Quantum Field Theory

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Chapter 1 Introduction

Necesscity of field theory in relativistic system

Schrodinger equation $\Rightarrow$ conservation of particle number.

$H\psi = i\hbar\frac{\partial\psi}{\partial t} \qquad \Rightarrow \qquad \frac{d}{dt}\int d^3x\psi^\dagger\psi = 0 \rightarrow \int d^3x(\psi^\dagger\psi)$ indep of time

If H is hermitian, $H = H^\dagger$. Then number of particles is conserved and no particle creation or annihilation.

Canonical commutation relation gives uncertainty relation,

$$[x, p] = -i\hbar, \qquad \Rightarrow \qquad \triangle x\triangle p \geqslant \hbar$$

From

$$p^2c^2 + m^2c^4 = E^2$$

get

$$\triangle E = \frac{p\triangle p}{E}c^2 \geqslant \frac{p\hbar c^2}{E\triangle x} \quad or \quad \triangle x \geqslant \frac{pc}{E}(\frac{\hbar c}{\triangle E})$$

To avoid new particle creation we require $\triangle E \leqslant mc^2$. Then we get a lower bound on $\triangle x$

$$\triangle x \geqslant \frac{pc}{E}\frac{\hbar}{mc} = (\frac{v}{c})(\frac{\hbar}{mc})$$

For relativistic particle $\frac{v}{c} \approx 1$, then

$$\triangle x \geqslant (\frac{\hbar}{mc}) \qquad \text{Compton wavelength}$$

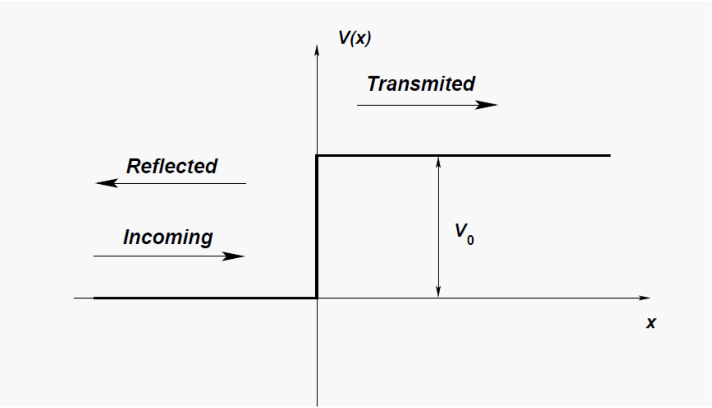
$\Rightarrow$ Particle can not be confined to a interval smaller than its Compton wavelength

Klein paradox

To illustrate this feature we will study Klein's paradox in the context of the Klein-Gordon equation given by

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 - m^2\right)\psi(x,t) = 0$$

- Let us consider a square potential with height $V_0 > 0$ as shown in the figure,



A solution to the wave equation in regions I and II is given by

$$\psi_I(x,t) = e^{-iEt-ip_1x} + R\, e^{-iEt+ip_1x}$$

$$\psi_{II}(x,t) = Te^{-iEt-ip_2x}$$

where

$$p_1 = \sqrt{E^2 - m^2}, \qquad p_2 = \sqrt{(E - V_0)^2 - m^2}$$

The constants R and T are computed by matching the two solutions across the boundary $x = 0$. The conditions $\psi_I(t,0) = \psi_{II}(t,0)$ and $\partial_x\psi_I(t,0) = \partial_x\psi_{II}(t,0)$ give

$$1 + R = T, \qquad (1 - R)\, p_1 = Tp_2$$

Solve for R and T

$$T = \frac{2p_1}{p_1 + p_2}, \qquad R = \frac{p_1 - p_2}{p_1 + p_2}$$

- if $E - m > V_0$ both p_1 and p_2 are real and there are both transmitted and reflected wave.
- If $E - m < V_0$ and $E - m < V_0 - 2m$, then p_2 is imaginary, we get a reflected wave, transmitted wave being exponentially damped within a distance of a Compton wavelength inside the barrier and there is total reflection.
- when $V_0 > 2m$ and $V_0 - 2m < E - m < V_0$ then both p_1 and p_2 are real and there are both reflected and transmitted waves . This implies that there is a nonvanishing probability of finding the particle at any point across the barrier with negative kinetic energy $(E - m - V_0 < 0)$!
  This weird result is known as **Klein's paradox**. This result can only be understood in terms of particle creation at sudden potential step.

Gauge Theory–Quantum Field Theory with Local Symmetry

Gauge principle

All fundamental Interactions are descibed in terms of gauge theories;

1. Strong Interaction-QCD;
   gauge theory based on SU(3) symmetry
2. Electromagnetic and Weak interaction-
   gauge theory based on $SU(2)\times\ U(1)$ symmetry
3. Gravitational interaction-
   Einstein's theory-gauge theory of local coordinate transformation.

Natural unit

$$\hbar = c = 1$$

In MKS units

$$\hbar = 1.055 \times 10^{-34} J\,\text{sec}, \quad c = 2.99 \times 10^{8} m/sec$$

In this unit, at the end of the calculation one puts back factors of $\hbar$ and c depending on the physical quantities in the problem.
For example, the quantity m_e can have following different meanings depending on the contexts;

1. Reciprocal length
$$m_e = \frac{1}{\frac{\hbar}{m_e c}} = \frac{1}{3.86 \times 10^{-11} cm}$$
2. Reciprocal time
$$m_e = \frac{1}{\frac{\hbar}{m_e c^2}} = \frac{1}{1.29 \times 10^{-21} sec}$$
3. Energy
$$m_e = m_e c^2 = 0.511 Mev$$
4. Momentum
$$m_e = m_e c = 0.511 Mev/c$$

The following conversion relations

$$\hbar = 6.58 \times 10^{-22} Mev - sec \qquad \hbar c = 1.973 \times 10^{-11} Mev - cm$$

are quite useful in getting the physical quantities in the right units.
Example: Thomson cross section

$$\sigma = \frac{8\pi\alpha^2}{3m_e^2} = \frac{8\pi\alpha^2(\hbar c)^2}{3m_e^2 c^4} = (\frac{1}{137})^2 \times \frac{(1.973 \times 10^{-11} Mev - cm)^2}{(0.5 Mev)^2} \times (\frac{8\pi}{3}) \simeq 6.95 \times 10^{-25} cm^2$$

Useful convertion factor

$$1ev = 1.6 \times 10^{-19} J, \qquad 1Gev = 1.6 \times 10^{-10} J \quad or \quad 1J = \frac{1}{1.6} \times 10^{10} Gev$$

Review of Special Relativity

Basic principles of special relativity :

1. The speed of light : same in all inertial frames.
2. Physical laws: same forms in all inertial frames.

Lorentz transformation–relate coordinates in different inertial frame

$$x' = \frac{x - vt}{\sqrt{1 - v^2}} \quad y' = y, \qquad z' = z, \qquad t' = \frac{t - vx}{\sqrt{1 - v^2}}$$

$\Rightarrow$

$$t^2 - x^2 - y^2 - z^2 = t'^2 - x'^2 - y'^2 - z'^2$$

Proper time $\tau^2 = t^2 - \overrightarrow{r}^2$ invariant under Lorentz transfomation.
Particle moves from $\overrightarrow{r_1}(t_1)$ to $\overrightarrow{r_2}(t_2)$ .The speed is

$$|\overrightarrow{v}| = \frac{1}{|t_2 - t_1|}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

For $|\overrightarrow{v}| = 1$,

$$(t_1 - t_2)^2 = |\overrightarrow{r_1} - \overrightarrow{r_2}|^2$$

this is invariant under Lorentz transformation $\quad \Rightarrow \quad$ speed of light same in all inertial frames.

Another form of the Lorentz transformation

$$x' = \cosh\omega\, x - \sinh\omega\; t, \quad y' = y, \quad z' = z, \quad t' = \sinh\omega\, x - \cosh\omega\, t$$

where

$$\tanh\omega = v$$

For infinitesmal interval (dt, dx, dy, dz), proper time is

$$(d\tau)^2 = (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

Minkowski space,

$$x^{\mu} = (t, x, y, z) = (x^0, x^1, x^2, x^3), \qquad 4 - vector$$

Lorentz invariant product can be written as

$$x^2 = (x_0)^2 - (x_1)^2 - (x_2)^2 - (x_3)^2 = x^{\mu} x^{\nu} g_{\mu\nu}$$

where

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Define another 4-vector

$$x_{\mu} = g_{\mu\nu} x^{\nu} = (t, -x^1, -x^2, -x^3) = (t, -\overrightarrow{r})$$

so that

$$x^2 = x^{\mu} x_{\mu}$$

For infinitesmal coordinates

$$(dx)^2 = (dx^{\mu})(dx_{\mu}) = dx^{\mu} dx^{\nu} g_{\mu\nu} = (dx^0)^2 - (d\overrightarrow{x})^2$$

Write the Lorentz transformation as

$$x^{\mu} \rightarrow x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$$

For example for Lorentz transformation in the $x-$direction, we have

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \frac{1}{\sqrt{1-\beta^2}} & \frac{-\beta}{\sqrt{1-\beta^2}} & 0 & 0 \\ \frac{-\beta}{\sqrt{1-\beta^2}} & \frac{1}{\sqrt{1-\beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Write

$$x'^2 = x'^{\mu} x'^{\nu} g_{\mu\nu} = \Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta}\, g_{\mu\nu} x^{\alpha} x^{\beta}$$

then $x^2 = x'^2$ implies

$$\Lambda^{\mu}_{\alpha} \Lambda^{\nu}_{\beta}\, g_{\mu\nu} = g_{\alpha\beta}$$

and is called **pseudo-orthogonality** relation.

Energy and Momentum

Start from

$$dx^{\mu} = (dx^0, dx^1, dx^2, dx^3)$$

Proper time is Lorentz invariant and has the form,

$$(d\tau)^2 = dx^{\mu}dx_{\mu} = (dt)^2 - (\frac{d\overrightarrow{x}}{dt})^2(dt)^2 = (1-\overrightarrow{v}^2)(dt)^2$$

$4-velocity$,

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} = (\frac{dx^0}{d\tau}, \frac{d\overrightarrow{x}}{d\tau})$$

there is a constraint

$$u^{\mu}u_{\mu} = \frac{dx^{\mu}}{d\tau}\frac{dx_{\mu}}{d\tau} = 1$$

Note that

$$\overrightarrow{u} = \frac{d\overrightarrow{x}}{d\tau} = \frac{d\overrightarrow{x}}{dt}(\frac{dt}{d\tau}) = \frac{1}{\sqrt{1-v^2}}\overrightarrow{v} \approx \overrightarrow{v}, \qquad \text{for } v \ll 1$$

$4-velocity \qquad \Longrightarrow \qquad 4-momentum$

$$p^{\mu} = mu^{\mu} = (\frac{m}{\sqrt{1-v^2}}, \frac{m\overrightarrow{v}}{\sqrt{1-v^2}})$$

For $v \ll 1$,

$$p^0 = \frac{m}{\sqrt{1-v^2}} = m(1 + \frac{1}{2}v^2 + ...) = m + \frac{m}{2}v^2 + ..., \qquad \text{energy}$$

$$\overrightarrow{p} = m\overrightarrow{v}\frac{1}{\sqrt{1-v^2}} = m\overrightarrow{v} + ... \qquad \text{momentum}$$

$$p^\mu = (E, \overrightarrow{p})$$

Note that

$$p^2 = E^2 - \overrightarrow{p}^2 = \frac{m^2}{1-v^2}[1-v^2] = m^2$$

Tensor analysis

Physical laws take the same forms in all inertial frames, if we write them in terms of tensors in Minkowski space.
Basically, tensors are

$$\text{tensors} \sim \text{product of vectors}$$

2 different types of vectors,

$$x'^{\mu} = \Lambda^{\mu}_{\ \nu} x^{\nu}, \qquad x'_{\mu} = \Lambda_{\mu}^{\ \nu} x_{\nu}$$

multiply these vectors to get $2nd$ rank tensors,

$$T'^{\mu\nu} = \Lambda^{\mu}_{\ \alpha} \Lambda^{\nu}_{\ \beta} T^{\alpha\beta}, \qquad T'_{\mu\nu} = \Lambda_{\mu}^{\ \alpha} \Lambda_{\nu}^{\ \beta} T_{\alpha\beta}, \qquad T'^{\mu}_{\nu} = \Lambda^{\mu}_{\ \alpha} \Lambda_{\nu}^{\ \beta} T^{\alpha}_{\beta}$$

In general,

$$T'^{\mu_1 \cdots \mu_n}_{\nu_1 \cdots \nu_m} = \Lambda^{\mu_1}_{\alpha_1} \cdots \Lambda^{\mu_n}_{\alpha_n} \Lambda^{\beta_1}_{\nu_1} \cdots \Lambda^{\beta_m}_{\nu_m} T^{\alpha_1 \cdots \alpha_n}_{\beta_1 \cdots \beta_m}$$

transformation of tensor components is **linear and homogeneous**.

Tensor operations; operation which preserves the tensor property

1. Multiplication by a constant, (cT) has the same tensor properties as T
2. Addition of tensor of same rank
3. Multiplication of two tensors
4. Contraction of tensor indices. For example, $T_{\mu}^{\mu\alpha\beta\gamma}$ is a tensor of rank 3 while $T_{\nu}^{\mu\alpha\beta\gamma}$ is a tensor or rank 5. This follows from the psudo-orthogonality relation
5. Symmetrization or anti-symmetrization of indices. This can be seen as follows. Suppose $T^{\mu\nu}$ is a second rank tensor,

   $$T'^{\mu\nu} = \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}T^{\alpha\beta}$$

   interchanging the indices

   $$T'^{\nu\mu} = \Lambda^{\nu}_{\ \alpha}\Lambda^{\mu}_{\ \beta}T^{\alpha\beta} = \Lambda^{\nu}_{\ \beta}\Lambda^{\mu}_{\ \alpha}T^{\beta\alpha}$$

   Then

   $$T'^{\mu\nu} + T'^{\nu\mu} = \Lambda^{\mu}_{\ \alpha}\Lambda^{\nu}_{\ \beta}\left(T^{\alpha\beta} + T^{\beta\alpha}\right)$$

   symmetric tensor transforms into symmetric tensor. Similarly, the anti-symmetric tensor transforms into antisymmetic one.
6. $g_{\mu\nu}$, and $\varepsilon^{\alpha\beta\gamma\delta}$ have the property

   $$\Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}\,g_{\mu\nu} = g_{\alpha\beta}, \qquad \varepsilon^{\alpha\beta\gamma\delta}\det(\Lambda) = \Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu}\Lambda^{\gamma}_{\rho}\Lambda^{\delta}_{\sigma}\varepsilon^{\mu\nu\rho\sigma}$$

   $g_{\mu\nu}$, and $\varepsilon^{\alpha\beta\gamma\delta}$ transform in the same way as tensors if $\det(\Lambda) = 1$.

Example: $M^{\mu\nu} = x^{\mu}p^{\nu} - x^{\nu}p^{\mu}$, $\qquad F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ $\qquad$ 2nd rank antisymmetric tensor.

Note that if all components of a tensor vanish in one inertial frame they vanish in all inertial frame. Suppose

$$f^{\mu} = ma^{\mu}$$

Define

$$t^{\mu} = f^{\mu} - ma^{\mu}$$

then $t^{\mu} = 0$ in this inertial frame. In another inertial frame,

$$t'^{\mu} = f^{\mu'} - ma'^{\mu} = 0$$

we get

$$f^{\mu'} = ma'^{\mu}$$

Thus physical laws in tensor form are same in all inertial frames .

Action principle: actual trajectory of a partilce minimizes the action

Particle mechanics

A particle moves from x_1 at t_1 to x_2 at t_2. Write the action as

$$S = \int_{t_1}^{t_2} L(x, \dot{x})\, dt \qquad L : \text{Lagrangian}$$

For the least action, make a small change $x(t)$,

$$x(t) \rightarrow x'(t) = x(t) + \delta x(t)$$

with end points fixed

$$i.e. \quad \delta x(t_1) = \delta x(t_2) = 0 \qquad \text{initial conditions}$$

Then

$$\delta S = \int_{t_1}^{t_2} [\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\delta(\dot{x})]\, dt$$

Note that

$$\delta \dot{x} = \dot{x}'(t) - \dot{x}(t) = \frac{d}{dt}[\delta(x)]$$

Integrate by parts and used the initial conditions

$$\delta S = \int_{t_1}^{t_2} [\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial \dot{x}}\frac{d}{dt}(\delta x)]\, dt = \int_{t_1}^{t_2} [\frac{\partial L}{\partial x} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}})]\delta x\, dt$$

Since $\delta x(t)$ is arbitrary, $\delta S = 0$ implies

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \qquad \text{Euler-Lagrange equation}$$

Conjugate momentum is

$$p \equiv \frac{\partial L}{\partial \dot{x}}$$

Hamiltonian is ,

$$H(p, q) = p\dot{x} - L(x, \dot{x})$$

Consider the simple case

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x}$$

Suppose

$$L = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x)$$

then

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right), \qquad \Rightarrow \qquad -\frac{\partial V}{\partial x} = m\frac{d^2x}{dt^2}$$

Hamiltonian

$$H = p\dot{x} - L = \frac{m}{2}(\dot{x})^2 + V(x) \quad \text{where} \quad p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

is just the total energy.

Generalization

$$x(t) \rightarrow q_i(t), \quad i = 1, 2, ..., n$$

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i)\, dt$$

Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad i = 1, 2, ..., n$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad H = \sum_i p_i \dot{q}_i - L$$

Example: harmonic oscillator in 3-dimensions
Lagrangian

$$L = T - V = \frac{m}{2}(\dot{x_1}^2 + \dot{x_2}^2 + \dot{x_3}^2) - \frac{mw^2}{2}(x_1^2 + x_2^2 + x_3^2)$$

and

$$\frac{\partial L}{\partial x_i} = -mw^2 x_i, \qquad \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$$

Euler-Langarange equation

$$m\ddot{x}_i = -mw^2 x_i$$

same as Newton's second law.
Remarks:

- We need action principle for quantization
- In action principle formulation, the discussion of symmetry is simpler
- Can take into account the constraints in the coordinates in terms of Lagrange multiplers

Field Theory

Field theory $\sim$ limiting case where number of degrees of freedom is infinite. $q_i(t) \to \phi(\overrightarrow{x}, t)$.
Action

$$S = \int L(\phi, \partial_\mu \phi)\, d^3x dt \qquad L : \text{Lagrangian density}$$

Variation of action

$$\delta S = \int [\frac{\partial L}{\partial \phi} \delta\phi + \frac{\partial L}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi)]\, dx^4 = \int [\frac{\partial L}{\partial \phi} - \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi)}] \delta\phi\, dx^4$$

Use $\delta(\partial_\mu \phi) = \partial_\mu(\delta\phi)$ and do the integration by part. then $\delta S = 0$ implies

$$\Longrightarrow \frac{\partial L}{\partial \phi} = \partial_\mu (\frac{\partial L}{\partial(\partial_\mu \phi)}) \qquad \text{Euler-Lagrange equation}$$

Conjugate momentum density

$$\pi(\overrightarrow{x}, t) = \frac{\partial L}{\partial(\partial_0 \phi)}$$

and Hamiltonian density

$$H = \pi\dot{\phi} - L$$

Generalization to more than one field

$$\phi(\overrightarrow{x}, t) \to \phi_i(\overrightarrow{x}, t), \qquad i = 1, 2, ..., n$$

Equations of motion are

$$\frac{\partial L}{\partial \phi_i} = \partial_\mu \left(\frac{\partial L}{\partial(\partial_\mu \phi_i)} \right) \quad i = 1, 2, ..., n$$

and conjugate momentum

$$\pi_i(\overrightarrow{x}, t) = \frac{\partial L}{\partial(\partial_0 \phi_i)}$$

Hamiltonian density is

$$H = \sum_i \pi_i \dot{\phi}_i - L$$

Symmetry and Noether's Theorem
Continuous symmetry $\Longrightarrow$ conservation law, e.g. invariance under time translation

$$t \to t + a, \quad \text{a} \quad \text{is arbitrary constant}$$

gives energy conservation. Newton's equation for a force derived from a potential $V(\overrightarrow{x}, t)$ is,

$$m\frac{d^2\overrightarrow{x}}{dt^2} = -\overrightarrow{\nabla} V(\overrightarrow{x}, t)$$

Suppose V($\overrightarrow{x}$,t)=V($\overrightarrow{x}$), then invariant under time translation and

$$m\frac{d\overrightarrow{x}}{dt} \cdot \left(\frac{d^2\overrightarrow{x}}{dt^2}\right) = -\left(\frac{d\overrightarrow{x}}{dt}\right) \cdot \overrightarrow{\nabla} V = -\frac{d}{dt}[V(\overrightarrow{x})]$$

Or

$$\frac{d}{dt}[\frac{1}{2}m(\frac{d\overrightarrow{x}}{dt})^2 + V(\overrightarrow{x})] = 0, \qquad \text{energy conservation}$$

Similarity, invariance under spatial translation

$$\overrightarrow{x} \to \overrightarrow{x} + \overrightarrow{a}$$

gives momentum conservation and invariance under rotations gives angular momentum conservation. Noether's theorem : unified treatment of symmetries in the Lagrangian formalism.
Particle mechanics

Action in classical mech

$$S = \int L(q_i, \dot{q}_i)\, dt$$

Suppose S is invariant under a continuous symmetry transformation,

$$q_i \longrightarrow q_i' = f_i\,(q_j)\,,$$

For infinitesmal change

$$q_i \longrightarrow q_i' \simeq q_i + \delta q_i$$

The change of S

$$\delta S = \int [\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i]\, dt \quad \text{where} \quad \delta \dot{q}_i \longrightarrow \frac{d}{dt}(\delta q_i)$$

Using the equation of motion,

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})$$

we can write δS as

$$\delta S = \int [\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i)]\, dt = \int [\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}\delta q_i)]\, dt$$

Thus $\delta S = 0 \Rightarrow$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}\delta q_i) = 0$$

This can be written as

$$\text{or} \quad \frac{dA}{dt} = 0, \quad A = \frac{\partial L}{\partial \dot{q}_i}\delta q_j$$

A is the conserved charge.

Note if $\delta L \neq 0$ but changes by a total time derivative $\delta L = \frac{d}{dt}K$,we still get the conservation law in the form,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i - K\right) = 0$$

because the action is still invariant. For example, for translation in time, $t \longrightarrow t + \varepsilon$,

$$q\left(t+\varepsilon\right) = q\left(t\right) + \varepsilon\frac{dq}{dt}, \qquad \Longrightarrow \delta q = \varepsilon\frac{dq}{dt}$$

Similarly,

$$\delta L = \frac{dL}{dt}$$

The conservation law is then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i - L\right) = 0$$

Or

$$\frac{dH}{dt} = 0, \qquad \text{with} \qquad H = \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$$

Example: rotational symmetry in 3-dimension
action

$$S = \int L(x_i, \dot{x}_i)\, dt$$

Suppose S is invariant under rotation,

$$x_i \to x_i' = R_{ij} x_j, \qquad RR^T = R^T R = 1 \quad or \quad R_{ij} R_{ik} = \delta_{jk}$$

For infinitesmal rotations

$$R_{ij} = \delta_{ij} + \varepsilon_{ij} \qquad\qquad |\varepsilon_{ij}| \ll 1$$

Orthogonality requires,

$$(\delta_{ij} + \varepsilon_{ij})(\delta_{ik} + \varepsilon_{ik}) = \delta_{jk} \Longrightarrow \varepsilon_{jk} + \varepsilon_{kj} = 0 \quad i, e, \quad \varepsilon_{jk} \quad \text{is} \quad \text{antisymmetric}$$

$$\delta x_i = \varepsilon_{ij} x_j$$

We can compute the conserved charges as

$$J = \frac{\partial L}{\partial \dot{x}} \varepsilon_{ij} x_j = \varepsilon_{ij} p_i x_j$$

If we write $\varepsilon_{ij} = -\varepsilon_{ijk} \theta_k$

$$J = -\theta_k \varepsilon_{ijk} p_i x_j = -\theta_k J_k \qquad\qquad J_k = \varepsilon_{ijk} x_i p_j$$

J_k k-th component of angular momentum.

Field Theory

Start from the action

$$S = \int L(\phi, \partial_\mu \phi)\, d^4x$$

Symmetry transformation,

$$\phi(x) \rightarrow \phi'(x'),$$

which includes the change of coordinates,

$$x^\mu \rightarrow x'^\mu \neq x^\mu$$

Infinitesmal transformation

$$\delta\phi = \phi'\left(x'\right) - \phi\left(x\right), \qquad \delta x'^\mu = x'^\mu - x^\mu$$

need to include the change in the volume element

$$d^4x' = Jd^4x \qquad \text{where} \quad J = \left|\frac{\partial(x_0', x_1', x_2', x_3')}{\partial(x_0, x_1, x_2, x_3)}\right|$$

J :Jacobian for the coordinate transformation. For infinitesmal transformation,

$$J = \left|\frac{\partial x'^\mu}{\partial x^\nu}\right| \approx \left|g^\mu_\nu + \frac{\partial(\delta x^\mu)}{\partial x^\nu}\right| \approx 1 + \partial_\mu(\delta x^\mu)$$

we have used the relation

$$det(1+\varepsilon) \approx 1 + Tr(\varepsilon) \qquad for \quad |\varepsilon| \ll 1$$

Then

$$d^4x' = d^4x(1 + \partial_\mu(\delta x^\mu))$$

change in the action is

$$\delta S = \int [\frac{\partial L}{\partial \phi}\delta\phi + \frac{\partial L}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) + L\partial_\mu(\delta x^\mu)]\, dx^4$$

Define the change of ϕ for fixed x^μ,

$$\bar{\delta}\phi(x) = \phi'(x) - \phi(x) = \phi'(x) - \phi'(x') + \phi'(x') - \phi(x) = -\partial^\mu \phi' \delta x_\mu + \delta\phi$$

$$or \quad \delta\phi = \bar{\delta}\phi + (\partial_\mu \phi)\delta x^\mu$$

Similarly,

$$\delta(\partial_\mu \phi) = \bar{\delta}(\partial_\mu \phi) + \partial_\nu(\partial_\mu \phi)\delta x^\nu$$

Operator $\bar{\delta}$ commutes with the derivative operator ∂_μ,

$$\bar{\delta}(\partial_\mu \phi) = \partial_\mu(\bar{\delta}\phi)$$

Then

$$\delta S=\int[\frac{\partial L}{\partial\phi}(\bar{\delta}\phi+(\partial_\mu\phi)\delta x^\mu)+\frac{\partial L}{\partial(\partial_\mu\phi)}(\bar{\delta}(\partial_\mu\phi)+\partial_\nu(\partial_\mu\phi)\delta x^\nu)+L\partial_\mu(\delta x^\mu)]\,dx^4$$

Use equation of motion

$$\frac{\partial L}{\partial\phi}=\partial^\mu(\frac{\partial L}{\partial(\partial_\mu\phi)})$$

we get

$$\frac{\partial L}{\partial\phi}\bar{\delta}\phi+\frac{\partial L}{\partial(\partial_\mu\phi)})\bar{\delta}(\partial_\mu\phi)=\partial^\mu(\frac{\partial L}{\partial(\partial_\mu\phi)}\bar{\delta}\phi+\frac{\partial L}{\partial(\partial_\mu\phi)}\partial_\mu(\bar{\delta}\phi)=\partial^\mu[\frac{\partial L}{\partial(\partial_\mu\phi)}\bar{\delta}\phi]$$

Combine other terms as

$$\begin{aligned}[\frac{\partial L}{\partial\phi}(\partial_\nu\phi)+\frac{\partial L}{\partial(\partial_\mu\phi)}\partial_\nu(\partial_\mu\phi)]\delta x^\nu+L\partial_\nu(\delta x^\nu) &= (\partial_\nu L)\delta x^\nu+L\partial_\nu(\delta x^\nu)\\ &= \partial_\nu(L\delta x^\nu)\end{aligned}$$

Then

$$\delta S=\int dx^4\partial_\mu[\frac{\partial L}{\partial(\partial_\mu\phi)}\bar{\delta}\phi+L\delta x^\mu]$$

and if $\delta S=0$ under the symmetry ransformation, then

$$\partial^{\mu} J_{\mu} = \partial^{\mu}[\frac{\partial L}{\partial(\partial_{\mu}\phi)}\bar{\delta}\phi + L\delta x^{\mu}] = 0 \qquad \text{current conservation}$$

Simple case: space-time translation

Here the coordinate transformation is,

$$x^{\mu} \to x'^{\mu} = x^{\mu} + a^{\mu} \Longrightarrow \phi'(x+a) = \phi(x)$$

then

$$\bar{\delta}\phi = -a^{\mu}\partial_{\mu}\phi$$

and the conservation laws take the form

$$\partial^{\mu}[\frac{\partial L}{\partial(\partial_{\mu}\phi)}(-a^{\nu}\partial_{\nu}\phi) + La^{\mu}] = -\partial^{\mu}(T_{\mu\nu}a^{\nu}) = 0$$

where

$$T_{\mu\nu} = \frac{\partial L}{\partial(\partial_{\mu}\phi)}\partial_{\nu}\phi - g_{\mu\nu}L \quad \text{energy momentum tensor}$$

In particular,

$$T_{0i} = \frac{\partial L}{\partial(\partial_0\phi)}\partial_i\phi$$

and

$$P_i = \int dx^3 T_{0i} \quad \text{momentum of the fields}$$