

- If ● is an eigenvalue of (I+B) with eigenstate ✿↗☉📭 then ●* is also an eigenvalue with eigenstate T|↗☉.

- < ↗✿❄✿↗☉▣💾❄📄 ↗✿❄↗☉▣ 📫 💾↗✿❄✿↗☉▣📁✍

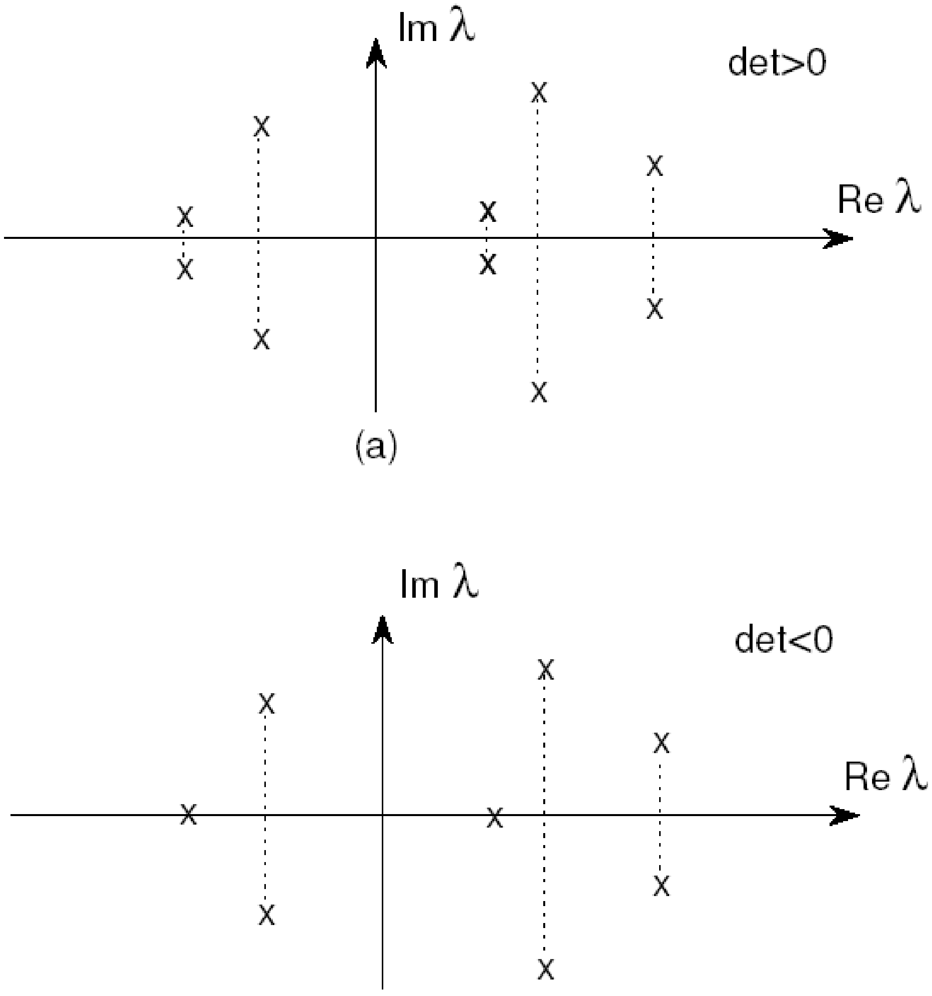
If ● is real, it is double degenerate. If ● is complex, it appears in pairs (●📭 ●*).

- $$\det(I + B) = (\lambda_1 \lambda_1^*)(\lambda_2 \lambda_2^*) \cdots (\lambda_n \lambda_n^*) \geq 0$$ QED.

- **T may be or not be the physical time reversal operator.**

C. J. Wu and SCZ, cond-mat/0407272.

Distribution of eigenvalues



(a)

(b)

Application in spin 3/2 system

- Another equivalent formulation:

$$H = \sum_{\langle ij\rangle,\sigma} -t\{c^+_{i,\alpha} c_{j,\alpha} + h.c.\} - \mu \sum_i c^+_{i,\alpha} c_{i,\alpha} \qquad V = -\frac{3U_0 + 5U_2}{16},$$

$$- \sum_{i,1\le a\le 5} \{V(n(i)-2)^2 + W n_a^2(i)\} \qquad W = \frac{U_2 - U_0}{4}$$

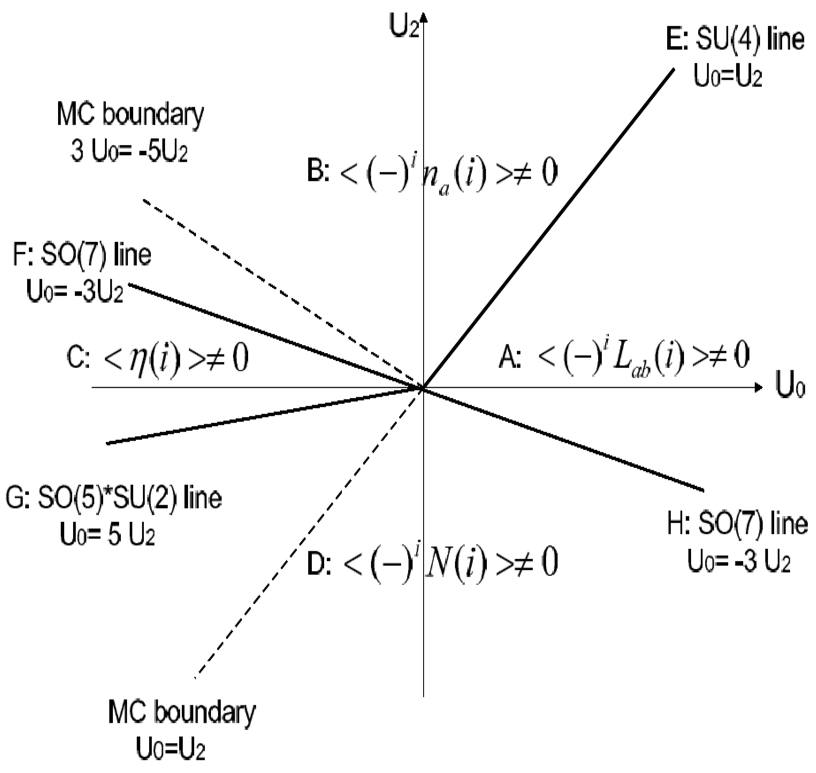
- Decompose in the density, vector channel (V,W>0):

$$H_I(\tau) = -\sum_{i,\sigma} V\{c^+_{i,\alpha}(\tau) c_{i,\alpha}(\tau)(n(i,\tau)-2)$$

$$+W\sum_i c^+_{i,\alpha}(\tau)\frac{\Gamma^a}{2} c_{i,\beta}(\tau) n^a(i,\tau)$$

$$I + B = I + \Im\exp\{-\int_0^\beta d\tau\, H_K + H_I(\tau)\}$$

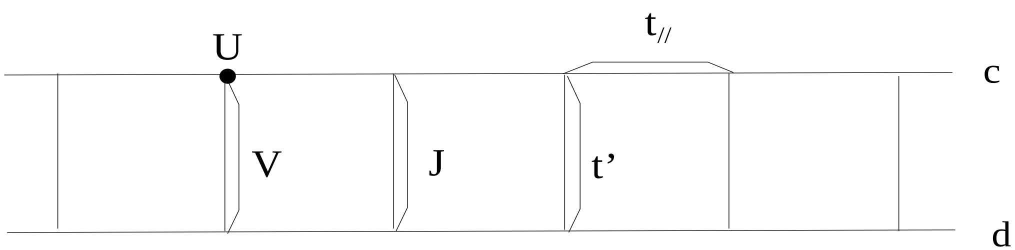
- Reliable determination of quantum phases.



The spin ½ bilayer system

$$H = -t_{//}\sum_{\langle ij\rangle}\{c^{+}_{i\sigma}c_{j\sigma} + d^{+}_{i\sigma}d_{j\sigma} + h.c\} - t_{\perp}\sum_{i}\{c^{+}_{i\sigma}d_{j\sigma} + h.c\} - \mu\sum_{i} n(i)$$

$$+ J\sum_{\langle ij\rangle}\vec{S}_{ic}\cdot\vec{S}_{id} + U\sum_{i}(n_{i,\uparrow,c} - \frac{1}{2})(n_{i,\downarrow,c} - \frac{1}{2}) + (c \to d) + V\sum_{i}(n_{i,c} - 1)(n_{i,d} - 1)$$



D. Scalapino, SCZ, W.Hanke, PRB 58, 443 (1998)

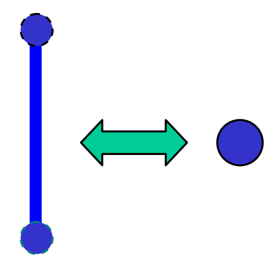
- Map to an anisotropic spin 3/2 system.
- n_1 bond strength, n_5 bond current, $n_{2\text{-}4}$ Neel order.

$$c_{\uparrow}, c_{\downarrow}, \qquad c_{\frac{3}{2}}, c_{\frac{1}{2}},$$

$$d_{\uparrow}, d_{\downarrow} \qquad c_{-\frac{1}{2}}, c_{-\frac{3}{2}}$$

$$n(i) = c^{+}_{i\sigma}c_{i\sigma} + d^{+}_{i\sigma}d_{i\sigma}, \qquad n_1(i) = -i(d^{+}_{i\sigma}c_{i\sigma} - h.c.)/2$$

$$n_5(i) = (d^{+}_{i\sigma}c_{i\sigma} + h.c.)/2, \; n_{2,3,4}(i) = c^{+}_{i}\frac{\vec{\sigma}}{2}c_{i} - d^{+}_{i}\frac{\vec{\sigma}}{2}d_{i},$$

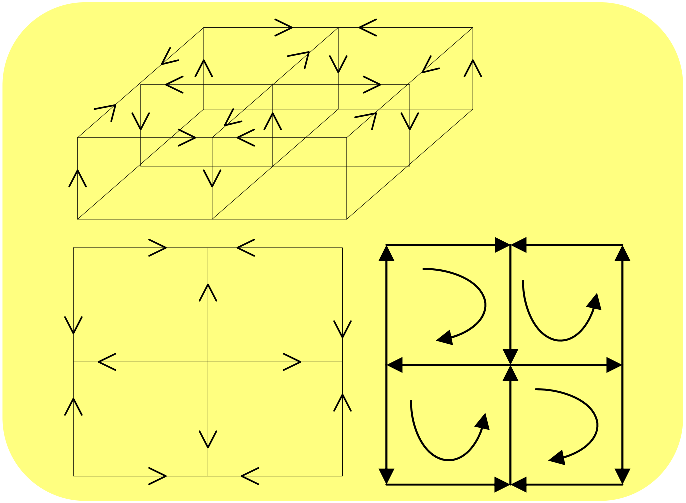


S. Capponi, C. J. Wu and SCZ, cond-mat/0403585

Current carrying Ground state

- Staggered inter-layer current.

  D-density wave phase in High T_c.

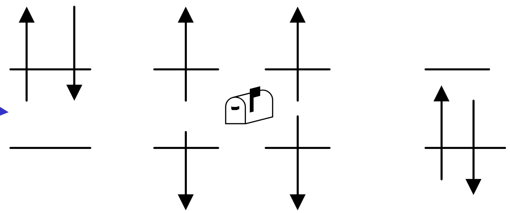


top view d-density wave

- Pseudo-spin SU(2) algebra:

$$n_5(i),\quad n_1(i),\quad Q(i)=\tfrac{1}{2}(c^{+}_{i\sigma}c_{i\sigma}-d^{+}_{i\sigma}d_{i\sigma})$$

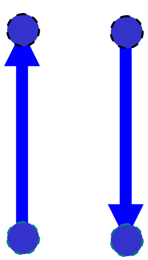
- Low energy rung states as spin-1 representation. (Q=1,0,-1).



a: U b: V-3/4J c: U

$$\Delta U = U - V + \tfrac{3}{4} J$$

- Rung current states as eigenstates of n_1.



$$|\,\mathrm{up,down}\rangle = \frac{1}{2}(|\,a\rangle - |\,c\rangle) \pm \frac{1}{\sqrt{2}}|\,b\rangle$$

$$n_1\,|\,\mathrm{up,down}\rangle = \pm\,|\,\mathrm{up,down}\rangle$$

S. Capponi, C. Wu and SCZ, cond-mat/0403585

s at T=0K

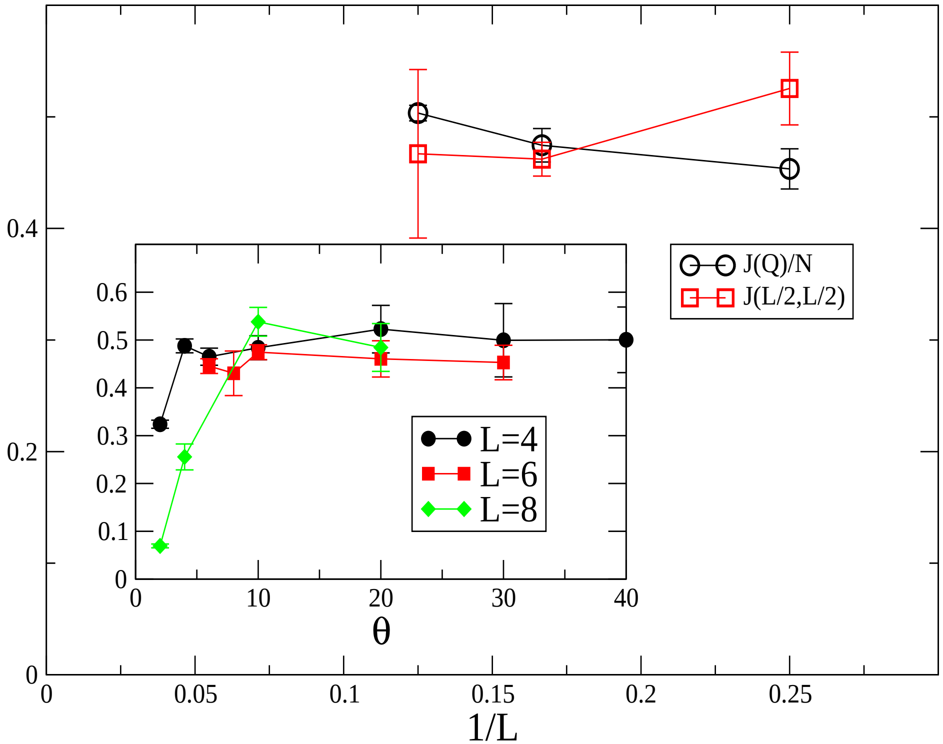


Fig. Parameters are t=1, $t_\perp$=0.1, U=0, V=0.5, J=2.0 and half-filling. Scaling of J(Q)/N and J(L/2,L/2) v.s. 1/L showing almost no finite-size effect and proving long-range order in the thermodynamic limit.

- T=time reversal operation* flipping two layers
- Anisotropic pseudo-spin 1 model

$\Delta U > 0$ favors the easy plane of CDW and current,

$t_\perp$ further favors the currents phase.

$$H_{ex} = J_p \sum_{\langle ij \rangle} \{ n_5(i) n_5(j) + n_1(i) n_1(j) + Q(i) Q(j) \}$$

$$+ \sum_i -2t_\perp n_5(i) + \Delta U (Q^2(i) - \frac{1}{2})$$

Conclusions

- Generic particle-hole channel SO(5) symmetry in spin 3/2 systems.
- SO(5) Landau-Fermi liquid theory and quintet BCS-pairing.
- Quantum phases of spin 3/2 Hubbard model.
- Non-Abelian holonomy of SO(5) spinors.
- Emergent non-Abelian gauge structure from Fermi liquid theory: dynamic generation of the spin-orbit interaction.
- New QMC algorithm without the sign problem: reliable determination of quantum phases.