Quantum Field Theory, Note 2

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Klein Gordon Equation

Classically,

$$E = \frac{\vec{p}^{\,2}}{2m} + V(\vec{r})$$

Quantization : $E \to i\frac{\partial}{\partial t}$, $\vec{p} \to -i\vec{\nabla}$ and act on ψ

$$i\frac{\partial \psi}{\partial t} = [-\frac{1}{2m}\nabla^2 + V(\vec{r})]\psi \quad \text{Schrodinger equation}$$

x and time t are not on equal footing.
For relativistic case, use

$$E^2 = \vec{p}^2 + m^2, \qquad \Longrightarrow \qquad (-\nabla^2 + m^2)\psi = -\partial_0^2\psi$$

Or

$$(\Box + m^2)\psi = 0, \text{ where } \quad \Box = \partial_0^2 - \nabla^2 = \partial^\mu\partial_\mu = \partial^2$$

This is known as Klein-Gordon equation.

Probablity interpretation

Klein-Gordon equation

$$(\partial_0^2 - \nabla^2 + m^2)\psi = 0$$

complex conjugate,

$$(\partial_0^2 - \nabla^2 + m^2)\psi^* = 0$$

gives the continuity equation,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$$

where

$$\rho = i(\psi \partial_0 \psi^* - \psi \partial_0 \psi^*), \qquad \vec{j} = i(\psi \vec{\nabla} \psi^* - \psi^* \vec{\nabla} \psi)$$

Define

$$P = \int d^3x \, \rho(x)$$

Then

$$\frac{dP}{dt} = \int_V \frac{\partial \rho}{\partial t} d^3x = -\int_V \vec{\nabla} \cdot \vec{j} \, d^3x = -\oint_S \vec{j} \cdot \vec{ds} = 0 \qquad \text{if} \ \ \vec{j} = 0, \ \ \text{on} \ \ S$$

P is conserved, probability ? But P is not positive, e.g.

$$\text{if} \ \ \psi = e^{iEt} \phi(x), \qquad \text{then} \qquad \rho = -2E \left| \phi(x) \right|^2 \leq 0$$

if we take $\rho = \psi \psi^*$ then it is not conserved,

$$\frac{d}{dt} \int \psi \psi^* d^3x \neq 0$$

Solutions to Klein-Gordon Equation

$$(\Box + m^2)\psi(x) = (-\nabla^2 + \partial_0^2 + m^2)\psi(x) = 0$$

plain wave solution,

$$\phi(x) = e^{-ipx} \quad if \qquad p_0^2 - P^2 - m^2 = 0 \quad \text{or} \quad p_0 = \pm\sqrt{\vec{p}^2 + m^2}$$

1. Positive energy solution: $P_0 = \omega_p = \sqrt{\vec{p}^2 + m^2}$, $\quad \vec{p}$ *arbitrary*

$$\phi^{(+)}(x) = \exp\left(-i\omega_p t + i\vec{p}\cdot\vec{x}\right) = e^{-ikx}$$

2. Negative energy solution: $P_0 = -\omega_p = -\sqrt{\vec{p}^2 + m^2}$

$$\phi^{(-)}(x) = \exp\left(i\omega_p t - i\vec{p}\cdot\vec{x}\right) = e^{ikx}$$

general solution is ,

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}[a(k)e^{-ikx} + a(k)^+ e^{ikx}] \quad , \qquad kx = \omega_k t - \vec{k}\cdot\vec{x}$$

Orthogonality relation

For any 2 solutions ϕ_1, ϕ_2 of Klein-Gordon equation,

$$(\partial_0^2 - \nabla^2 + m^2)\phi_1 = 0$$

and

$$(\partial_0^2 - \nabla^2 + m^2)\phi_2^* = 0$$

From these we get

$$\int d^3x \left\{ \left[\phi_2^* \partial_0^2 \phi_1 - \phi_1 \partial_0^2 \phi_2^*\right] - \left[\phi_2^* \nabla^2 \phi_1 - \phi_1 \nabla^2 \phi_2^*\right] \right\} = 0$$

Or

$$\int d^3x \left\{ \partial_0 \left[\phi_2^* \partial_0 \phi_1 - \phi_1 \partial_0 \phi_2^*\right] - \vec{\nabla} \cdot \left[\phi_2^* \vec{\nabla} \phi_1 - \phi_1 \vec{\nabla} \phi_2^*\right] \right\} = 0$$

Use Gauss' theorem and dropping the surface terms at spatial infinity,

$$\frac{d}{dt} \int d^3x \left[\phi_2^* \partial_0 \phi_1 - \phi_1 \partial_0 \phi_2^*\right] = 0$$

So we define "scalar product" as

$$\langle \phi_2 | \phi_1 \rangle = \int d^3x \left[\phi_2^* \partial_0 \phi_1 - \phi_1 \partial_0 \phi_2^*\right]$$

It is straightforward to derive the orthogonality relations as

$$\left\langle \phi_{p'}^{(+)} | \phi_{p}^{(+)} \right\rangle = \delta^3 \left(p - p' \right)$$

$$\left\langle \phi_{p'}^{(-)} | \phi_{p}^{(-)} \right\rangle = -\delta^3 \left(p - p' \right)$$

$$\left\langle \phi_{p'}^{(+)} | \phi_{p}^{(-)} \right\rangle = 0$$

Dirac Equation

Dirac(1928) wants first order derivative in t and in x, y, z. Ansatz

$$E = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \beta m = \vec{\alpha} \cdot \vec{p} + \beta m \tag{1}$$

where α_i, β are matrices. Then

$$E^2 = \frac{1}{2}(\alpha_i \alpha_j + \alpha_j \alpha_i) p_i p_j + \beta^2 p^2 + (\alpha_i \beta + \beta \alpha_i) m$$

To get energy momentum relation, need

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \tag{2}$$

$$\alpha_i \beta + \beta \alpha_i = 0 \tag{3}$$

$$\beta^2 = 1 \tag{4}$$

From Eq(2) we get

$$\alpha_i^2 = 1 \tag{5}$$

Togather with Eq(4) α_i, β all have eigenvalues ± 1. s

$$\alpha_1 \alpha_2 = -\alpha_2 \alpha_1 \implies \alpha_2 = -\alpha_1 \alpha_2 \alpha_1$$

Taking the trace

$$Tr\alpha_2 = -Tr\left(\alpha_1\alpha_2\alpha_1\right) = -Tr\left(\alpha_2\alpha_1^2\right) = -Tr\left(\alpha_2\right)$$

Thus

$$Tr\left(\alpha_i\right) = 0 \tag{6}$$

Similarly,

$$Tr\left(\beta\right) = 0$$

α_i, β even dimension. Pauli matrices $\sigma_1, \sigma_2, \sigma_3$,are all traceless and anti-commuting. But need 4 such matrices.$\Longrightarrow$ α_i, β, 4×4 matrices. Bjoken and Drell representation,

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac equation ; $E \to i\frac{\partial}{\partial t}$, $\vec{p} \to -i\vec{\nabla}$

$$(-i\vec{\alpha}\cdot\nabla + \beta m)\psi = i\frac{\partial\psi}{\partial t}$$

For convenience, define a new set of matrices

$$\gamma^0 = \beta, \qquad \gamma^i = \beta\alpha_i$$

and in Bjorken and Drell notation,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{7}$$

Dirac equation

$$(-i\gamma^i \partial_i - i\gamma^0 \partial_0 + m)\psi = 0, \qquad \text{or} \qquad (-i\gamma^\mu \partial_\mu + m)\psi = 0$$

Note that the anti-commutations are

$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$$

Probability interpretation

From Dirac equation

$$-i\frac{\partial\psi^\dagger}{\partial t} = \{-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi\}^\dagger$$

and

$$i(\frac{\partial\psi^\dagger}{\partial t}\psi + \psi^\dagger\frac{\partial\psi}{\partial t}) = \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi - \{(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi\}^\dagger\psi$$

Integrate over space,

$$i\frac{d}{dt}\int d^3x(\psi^\dagger\psi) = -i\int\vec{\nabla}\cdot\left(\psi^\dagger\vec{\alpha}\psi\right)d^3x = 0$$

The probability $\int d^3x(\psi^\dagger\psi)$ is conserved and positive.

Solution to Dirac equation

The Dirac equation is,

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

solution in the plane wave,

$$\psi(x) = e^{-ip\cdot x}\omega(p)$$

Then

$$(\not{p} - m)\omega(p) = 0 \qquad \text{where} \qquad \not{p} = \gamma^{\mu}p_{\mu} = \gamma^{0}p_{0} - \vec{\gamma}\cdot\vec{p}$$

and

$$\left(p_0 - \vec{\alpha}\cdot\vec{p} - \beta m\right)\omega(p) = 0, \qquad \text{where} \qquad \vec{\alpha} = \gamma_0\vec{\gamma}, \quad \beta = \gamma_0$$

rewrite this in terms of Hamiltonian

$$H\omega(p) = p_0\omega(p), \qquad \text{with} \qquad H = \vec{\alpha}\cdot\vec{p} + \beta m$$

This is an eigenvalue equation. Eigenvectors for different eigenvalues are orthogonal to each other,

$$\omega^{(i)\dagger}(p)\,\omega^{(j)}(p) = \delta_{ij}, \qquad \text{where} \qquad H\omega^{(i)}(p) = p_0^{(i)}\omega^{(i)}(p)$$

To find the eigenvalues and eigen vectors, we write

$$H = \vec{\alpha}\cdot\vec{p} + \beta m = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}, \qquad \omega(p) = \begin{pmatrix} u \\ l \end{pmatrix}$$

where u(upper components) and l (lower componets) are 2 components column vectors. Then we have

$$\begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \begin{pmatrix} u \\ l \end{pmatrix} = p_0 \begin{pmatrix} u \\ l \end{pmatrix}$$

Or

$$\begin{cases} (p_0 - m)u - (\vec{\sigma}\cdot\vec{p})l = 0 \\ -(\vec{\sigma}\cdot\vec{p})u + (p_0 + m)l = 0 \end{cases} \tag{8}$$

These are homogeneous linear equations. Non-trivial solution exists if

$$\begin{vmatrix} p_0 - m & -\vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & (p_0 + m) \end{vmatrix} = 0$$

This condition gives

$$p_0^2 = \vec{p}^2 + m^2 \qquad \text{or} \qquad p_0 = \pm\sqrt{\vec{p}^2 + m^2}$$

1. Positive energy solution $p_0 = E = \sqrt{\vec{p}^2 + m^2}$,
   Substitute this into Eqs(**??**),

$$l = \frac{\vec{\sigma} \cdot \vec{p}}{E + m} u$$

   Write the solution in the form,

$$\omega^{(s)}(p) = N \left(\begin{array}{c} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{array} \right) \chi_s, \qquad s = 1, 2 \qquad \chi_1 = \left(\begin{array}{c} 1 \\ 0 \end{array} \right), \qquad \chi_1 = \left(\begin{array}{c} 0 \\ 1 \end{array} \right)$$

   Here N is a normalization constant. The solution in coordinate space is

$$\psi = e^{-ipx} \omega^{(s)}(p) = e^{-iEt} e^{i \vec{p} \cdot \vec{x}} \left(\begin{array}{c} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{array} \right) \chi_s$$

   In the non-relativistic limit $|\vec{p}| \ll E$, lower componet much smaller than the upper component.
2. Negative energy solution $p_0 = -E = -\sqrt{\vec{p}^2 + m^2}$,

Similarly, the solution can be written as,

$$u = \frac{-\left(\vec{\sigma}\cdot\vec{p}\right)}{E+m} l$$

We write the solution as,

$$\omega^{(3)}(p) = N\begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \omega^{(4)}(p) = N\begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and in the coordinate space we get

$$\psi = e^{iEt} e^{i\vec{p}\cdot\vec{x}} N\begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_s$$

Orthogonallity of different eigenvectors then implies that

$$\omega^{(3)}(p)^{\dagger}\,\omega^{(1)}(p) = N^2\chi_1^{\dagger}\begin{pmatrix} \frac{-\vec{\sigma}\cdot\vec{p}}{E+m} & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s = 0$$

The standard notation for these 4-component column vector, *spinors* are,

$$u(p.s) = \omega^{(s)}(p) = N\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix}\chi_s, \quad s = 1,2$$

$$v(p,s) = N\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_s \quad N = \sqrt{E+m}$$

Note that $v - spinor$ is defined with $\vec{p}$ reversed and the plane wave factor becomes $e^{iEt}e^{-i\overrightarrow{p}\cdot\overrightarrow{x}} = e^{ipx}$.

The orthogonality for these spinors are

$$u^{\dagger}(p.s')v(-p,s) = 0$$

In the expansion of general solution to the Dirac equation, we write

$$\psi\left(\overrightarrow{x},t\right) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3\,2E_p}}\left[b\left(p,s\right)u\left(p,s\right)e^{-ip\cdot x} + d^{\dagger}\left(p,s\right)\upsilon\left(p,s\right)e^{ip\cdot x}\right]$$

To solve for $b\left(p,s\right)$,we multiply this by $u^{\dagger}\left(p',s'\right)e^{-p'\cdot x}$ and integrate over x,

$$\int u^{\dagger}\left(p',s'\right)e^{-p'\cdot x}\psi\left(\overrightarrow{x},t\right)d^3x = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3\,2E_p}}\left[\begin{array}{c} b\left(p,s\right)u^{\dagger}\left(p',s'\right)u\left(p,s\right)\delta^3\left(p-p'\right) \\ +d^{\dagger}\left(p,s\right)u^{\dagger}\left(p',s'\right)\upsilon\left(p,s\right)\delta^3\left(p+p'\right) \end{array}\right]$$

The last term vanishes because, $u^{\dagger}\left(p',s'\right)\upsilon\left(p,s\right) = u^{\dagger}\left(-p,s\right)\upsilon\left(p,s\right) = 0$ and we get

$$b\left(p,s\right) = \int \frac{d^3x e^{ip\cdot x}}{\sqrt{(2\pi)^3\,2E_p}}u^{\dagger}\left(p,s\right)\psi\left(\overrightarrow{x},t\right).$$

Dirac conjugate

Dirac equation in momentum space

$$(\not{p} - m)\psi(p) = 0$$

is not hermitian. In the Hermitian conjugate

$$\psi^\dagger(p)(\not{p}^\dagger - m) = 0$$

$\gamma_\mu' s$ are not hermitian,

$$\gamma_0^\dagger = \gamma_0 \quad \gamma_i^\dagger = -\gamma_i$$

But we can write

$$\gamma_\mu^\dagger = \gamma_0 \gamma_\mu \gamma_0$$

Then

$$\psi^\dagger(p)(\gamma_0\gamma_\mu\gamma_0 p^\mu - m) = 0 \qquad \text{or} \qquad \psi^\dagger(p)\gamma_0(\gamma_\mu p^\mu - m) = 0$$

Or

$$\bar{\psi}(\not{p} - m) = 0 \quad \text{where} \quad \bar{\psi} = \psi^\dagger\gamma_0 \quad \text{Dirac conjugate}$$

Dirac equation under Lorentz transformation

How Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

behaves under Lorentz transformation?

$$x^\mu \to x'^\mu = \Lambda^\mu_\nu x^\nu$$

In the new coordinate system, Dirac equation is

$$(i\gamma^\mu \partial'_\mu - m)\psi'(x') = 0 \tag{9}$$

Assume

$$\psi'(x') = S\psi(x)$$

Invert the Lorentz transformation

$$x^\gamma = \Lambda^\gamma_\mu x'^\mu \qquad \Longrightarrow \qquad \frac{\partial}{\partial x'^\mu} = \frac{\partial}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x'^\mu} = \Lambda^\gamma_\mu \frac{\partial}{\partial x^\gamma}$$

Then Eq(9) becomes

$$(i\gamma^\mu \Lambda^\alpha_\mu \partial_\alpha - m)S\psi(x) = 0 \qquad \text{or} \qquad (i(S^{-1}\gamma^\mu S)\Lambda^\alpha_\mu \partial_\alpha - m)\psi(x) = 0$$

equivalent to the original Dirac equation, if

$$(S^{-1}\gamma^{\mu}S)\Lambda^{\alpha}_{\mu} = \gamma^{\alpha} \quad \text{or} \quad (S^{-1}\gamma^{\mu}S) = \Lambda^{\mu}_{\alpha}\gamma^{\alpha} \tag{10}$$

infinitesimal transformation

$$\Lambda^{\mu}_{\nu} = g^{\mu}_{\nu} + \epsilon^{\mu}_{\nu} + O(\epsilon^2) \qquad \text{with} \quad \left|\epsilon^{\mu}_{\nu}\right| << 1$$

Pseudo-othogonality implies

$$g_{\mu\nu}(g^{\mu}_{\alpha} + \epsilon^{\mu}_{\alpha})(g^{\nu}_{\beta} + \epsilon^{\nu}_{\beta}) = g_{\alpha\beta}$$

Or

$$\epsilon_{\alpha\beta} + \epsilon_{\beta\alpha} = 0, \qquad \Longrightarrow \quad \epsilon_{\alpha\beta} \text{ antisymmetric}$$

Write

$$S = 1 - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu} + O(\epsilon^2) then S^{-1} = 1 + \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}$$

$\sigma_{\mu\nu}$: 4×4 matrices. Then Eq(10) yields,

$$(1 + \frac{i}{4}\sigma_{\alpha\beta}\epsilon^{\alpha\beta})\gamma^{\mu}(1 - \frac{i}{4}\sigma_{\alpha\beta}\epsilon^{\alpha\beta}) = (g^{\mu}_{\alpha} + \epsilon^{\mu}_{\alpha})\gamma^{\alpha}$$

Or

$$\epsilon^{\alpha\beta}\frac{i}{4}[\sigma_{\alpha\beta}, \gamma^{\mu}] = \epsilon^{\mu}_{\alpha}\gamma^{\alpha} = \frac{1}{2}\epsilon^{\alpha\beta}(g^{\mu}_{\alpha}\gamma_{\beta} - g^{\mu}_{\beta}\gamma_{\alpha})$$

coefficient of $\varepsilon^{\alpha\beta}$

$$[\sigma_{\alpha\beta}, \gamma_\mu] = 2i(g_{\beta\mu}\gamma_\alpha - g_{\alpha\mu}\gamma_\beta) \tag{11}$$

Solution

$$\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta]$$

satisfy Eq(11). To see this, we need to use the identiy,

$$[AB,\ C] = A\{B,\ C\} - \{A,\ C\}B$$

Then

$$\begin{aligned}
[\sigma_{\alpha\beta}, \gamma_\mu] &= \frac{i}{2}[\left(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha\right), \gamma_\mu] = \frac{i}{2}\left(\gamma_\alpha\{\gamma_\beta,\ \gamma_\mu\} - \{\gamma_\alpha, \gamma_\mu\}\gamma_\beta - (\alpha \leftrightarrow \beta)\right) \\
&= \frac{i}{2}\left(2\gamma_\alpha g_{\beta\mu} - 2g_{\alpha\mu}\gamma_\beta\right) \times 2 = 2i(g_{\beta\mu}\gamma_\alpha - g_{\alpha\mu}\gamma_\beta)
\end{aligned}$$

Finite Lorentz transformation,

$$\psi'(x') = S\psi(x), \qquad \text{with} \quad S = exp[-\frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu}] \tag{12}$$

$$\sigma^\dagger_{\mu\nu} = \gamma_0\sigma_{\mu\nu}\gamma_0 \qquad \text{and} \qquad S^\dagger = \gamma^0 S^{-1}\gamma^0$$

S is not unitary. From $\psi'(x') = S\psi$ we get

$$\psi^{\dagger'}(x') = \psi^\dagger S^\dagger = \psi^\dagger\gamma^0 S^{-1}\gamma^0, \qquad \text{or} \qquad \bar{\psi}'(x') = \bar{\psi}(x)S^{-1}$$

$\bar{\psi}$ Dirac conjugate

Fermion bilinears

The fermion bi-linears $\bar{\psi}_\alpha(x)\psi_\beta(x)$ has simple transformation. For example,

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)S^{-1}S\psi(x) = \bar{\psi}(x)\psi(x)$$

$\bar{\psi}(x)\psi(x)$ is Lorentz invariant. Similarly, .

$\bar{\psi}\gamma_\mu\psi$ 4-vector
$\bar{\psi}\gamma_\mu\gamma_5\psi$ axial vector
$\bar{\psi}\sigma_{\mu\nu}\psi$ 2nd rank antisymmetric ensor
$\bar{\psi}\gamma_5\psi$ pseudo scalar

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

Hole Theory (Dirac 1930)

Dirac proposed

$$\text{vaccum} = (E < 0 \text{ states all filled and } E > 0 \text{ states are empty })$$

Pauli exclusion principle makes vacuum stable.
In this picture hole in the negative sea,

$$\text{absence of an electron } -|e| \text{ and } -|E| \equiv \text{presence of particle } |E| \text{ and } +|e|$$

new particle is called "positron" also called *anti − particle*. This correspondence of particle and anti-particle is called *charge conjugation*.

Lorentz group
In Dirac equation, it is not clear what is the origin of Dirac γ matrices. It turns out that they are related to representations of Lorentz group. The Lorentz group is a collection of linear transformations of space-time coordinates

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$$

which leaves the proper time

$$\tau^2 = (x^o)^2 - (\vec{x})^2 = x^\mu x^\nu g_{\mu\nu} = x^2$$

invariant. This requires $\Lambda^\mu \nu$ satisfies the pseudo-orthogonality relation

$$\Lambda^\mu_\alpha \Lambda^\nu_\beta g_{\mu\nu} = g_{\alpha\beta}$$

Generators
For infinitesmal transformation, write

$$\Lambda^\mu_\alpha = g^\mu_\alpha + \epsilon^\mu_\alpha \qquad \text{with} \quad |\varepsilon^\mu_\alpha| \ll 1$$

As before, the pseudo-orthogonality relation implies, $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$. Consider $f(x^\mu)$, an arbitrary function of x^μ. Under infinitesimal Lorentz transformation, the change in f is

$$\begin{aligned} f(x^\mu) &\rightarrow f(x'^\mu) = f(x^\mu + \varepsilon^\mu_\alpha x^\alpha) \approx f(x^\mu) + \varepsilon_{\alpha\beta} x^\beta \partial_\alpha f + \cdots \\ &= f(x^\mu) + \frac{1}{2}\varepsilon_{\alpha\beta}[x^\beta \partial^\alpha - x^\alpha \partial^\beta] f(x) + \cdots \end{aligned}$$

Introduce operator $M_{\mu\nu}$ to represent this change,

$$f(x') = f(x) - \frac{i}{2}\varepsilon_{\alpha\beta}M^{\alpha\beta}f(x) + \cdots$$

then

$$M^{\alpha\beta} = -i(x^{\alpha}\partial^{\beta} - x^{\beta}\partial^{\alpha}) \tag{13}$$

generators $M_{\mu\nu}$ are called the generators of Lorentz group operating on functions of coordinates. Note that for $\alpha, \beta = 1, 2, 3$ these are just the angular momentum operator. Using the generators given in Eq(13) we can work out commutators of these generators,

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i\{g_{\beta\gamma}M_{\alpha\delta} - g_{\alpha\gamma}M_{\beta\delta} - g_{\beta\delta}M_{\alpha\gamma} + g_{\alpha\delta}M_{\beta\gamma}\}$$

Define

$$M_{ij} = \epsilon_{ijk}J_k, \quad M_{oi} = K_i$$

where $J_k's$ correspond to the usual rotations and K_i the Lorentz boost operators. We can solve for J_i to get

$$J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$$

We can compute the commutator of $J_i's$,

$$[J_i, J_j] = \left(\frac{1}{2}\right)^2 \varepsilon_{ikl}\varepsilon_{jmn}[M_{kl}, M_{mn}] = (-i)\,(\frac{1}{2})^2\varepsilon_{ikl}\varepsilon_{jmn}(g_{lm}M_{kn} - g_{km}M_{ln} - g_{ln}M_{km} + g_{kn}M_{lm})$$

$$= (\frac{1}{2})^2 (-i) [-\epsilon_{ikl}\varepsilon_{jln} M_{kn} + \epsilon_{ikl}\varepsilon_{jkn} M_{ln} + \epsilon_{ikl}\varepsilon_{jml} M_{km} - \epsilon_{ikl}\varepsilon_{jmk} M_{lm}]$$

Using identity

$$\epsilon_{abc}\varepsilon_{alm} = (\delta_{bl}\delta_{cm} - \delta_{bm}\delta_{cl})$$

we get

$$[J_i, J_j] = i\epsilon_{ijk} J_k \qquad (14)$$

Thus we can identify J_i as the angular momentum operator.
Similarly, we can derive

$$[K_i, K_j] = -i\epsilon_{ijk} J_k, \qquad [J_i, K_j] = i\epsilon_{ijk} K_k \qquad (15)$$

Eqs(14,15) are called the Lorentz algebra.
To simplify the Lorentz algebra, we define the combinations

$$A_i = \frac{1}{2}(J_i + iK_i) \quad , B_i = \frac{1}{2}(J_i - iK_i)$$

Then we get following commutation relations,

$$[A_i, A_j] = i\epsilon_{ijk} A_k, \qquad [B_i, B_j] = i\epsilon_{ijk} B_k, \qquad [A_i, B_j] = 0$$

For example,

$$\begin{aligned}[A_1, A_2] &= \frac{1}{4}[J_1 + iK_1, J_2 + iK_2] = \frac{1}{4}\left([J_1, J_2] + i[J_1, K_2] + i[K_1, J_2] + i^2[K_1, K_2]\right) \\ &= \frac{1}{4}\left(iJ_3 + i^2K_3 + i^2K_3 - i^3J_3\right) = \frac{1}{2}i(J_3 + iK_3) = iA_3\end{aligned}$$

Thus algebra of Lorentz generators factorizes into 2 independent $SU(2)$ algebra. The representations are just the tensor products of the representation of $SU(2)$ algebra. Thus we label the irreducible representation by (j_1, j_2) which transforms as $(2j_1 + 1)$-dim representation under A_i algebra and $(2j_2 + 1)$-dim representation under B_i algebra.

Simple representations

1. $(\frac{1}{2}, 0)$ representation χ_a
   This 2-component object has the following properties,

$$A_i \chi_a = (\frac{\sigma_i}{2})_{ab} \chi_b \qquad \Longrightarrow \qquad \frac{1}{2}(J_i + iK_i)\chi_a = (\frac{\sigma_i}{2})_{ab}\chi_b$$

$$B_i \chi_a = 0 \qquad \Longrightarrow \qquad \frac{1}{2}(J_i - iK_i)\chi_a = 0$$

   Combining these realtions

$$\vec{J}\chi = (\frac{\vec{\sigma}}{2})\chi, \qquad \vec{K}\chi = -i(\frac{\vec{\sigma}}{2})\chi$$

2. $(0, \frac{1}{2})$ representation η_a

Similarly, we can get

$$A_i \eta_a = 0 \qquad \Rightarrow \qquad \frac{1}{2}(J_i + iK_i)\eta_a = 0$$

$$B_i \eta_a = (\frac{\sigma_i}{2})_{ab} \qquad \Longrightarrow \quad \frac{1}{2}(J_i - iK_i)\eta_a = (\frac{\sigma_i}{2})_{ab}\eta_b$$

$$\vec{J}\eta = (\frac{\vec{\sigma}}{2})\eta, \qquad \vec{K}\eta = i(\frac{\vec{\sigma}}{2})\eta$$

If we define a 4-component ψ by putting togather these 2 representations,

$$\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$$

Then action of the Lorentz generators are

$$\vec{J}\psi = \begin{pmatrix} \frac{\vec{\sigma}}{2} & 0 \\ 0 & \frac{\vec{\sigma}}{2} \end{pmatrix}\psi, \qquad \vec{K}\psi = \begin{pmatrix} -i\frac{\vec{\sigma}}{2} & 0 \\ 0 & i\frac{\vec{\sigma}}{2} \end{pmatrix}\psi \tag{16}$$

ψ are related to the 4-component Dirac field we studied before, but with different representation for the γ matrices. This can be seen as follows.

Consider Dirac matrices in the following form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \text{where} \quad \sigma^\mu = (1, \vec{\sigma}) \quad , \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

More explicitly,

$$\gamma^o = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

It is straightforward to check that in this case.

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

This means that in 4-component field $\psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix}$, χ is right-handed and η is left-handed. In this representation, it is easy to check that

$$\sigma_{0i} = i\gamma_0\gamma_1 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} -i\sigma^i & 0 \\ 0 & i\sigma^i \end{pmatrix}$$

$$\sigma_{ij} = i\gamma_i\gamma_j = i \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}$$

In the Lorentz transformation of Dirac field,

$$\psi'(x') = S\psi = exp\{-\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}\} = exp\{-\frac{i}{4}(2\sigma_{0i}\varepsilon^{0i} + \sigma_{ij}\varepsilon^{ij})\}$$

Write $\varepsilon^{0i} = \beta^i, \quad \varepsilon^{ij} = \varepsilon^{ijk}\theta^k$

$$\sigma_{ij}\epsilon^{ij} = \varepsilon^{ijk}\theta^k\epsilon_{ijl}\begin{pmatrix} \sigma_l & 0 \\ 0 & \sigma_l \end{pmatrix} = 2\begin{pmatrix} \vec{\sigma}\cdot\vec{\theta} & 0 \\ 0 & \vec{\sigma}\cdot\vec{\theta} \end{pmatrix}$$

$$\sigma_0 i\varepsilon^0 i = \begin{pmatrix} -i\vec{\sigma}\cdot\vec{\beta} & 0 \\ 0 & i\vec{\sigma}\cdot\vec{\beta} \end{pmatrix}$$

$\Rightarrow$

$$-\frac{i}{4}(2\sigma_{0i}\varepsilon^{0i} + \sigma_{ij}\varepsilon^{ij}) = \frac{-i}{2}\begin{pmatrix} \vec{\sigma}\cdot\overrightarrow{\theta} - i\vec{\sigma}\cdot\vec{\beta} & 0 \\ 0 & \vec{\sigma}\cdot\overrightarrow{\theta} + i\vec{\sigma}\cdot\vec{\beta} \end{pmatrix}$$

More precisely,

$$\psi'(x') = S\psi = exp\{-\frac{i}{4}\sigma_{\mu\nu}\varepsilon^{\mu\nu}\}\psi = \exp\left[\frac{-i}{2}\begin{pmatrix} \vec{\sigma}\cdot\overrightarrow{\theta} - i\vec{\sigma}\cdot\vec{\beta} & 0 \\ 0 & \vec{\sigma}\cdot\overrightarrow{\theta} + i\vec{\sigma}\cdot\vec{\beta} \end{pmatrix}\right]\psi \qquad (17)$$

If we write the Lorentz transformations in terms of generators,

$$L = \exp(-iM_{\mu\nu}\varepsilon^{\mu\nu})$$

then in terms of the generators $\overrightarrow{J}$, $\overrightarrow{K}$

$$L = \exp\left[(-i)\left(\overrightarrow{J}\cdot\overrightarrow{\theta} + \overrightarrow{K}\cdot\overrightarrow{\beta}\right)\right]$$

We then see from Eq(17) that for this ψ, $\overrightarrow{J}$, $\overrightarrow{K}$ are of the form,

$$\overrightarrow{J} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \qquad \overrightarrow{K} = \frac{1}{2}\begin{pmatrix} -i\vec{\sigma} & 0 \\ 0 & i\vec{\sigma} \end{pmatrix}$$

These are the same as those in Eq(16).
Thus the wavefunction which satisfies Dirac equation is just the representation $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ under Lorentz group. Futhermore, the right-handed components transform as $\left(\frac{1}{2}, 0\right)$ represenation while left-handed components transform as $\left(0, \frac{1}{2}\right)$ representation.

Alternative choice is to use ψ_R and the complex conjugate ψ_R^* (sometime dotted indice are used for this basis) instead of ψ_R and ψ_L. Since

$$\vec{J}\psi_R = (\frac{\vec{\sigma}}{2})\psi_R, \qquad \vec{K}\psi_R = -i(\frac{\vec{\sigma}}{2})\psi_R$$

we get for the complex conjuate

$$\vec{J}\psi_R^* = (\frac{\vec{\sigma}^*}{2})\psi_R^*, \qquad \vec{K}\psi_R^* = i(\frac{\vec{\sigma}^*}{2})\psi_R^*$$

It is probably more clearer to use some other notation for ψ_R^*,

$$\vec{J}\chi = (\frac{\vec{\sigma}^*}{2})\chi, \qquad \vec{K}\chi = i(\frac{\vec{\sigma}^*}{2})\chi$$

Then

$$\overrightarrow{A}\chi = \frac{1}{2}(\overrightarrow{J} + i\overrightarrow{K})\chi = 0, \qquad \overrightarrow{B}\chi = \frac{1}{2}(\overrightarrow{J} - i\overrightarrow{K})\chi = (\frac{\vec{\sigma}^*}{2})\chi$$

and indeed χ belongs to the irrep $(0, \frac{1}{2})$.