QFT-Canonical Quantization

Ling-Fong Li

Chapter 3 Canonical Quantization

Quantization of Free Fields

The quantization of field is a generalization of the quantization in the non-relativistic quantum mechanics where we impose the commutaiton relations for coordinates $q_i, i = 1, 2 \cdots, n$ and their conjugate momenta p_j,

$$[q_i, p_j] = i\delta_{ij}$$

where p_j is defined by

$$p_j = \frac{\partial L}{\partial \dot{q}_j}, \qquad L : \text{Lagrangian}$$

The Hamiltonian is

$$H = \sum_i p_i \dot{q}_i - L$$

The dynamics is determined by the Schrodinger equation,

$$H\Psi = i\frac{\partial \Psi}{\partial t}$$

Here wave function $\Psi(t)$ gives time evolution while operators $p_{i,} q_j$ are time independent.Thisρ is known as the **Schrodinger picture.** Alternatively, we can go to **Heisenberg picture** where

$p_i(t)$ and $q_j(t)$ carry the time dependence instead of state vector Ψ.This is known as the Hsienberg picture which is related to the Schrodinger picture by unitary transformation,

$$\Psi_S(t) = e^{-iHt}\Psi_H$$

and

$$O_H(t) = e^{iHt}O_S e^{-iHt}$$

In this picture the canonical commutation relation is then

$$[q_i(t), p_j(t)] = i\delta_{ij}$$

In relativistic field theory we will use Heisenberg picture so that both spatial coordinate $\vec{x}$ and t both appear as arguments of the field operator $\phi\left(\vec{x}, t\right)$

Thus in field theory we replace $q_i(t)$ by $\phi\left(\vec{x}, t\right)$. To make this correspondence more transparent, divide the 3-dim space into cells of volume ΔV_i and define the ith coordinate $\phi_i(t)$ by averaging $\phi\left(\vec{x}, t\right)$ over the ith cell

$$\phi_i(t) = \frac{1}{\Delta V_i}\int_{\Delta V_i} d^3x\phi\left(\vec{x}, t\right)$$

Similarly, $\partial_0 \phi_i(t)$ is the averge of $\partial \phi\left(\overrightarrow{x}, t\right)/\partial t$ over the $i$the cell. Write the Lagrangian L as integration of Lagrangain density $\mathcal{L}$,

$$L = \int d^3x \ \mathcal{L}$$

and let $\mathcal{L}_i$ be the average of $\mathcal{L}$ in the $i$the cell. We define the conjugate momenta as

$$p_i(t) = \frac{\partial L}{\partial\left(\partial_0 \ \phi_i(t)\right)} = \Delta V_i \frac{\partial \mathcal{L}_i}{\partial\left(\partial_0 \ \phi_i(t)\right)} \equiv \Delta V_i \pi_i(t)$$

The Hamiltonian is then defined as

$$H = \sum_i p_i(t)\,\partial_0 \ \phi_i(t) - L = \sum_i \Delta V_i\left(\pi_i \partial_0 \ \phi_i(t) - \mathcal{L}_i\right) \longrightarrow \int d^3x \mathcal{H}$$

and

$$\mathcal{H} = \pi_i \partial_0 \ \phi_i(t) - \mathcal{L}$$

Canonical commutation relations are

$$\left[\phi_i(t), p_j(t)\right] = i\delta_{ij}, \qquad \left[\phi_i(t), \phi_j(t)\right] = 0, \qquad \left[p_i(t), p_j(t)\right] = 0$$

Or in terms of π_i

$$\left[\phi_i(t), \ \pi_j(t)\right] = i\frac{\delta_{ij}}{\Delta V_i}$$

These become in the continuum language,

$$\left[\phi\left(\vec{x},t\right),\pi\left(\vec{x}',t\right)\right]=i\delta^3\left(\vec{x}-\vec{x}'\right),\qquad\left[\phi\left(\vec{x},t\right),\phi\left(\vec{x}',t\right)\right]=0,$$

$$\left[\pi\left(\vec{x},t\right),\pi\left(\vec{x}',t\right)\right]=0$$

where the Dirac delta function emerges as the limit of $\frac{\delta_{ij}}{\Delta V_i}$ as $\Delta V_i\longrightarrow 0$, according to

$$\int d^3x'\delta^3\left(\vec{x}-\vec{x}'\right)f(\vec{x'})=f(\vec{x})$$

Also we have

$$\pi\left(\vec{x},t\right)=\frac{\partial\mathcal{L}}{\partial\left(\partial_0\phi\right)}$$

Scalar field

Consider a scalar field ϕ satifies the Klein-Gordon equation

$$\left(\partial^\mu\partial_\mu+\mu^2\right)\phi=0$$

Lagrangian density is

$$\mathcal{L}=\frac{1}{2}\left(\partial^\mu\phi\right)\left(\partial_\mu\phi\right)-\frac{\mu^2}{2}\phi^2$$

Euler-Lagrange equation for this $\mathcal{L}$

$$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial^\mu \phi \right)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

gives the Klein-Gordon equation.

$$\partial^\mu \partial_\mu \phi + \mu^2 \phi = 0$$

Canonical quantization

Conjugate momentum

$$\pi\left(\overrightarrow{x}, t\right) = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \phi\right)} = \left(\partial_0 \phi\right)$$

Impose commutation relations,

$$\left[\phi\left(\overrightarrow{x}, t\right), \pi\left(\overrightarrow{y}, t\right)\right] = i\delta^3\left(\overrightarrow{x} - \overrightarrow{y}\right), \qquad \left[\phi\left(\overrightarrow{x}, t\right), \phi\left(\overrightarrow{y}, t\right)\right] = 0, \tag{1}$$

$$\left[\pi\left(\overrightarrow{x}, t\right), \pi\left(\overrightarrow{y}, t\right)\right] = 0$$

Hamiltonian density is

$$\mathcal{H} = \pi\partial_0\phi - \mathcal{L} = \frac{1}{2}\left[\left(\partial^0\phi\right)^2 + \left(\vec{\nabla}\phi\right)^2\right] + \frac{1}{2}\mu^2\phi^2$$

We can compute the commutator

$$\left[H, \phi(\vec{x}, t)\right] = \int d^3y \left[\mathcal{H}, \phi(\vec{x}, t)\right] = -i\partial_0\phi$$

Thus Hamiltonian generates the time translation.

Mode expansion

To find physical contents, expand in classical solutions,

$$\phi(\overrightarrow{x}, t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2w_k}} \left[a(\overrightarrow{k})e^{-ik\cdot x} + a^{\dagger}(\overrightarrow{k})e^{ik\cdot x} \right], \quad k_0 = \sqrt{\overrightarrow{k}^2 + \mu^2}$$

$a(k)$ and $a^{\dagger}(k)$ are operators.Note that term $a^{\dagger}(\overrightarrow{k})e^{ik\cdot x}$ corresponds to the negative energy solution. This will become the creation operator while the first term $a(\overrightarrow{k})e^{-ikx}$ correspond to destruction operator.

Solve $a(k)$ and $a^\dagger(k)$ in ϕ and $\partial_0\phi$.This can be carried out as follows. The derivative of ϕ is

$$\partial_0\phi\left(\overrightarrow{x},t\right)=\int\frac{d^3k}{\sqrt{(2\pi)^3\,2w_k}}\left(-ik_0\right)\left[a(\overrightarrow{k})e^{-ik\cdot x}-a^\dagger(\overrightarrow{k})e^{ik\cdot x}\right],\qquad k_0=\sqrt{\overrightarrow{k}^2+\mu^2}=w_k$$

Combining these two relations and integrating over x after multiplying $e^{ik'x}$, we get

$$\int e^{ik'x}d^3x\left(\partial_0\phi-ik_0\phi\right)=\int\frac{d^3k}{\sqrt{(2\pi)^3\,2w_k}}\left(-2ik_0\right)\delta^3\left(k-k'\right)a\left(k\right)$$

From this we get

$$a(k)=i\int d^3x\frac{1}{\sqrt{(2\pi)^3\,2w_k}}\left[e^{ikx}\partial_0\phi-\left(\partial_0e^{ik\cdot x}\right)\right]$$

If we introduce the notation

$$f\overleftrightarrow{\partial_0}g\equiv f\partial_0g-\left(\partial_0f\right)g$$

we can write

$$a(k)=i\int d^3x\frac{e^{ik\cdot x}}{\sqrt{(2\pi)^3\,2w_k}}\overleftrightarrow{\partial_0}\phi\left(x\right)$$

Hermitian conjugate

$$a^{\dagger}(k) = -i \int \frac{d^3x \ e^{-ik\cdot x}}{\sqrt{(2\pi)^3 \, 2w_k}} \overleftrightarrow{\partial_0} \phi(x)$$

where

$$f \overleftrightarrow{\partial_0} g \equiv f \partial_0 g - (\partial_0 f) \, g$$

Commutators can be calculated as

$$\left[a(\overrightarrow{k}), a^{\dagger}(\overrightarrow{k'})\right] = \delta^3(\overrightarrow{k} - \overrightarrow{k'}) \ , \qquad \left[a(\overrightarrow{k}), a(\overrightarrow{k}')\right] = 0$$

For example,

$$\begin{aligned}
\left[a(\overrightarrow{k}), \ a^{\dagger}(\overrightarrow{k'})\right] &= \int \frac{d^3x d^3x' e^{ikx} e^{-ik'x'}}{\sqrt{(2\pi)^3 \, 2w_k \, (2\pi)^3 \, 2w_{k'}}} \left[\partial_0\phi(x) - ik_0\phi(x) , \ \partial_0\phi(x') - ik_0'\phi(x')\right] \\
&= \int \frac{d^3x d^3x' e^{ikx} e^{-ik'x'}}{\sqrt{(2\pi)^3 \, 2w_k \, (2\pi)^3 \, 2w_{k'}}} \left(ik_0'(-i) - ik_0 i\right) \delta^3(x - x') \\
&= \delta^3(\overrightarrow{k} - \overrightarrow{k'}) \, \delta^3(\overrightarrow{k} - \overrightarrow{k'})
\end{aligned}$$

Same as harmonic oscillators.

The Hamiltonian is

$$H = \int d^3k \mathcal{H}_k = \frac{1}{2} \int d^3k w_k \left[a^\dagger(\overrightarrow{k}) a(\overrightarrow{k}) + a(\overrightarrow{k}) a^\dagger(\overrightarrow{k}) \right]$$

superposition of oscillators with frequency w_k.

We can compute the commutator

$$\left[H,\ a^\dagger(k) \right] = \int d^3k'\ w_{k'}\ \left[a^\dagger(k') a(k'),\ a^\dagger(k) \right] = w_k\ a^\dagger(k)$$

If we have an eigenstate of H with eigenvalue E,

$$H|E\rangle = E|E\rangle,$$

then applying the commutator, we get

$$\left(Ha^\dagger(k) - a^\dagger(k)H \right) |E\rangle = w_k\ a^\dagger(k)|E\rangle$$

which gives

$$Ha^\dagger(k)|E\rangle = (E + w_k\ )\ a^\dagger(k)|E\rangle$$

Thus the operator $a^\dagger(k)$ will increase the energy eigenvalue by w_k, **creation operator.**

Similarly,

$$[H,\ a(k)] = \int d^3k'\ w_{k'}\ \left[a^{\dagger}(k')a(k'),\ a(k)\right] = -w_k\ a(k)$$

and $a(k)$ will decrease the energy eigenvalue by w_k,**destruction operator**.
From Noether's theorem, momentum operator is,

$$P_i = \int d^3x T_{0i} = \int d^3x \frac{\partial \mathcal{L}}{\partial \left(\partial_0 \phi\right)} \partial_i \phi = \int d^3x \pi \partial_i \phi$$

and we have the commutator,

$$\begin{aligned} \left[P_i, \phi(\vec{x}, t)\right] &= \int d^3y \left[\pi(\vec{y}, t)\partial_i\phi(\vec{y}, t), \phi(\vec{x}, t)\right] \\ &= \int d^3y \partial_i\phi(\vec{y}, t)\left(-i\right)\delta^3(\vec{x} - \vec{y}) = -i\partial_i\phi(\vec{x}, t) \end{aligned}$$

In terms of creation and annihilation operators,

$$\vec{p} = \frac{1}{2}\int d^3k\,\vec{k}\left[a^{\dagger}\left(k\right)a\left(k\right) + a\left(k\right)a^{\dagger}\left(k\right)\right] = \int d^3k\,\vec{p_k}$$

with

$$\vec{p_k} = \frac{\vec{k}}{2}\left[a^{\dagger}\left(k\right)a\left(k\right) + a\left(k\right)a^{\dagger}\left(k\right)\right]$$

Note

$$a(k)\, a^{\dagger}(k) = a^{\dagger}(k)\, a(k) + \delta^3(0)$$

Interpret $\delta^3(0)$ as

$$\delta^3(\overrightarrow{k}) = \int \frac{d^3x}{(2\pi)^3} e^{i\overrightarrow{k}\cdot\overrightarrow{x}}$$

as $\overrightarrow{k} \to 0$

$$\delta^3(0) = (2\pi)^{-3} \int d^3x = \frac{V}{(2\pi)^3}$$

V total volume of the system. Then

$$H = \int d^3k w_k \left[a^{\dagger}(k)\, a(k) + \frac{(2\pi)^{-3}}{2} V \right]$$

Last term will be dropped.

To achieve this more formally, use normal ordering.

Normal ordering
In normal ordering : $(\cdots)$: move all $a^\dagger(k)$ to the left of $a(k)$.
For example,
.

$$:\quad a(k)a^\dagger(k) := a^\dagger(k)a(k)$$
$$:\quad a^\dagger(k)a(k) := a^\dagger(k)a(k)$$

Vaccum is defined by

$$a(k)|0\rangle = 0 \qquad \forall\ \overrightarrow{k} \qquad \Longrightarrow \langle 0|a^\dagger(k) = 0$$

Then

$$\langle 0| : f\left(a, a^\dagger\right) : |0\rangle = 0$$

Define Hamiltonican by normaling ordering

$$H = \frac{1}{2}\int d^3k w_k : \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] := \int d^3k w_k a^\dagger(k)a(k)$$

Similarly,

$$\overrightarrow{p} = \frac{1}{2}\int d^3k \overrightarrow{p_k} : \left[a^\dagger(k)a(k) + a(k)a^\dagger(k)\right] := \int d^3k \overrightarrow{p_k} a^\dagger(k)a(k)$$

Then vacuum has zero energy and momentum.

Particle interpretation

State defined by

$$|\overrightarrow{k}\rangle = \sqrt{(2\pi)^3 2w_k} a^\dagger(k)|0\rangle$$

is eigenstate of H & $\overrightarrow{p}$,

$$H|\overrightarrow{k}\rangle = w_k|\overrightarrow{k}\rangle, \qquad \overrightarrow{p}|\overrightarrow{k}\rangle = \overrightarrow{k}|\overrightarrow{k}\rangle \qquad \text{where } w_k = \sqrt{\overrightarrow{k}^2 + \mu^2}$$

Interpret this as one-particle state because eigenvalues are related by

$$w_k^2 + \overrightarrow{k}^2 = \mu^2$$

Similarly,we can define 2 particle satate by

$$|\overrightarrow{k}_1, \overrightarrow{k}_2\rangle = \sqrt{(2\pi)^3 2w_{k_1}} \sqrt{(2\pi)^3 2w_{k_2}} a^\dagger(\overrightarrow{k_1}) a^\dagger(\overrightarrow{k_2})|0\rangle$$

Generlization to multiparticle states,

$$|\overrightarrow{k}_1, \cdots \overrightarrow{k}_n\rangle = \sqrt{(2\pi)^3 2w_{k_1}} \cdots \sqrt{(2\pi)^3 2w_{k_n}} a^\dagger(\overrightarrow{k_1}) \cdots a^\dagger(\overrightarrow{k_2})|0\rangle$$

Bose statistics

Expand arbitrary state

$$|\Phi\rangle = \left[C_0 + \sum_{i=1}^{\infty} \int d^3k_1 ... d^3k_n C_n\left(k_1, k_2, ..., k_n\right) a^\dagger(\overrightarrow{k_1})...a^\dagger(\overrightarrow{k_n})|0\rangle \right]$$

$C_n\left(k_1, k_2, ..., k_n\right)$ the momentum space wavefunction.
Since

$$\left[a^\dagger\left(k_i\right), a^\dagger\left(k_j\right)\right] = 0$$

$$C_n\left(k_1, ..., k_i, ..., k_j ..., k_n\right) = C_n\left(k_1, ..., k_j, ..., k_i ..., k_n\right)$$

$C_n\left(k_1, k_2, ..., k_n\right)$ satisfies Bose statistics

Fermion fields

To quantize fermion field we can proceed the same way as the scalar field.
Start with Dirac equation for free particles,

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0 \qquad \text{or} \qquad \overline{\psi}\left(-i\gamma^\mu \overleftarrow{\partial_\mu} - m\right) = 0$$

Lagrangian density for this equation is

$$\mathcal{L} = \bar{\psi}_\alpha \left(i\gamma^\mu \partial_\mu - m\right)_{\alpha\beta} \psi_\beta$$

Then

$$\frac{\partial \mathcal{L}}{\partial \psi^\dagger_\gamma} = \left(\gamma^0\right)_{\gamma\alpha} \left(i\gamma^\mu \partial_\mu - m\right)_{\alpha\beta} \psi_\beta, \qquad \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi_\gamma\right)} = 0$$

and Euler-Lagrange equation gives,

$$\left(i\gamma^\mu \partial_\mu - m\right)_{\alpha\beta} \psi_\beta = 0$$

Conjugate momentum density is

$$\pi_\alpha = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \psi_\alpha\right)} = i\psi^\dagger_\alpha$$

If we impose the commutation relation like scalar field, will get Dirac particles satisfying Bose statistics which is not correct physically.

Impose anticommutation relations to get Fermi-Dirac statistics,

$$\left\{ \pi_\alpha \left(\overrightarrow{x}, t \right), \psi_\beta \left(\overrightarrow{y}, t \right) \right\} = i\delta^3 \left(\overrightarrow{x} - \overrightarrow{y} \right) \delta_{\alpha\beta} \ , \quad \left\{ \psi_\alpha \left(\overrightarrow{x}, t \right), \psi_\beta \left(\overrightarrow{y}, t \right) \right\} = 0$$

Hamiltonian density

$$\mathcal{H} = \sum_\alpha \pi_\alpha \dot{\psi}_\alpha - \mathcal{L} = i\psi^\dagger \gamma_0 \gamma_0 \partial_0 \psi - \overline{\psi} \left(i\gamma^\mu \partial_\mu - m \right) \psi = \overline{\psi} \left(i \overrightarrow{\gamma} \cdot \overrightarrow{\nabla} + m \right) \psi$$

Mode expansion

Expansion in terms of classical solutions,

$$\psi\left(\overrightarrow{x},t\right) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3\, 2E_p}} \left[b\left(p,s\right) u\left(p,s\right) e^{-ip\cdot x} + d^{\dagger}\left(p,s\right) \upsilon\left(p,s\right) e^{ip\cdot x} \right]$$

$$\psi^{\dagger}\left(\overrightarrow{x},t\right) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3\, 2E_p}} \left[b^{\dagger}\left(p,s\right) u^{\dagger}\left(p,s\right) e^{ip\cdot x} + d\left(p,s\right) \upsilon^{\dagger}\left(p,s\right) e^{-ip\cdot x} \right]$$

Invert these relations to get the field operators in the momentum space. Multiply ψ by $u^{\dagger}\left(p',s'\right) e^{ip'x}$ and integrate over x,

$$\int d^3x e^{ip'x} u^{\dagger}\left(p',s'\right) \psi\left(\vec{x},t\right) = \sum_s \int \frac{d^3p}{\sqrt{(2\pi)^3\, 2E_p}} b\left(p,s\right) u^{\dagger}\left(p',s'\right) u\left(p,s\right) (2\pi)^3 \delta^3\left(p-p'\right)$$

where we have used the relation,

$$u^{\dagger}\left(-p,s'\right) v\left(p,s\right) = 0$$

From the Dirac equation we have

$$\bar{u}\left(p,s'\right) \gamma^{\mu} \left(\not{p} - m\right) u\left(p,s\right) = 0$$

and

$$\bar{u}\left(p,s'\right)\left(\not{p}-m\right)\gamma^{\mu}u\left(p,s\right)=0$$

Add these two equations we get

$$p^{\mu}\bar{u}\left(p,s'\right)u\left(p,s\right)=m\bar{u}\left(p,s'\right)\gamma^{\mu}u\left(p,s\right)$$

Take the time component,

$$u^{\dagger}\left(p',s'\right)u\left(p,s\right)=2p^{0}$$

Using this relation, we get

$$b\left(p,s\right)=\int\frac{d^{3}xe^{ip\cdot x}}{\sqrt{\left(2\pi\right)^{3}2E_{p}}}u^{\dagger}\left(p,s\right)\psi\left(\overrightarrow{x},t\right)$$

The Hermitian conjugate yields

$$b^{\dagger}\left(p,s\right)=\int\frac{d^{3}xe^{-ip\cdot x}}{\sqrt{\left(2\pi\right)^{3}2E_{p}}}\psi^{\dagger}\left(\overrightarrow{x},t\right)u\left(p,s\right)$$

From these, we can compute the anti-commutation relations for b, d,

$$\begin{aligned}
\left\{b\left(p,s\right), b^{\dagger}\left(p',s'\right)\right\} &= \int \frac{d^3x' d^3x e^{ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}} \frac{e^{-ip'\cdot x'}}{\sqrt{(2\pi)^3 2E_{p'}}} \left\{u^{\dagger}\left(p,s\right)\psi\left(\overrightarrow{x},t\right),\ \psi^{\dagger}(\overrightarrow{x}',t)u\left(p',s'\right)\right\} \\
&= \int \frac{d^3x e^{ip\cdot x}}{\sqrt{(2\pi)^3 2E_p}} \int \frac{d^3x' e^{-ip'\cdot x'}}{\sqrt{(2\pi)^3 2E_{p'}}} (2\pi)^3 \delta^3\left(x-x'\right) u^{\dagger}\left(p,s\right) u\left(p',s'\right) \\
&= \delta_{ss'}\delta^3(\overrightarrow{p}-\overrightarrow{p}'),
\end{aligned}$$

SImilarly

$$\left\{d\left(p,s\right), d^{\dagger}\left(p',s'\right)\right\} = \delta_{ss'}\delta^3(\overrightarrow{p}-\overrightarrow{p}')$$

and all other anticommutators vanish.
Hamiltonian

$$H = \sum_s \int d^3p \mathcal{H}_{ps}$$

where

$$\mathcal{H}_{ps} = E_p\left[b^{\dagger}\left(p,s\right)b\left(p,s\right) - d\left(p,s\right)d^{\dagger}\left(p,s\right)\right]$$

Similarly,

$$\overrightarrow{p} = \sum_s d^3p\,\overrightarrow{p}_p$$

where

$$\overrightarrow{p}_p = \overrightarrow{p} \left[b^\dagger(p,s)\, b(p,s) - d(p,s)\, d^\dagger(p,s) \right]$$

Commutators of H with $b^\dagger(p,s)$

$$\left[H, b^\dagger(p,s) \right] = \sum_{s'} d^3p' \left[b^\dagger(p',s')\, b(p',s'), b^\dagger(p,s) \right] E_p = b^\dagger(p,s)\, E_p$$

$$\left[\overrightarrow{p}, b^\dagger(p,s) \right] = \overrightarrow{p}\, b^\dagger(p,s)$$

where we have used the identity

$$[AB,\ C] = A\{B, C\} - \{A, C\}B$$

$b^\dagger(p,s)$ creats a particle with E_p and $\overrightarrow{p}$ with relation $E_p = \sqrt{\overrightarrow{p}^{\,2} + m^2}$.
$d^\dagger(p,s)$ creates a particle with same mass but opposite charge as $b^\dagger(p,s)$.

Symmetry

$$\mathcal{L} = \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi$$

is invariant under,

$$\psi(x) \rightarrow e^{i\alpha}\psi(x) \Longrightarrow \psi^{\dagger}(x) \rightarrow \psi^{\dagger}(x)\, e^{-i\alpha} \qquad \alpha : \text{some real constant}$$

Noether's theorem,$\Longrightarrow$ conserved current,

$$\partial^{\mu} j_{\mu} = 0, \qquad \text{where} \qquad j_{\mu} = \overline{\psi}\gamma_{\mu}\psi$$

To see this consider the infinitesmal transformation

$$\delta\psi = i\alpha\psi, \qquad \delta\psi^{\dagger} = -i\alpha\psi^{\dagger}$$

Then from Noether's theorem, the conserved current is

$$j_{\mu} = \frac{\partial \mathcal{L}}{\partial\left(\partial_{\mu}\psi_{\alpha}\right)}\delta\psi_{\alpha} = -i\alpha\overline{\psi}\gamma_{\mu}\psi$$

We can compute the conserved charge

$$
\begin{aligned}
Q &= \int j_0(x)\, d^3x = \int d^3x : \psi^\dagger\left(\vec{x}, t\right)\psi\left(\vec{x}, t\right) : \\
&= \int d^3x \sum_{ss'} \int \frac{d^3p'}{\sqrt{(2\pi)^3\, 2E_{p'}}} : \left[b^\dagger(p', s')\, u^\dagger(p', s')\, e^{ip'\cdot x} + d(p', s')\, v^\dagger(p', s')\, e^{-ip'\cdot x}\right] \\
&\quad \times \int \frac{d^3p}{\sqrt{(2\pi)^3\, 2E_p}} \left[b(p, s)\, u(p, s)\, e^{-ip\cdot x} + d^\dagger(p, s)\, v(p, s)\, e^{ip\cdot x}\right] : \\
&= \sum_s \int d^3p : \left[b\dagger(p, s)\, b(p, s) + d(p, s)\, d^\dagger(p, s)\right] := \sum \int d^3p \left[N^+(p, s) - N^-(p, s)\right]
\end{aligned}
$$

where

$$
N_{ps}^+ = b^\dagger(p, s)\, b(p, s) \qquad N_{ps}^- = d^\dagger(p, s)\, d(p, s)
$$

are the number operators $\implies$ particle and anti-particle have opposite "charge".

Electromagnetic fields

Maxwell's equations,

$$\nabla \cdot \overrightarrow{B} = 0, \quad \nabla \times \overrightarrow{E} + \frac{\partial \overrightarrow{B}}{\partial t} = 0, \tag{2}$$

$$\nabla \cdot \vec{E} = 0, \qquad \frac{1}{\mu_0} \nabla \times \overrightarrow{B} - \epsilon_0 \frac{\partial \overrightarrow{E}}{\partial t} = 0 \tag{3}$$

Introduce $\overrightarrow{A}, \phi$ by

$$\overrightarrow{B} = \nabla \times \overrightarrow{A}, \qquad \overrightarrow{E} = -\nabla\phi - \frac{\partial \overrightarrow{A}}{\partial t} \tag{4}$$

These solve equations in Eq(2). Write relations in Eq(4) as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \text{with} \quad F^{0i} = \partial^0 A^i - \partial^i A^0 = -E^i, \quad F^{ij} = \partial^i A^j - \partial^j A^i = -\epsilon_{ijk} B_k$$

Other two sets of equations in Eq(3)

$$\partial_\nu F^{\mu\nu} = 0, \quad \mu = 0, 1, 2, 3$$

For example

$$\mu = 0, \quad \partial_i F^{0i} = 0 \quad \Rightarrow \quad \nabla \cdot \overrightarrow{E} = 0$$

$$\mu = i, \quad \partial_\nu F^{i\nu} = 0 \quad \Rightarrow \quad \nabla \times \overrightarrow{B} - \frac{\partial \overrightarrow{E}}{\partial t} = 0$$

Note $c^2 = \frac{1}{\mu_0 \epsilon_0} = 1$. $F^{\mu\nu}$ is invariant under the transformation,

$$A^\mu \longrightarrow A^\mu + \partial^\mu \alpha \qquad \alpha = \alpha(x)$$

$\alpha(x)$ is arbitrary function. This is called gauge transformation. Given a set of $\overrightarrow{B}$ and $\overrightarrow{E}$ fields, $\overrightarrow{A}$,and ϕ are not unique. Different $\alpha(x)$ gives same $\overrightarrow{B}$ and $\overrightarrow{E}$ fields This property is usually called the **gauge invariance**.

Lagrangian density given by,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (\overrightarrow{E}^2 - \overrightarrow{B}^2)$$

will give Maxwell equations a la Euler-Lagrange equations.. To see this we compute

$$\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu\right)} = -\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right), \qquad \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

Then

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A_\nu\right)} = \frac{\partial \mathcal{L}}{\partial A_\nu}, \qquad \Longrightarrow \partial_\mu \left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \partial_\mu F^{\mu\nu} = 0$$

These are indeed the Maxwell equations as we discussed before.
Conjugate momenta

$$\pi_0 = \frac{\partial L}{\partial(\partial_0 A_0)} = 0, \qquad \pi^i(x) = \frac{\partial L}{\partial(\partial_0 A_i)} = -F^{0i} = E^i$$

No conjugate momenta for $A_0 \Longrightarrow$ not a dynamical degree of freedom.
Hamiltanian density,

$$\mathcal{H} = \pi^k \dot{A}_k - \mathcal{L} = \frac{1}{2} (\overrightarrow{E}^2 + \overrightarrow{B}^2) + (\overrightarrow{E} \cdot \nabla) A_0$$

Using $\overrightarrow{\nabla} \cdot \overrightarrow{E} = 0$,Hamiltonian becomes,

$$H = \int d^3x \mathcal{H} = \frac{1}{2} \int d^3x (\overrightarrow{E}^2 + \overrightarrow{B}^2)$$

Impose commutation relation,

$$[\pi^i(\overrightarrow{x}, t),\ A^j(\overrightarrow{y}, t)] = -i\delta_{ij}\delta^3(\overrightarrow{x} - \overrightarrow{y}), \quad \ldots$$

But this is not consistent with $\overrightarrow{\nabla} \cdot \vec{E} = 0$ because

$$[\nabla \cdot E(x, t),\ A_j(x, t)] = -i\partial_j \delta^3(x - y) \neq 0$$

$\delta-$function in momentum space

$$\partial_j \delta^3(\overrightarrow{x} - \overrightarrow{y}) = i \int \frac{d^3k}{(2\pi)^3} e^{i\overrightarrow{k} \cdot (\overrightarrow{x} - \overrightarrow{y})} k_j$$

To get zero for the commutator of $\nabla \cdot E$, replace,

$$\delta_{ij}\delta^3(\vec{x} - \vec{y}) \longrightarrow \delta_{ij}^{tr}(\vec{x} - \vec{y}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot (\vec{x} - \vec{y})} (\delta_{ij} - \frac{k_i k_j}{k^2})$$

then

$$\partial_i \delta_{ij}^{tr} \delta^3(\vec{x} - \vec{y}) = i \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}(\cdot\vec{x} - \vec{y})} k_i (\delta_{ij} - \frac{k_i k_j}{k^2}) = 0$$

So commutator is modified to,

$$[E^i(x,t),\ A_j(y,t)] = -i\delta^{tr}_{ij}(\vec{x}-\vec{y})$$

which implies

$$[E^i(x,t),\ \overrightarrow{\nabla}\cdot\vec{A}(y,t)] = 0$$

Now that A_0 and $\overrightarrow{\nabla}\cdot\vec{A}$ commute with all operators, they must be C-number. Choose a gauge such that

$$A_0 = 0 \; and \; \nabla\cdot\vec{A} = 0 \qquad \text{radiation gauge}$$

In this gauge

$$\pi^i = \partial^i A^0 - \partial^0 A^i = -\partial^0 A^i$$

$$[\partial_0 A^i(\vec{x},t),\ A^j(\vec{y},t)] = i\delta^{tr}_{ij}(\vec{x}-\vec{y})$$

Mode expansion

Equation of motion $\partial_\nu F^{\mu\nu} = 0$ gives

$$\partial_\nu \left(\partial^\nu A^\mu - \partial^\mu A^\nu \right) = \Box A^\mu - \partial^\mu \left(\partial_\upsilon A^\upsilon \right) = 0$$

In radiaiton gauge ,

$$A_0 = 0, \qquad \overrightarrow{\nabla} \cdot \overrightarrow{A} = 0$$

wave equation becomes

$$\Box \, \overrightarrow{A} = 0 \qquad \text{massless Klein-Gordon equation}$$

General solution

$$\overrightarrow{A}(\vec{x}, t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_\lambda \overrightarrow{\epsilon}(\vec{k}, \lambda)[a(k, \lambda)e^{-ikx} + a^\dagger(k, \lambda)e^{ikx}] \qquad w = k_0 = |\overrightarrow{k}|$$

Only two degrees of freedom

$$\vec{\epsilon}(k, \lambda), \lambda = 1, 2 \quad \text{with } \vec{k} \cdot \vec{\epsilon}(k, \lambda) = 0$$

Standard choice

$$\vec{\epsilon}(k, \lambda) \cdot \vec{\epsilon}(k, \lambda') = \delta_{\lambda\lambda'}, \quad \vec{\epsilon}(-k, 1) = -\vec{\epsilon}(k, 1), \quad \vec{\epsilon}(-k, 2) = \vec{\epsilon}(-k, 2)$$

Solve for $a(k,\lambda)$ and $a^+(k,\lambda)$

$$a(k,\lambda) = i\int \frac{d^3x}{\sqrt{(2\pi)^3 2\omega}}[e^{ik\cdot x}\overleftrightarrow{\partial_0}\vec{\epsilon}(k,\lambda)\cdot\vec{A}(x)]$$

$$a^\dagger(k,\lambda) = -i\int \frac{d^3x}{\sqrt{(2\pi)^3 2\omega}}[e^{-ik\cdot x}\overleftrightarrow{\partial_0}\vec{\epsilon}(k,\lambda)\cdot\vec{A}(x)]$$

Commutation relations,

$$[a(k,\lambda),\ a^\dagger(k',\lambda')] = \delta_{\lambda\lambda'}\delta^3(\vec{k}-\vec{k}'), \quad [a(k,\lambda),\ a(k',\lambda')] = 0,$$

Hamiltonian and momentum operators are

$$H = \frac{1}{2}\int d^3x : (E^2+B^2) := \int d^3k\omega\sum_\lambda a^+(k,\lambda)a(k,\lambda)$$

$$\vec{P} = \int d^3x : E\times B := \int d^3k\vec{k}\sum_\lambda a^+(k,\lambda)a(k,\lambda)$$

The vaccum is defined by

$$a(\vec{k},\lambda)|0>=0 \quad \forall\vec{k},\lambda$$

and one photon state with momentum k polarization λ is given by, $a^\dagger(\vec{k},\lambda)|0>$.

Appendix 1– Simple Harmonic Oscillator

Here we review the creation and annihilation operators in the simple harmonic oscillator in one dimension. The Hamiltonian is

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2$$

where for convenience we have set $m = 1$. Here p, x satisfy the comutation relation,

$$[x, p] = i$$

Define

$$a = \sqrt{\frac{1}{2\omega}}\left(\omega x + ip\right), \qquad a^{\dagger} = \sqrt{\frac{1}{2\omega}}\left(\omega x - ip\right)$$

The commutator is ,

$$\left[a, a^{\dagger}\right] = \frac{1}{2\omega}\left[\omega x + ip,\ \omega x - ip\right] = 1$$

From

$$x = \frac{1}{\sqrt{2\omega}}\left(a + a^{\dagger}\right), \qquad p = -i\sqrt{\frac{\omega}{2}}\left(a - a^{\dagger}\right)$$

we get for the Hamiltonian

$$H = \frac{1}{2}[-\frac{\omega}{2}\left(a - a^{\dagger}\right)^2 + \frac{\omega^2}{2\omega}\left(a + a^{\dagger}\right)^2 = \frac{\omega}{2}\left(a^{\dagger}a + aa^{\dagger}\right)$$

Using the commutation relation we can write H as

$$H = \omega \left(a^\dagger a + \frac{1}{2} \right)$$

The second term here is called the zero-point energy.

We can compute the commutator of H with a or $a^\dagger$,

$$[H,\ a] = -\omega a, \qquad \left[H,\ a^\dagger\right] = \omega a^\dagger$$

Suppose $|E\rangle$ is eigenstate of Hamiltoian with eigenvalue E,

$$H\,|E\rangle = E\,|E\rangle$$

Then we get

$$\left(Ha^\dagger - a^\dagger H\right)|E\rangle = \omega a^\dagger\,|E\rangle\,, \qquad \Longrightarrow \qquad H\left(a^\dagger\,|E\rangle\right) = (E+\omega)\left(a^\dagger\,|E\rangle\right)$$

Thus $a^\dagger$ increases the energy eigenvalue by ω and is called **raising operator** (or **creation operator**). Similarly,

$$(Ha - aH)\,|E\rangle = -\omega a\,|E\rangle\,, \qquad \Longrightarrow \qquad H\,(a\,|E\rangle) = (E-\omega)\,(a\,|E\rangle)$$

which implies that the operator a decreaes the energy eigenvalue by ω.Since H is bounded below, there must exist a state with lowest energy eigen value, the ground state $|0\rangle$, defined by

$$a\,|0\rangle = 0$$

will have energy eigen value

$$H|0\rangle = \frac{1}{2}\omega|0\rangle$$

It is clear that the excited states are related to $|0\rangle$ by the action of $a^\dagger$. For example,

$$H|n\rangle = \left(n + \frac{1}{2}\right)\omega|n\rangle, \qquad \text{where} \qquad |n\rangle = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}|0\rangle$$

The state $|n\rangle$ can be interpreted as state with n quanta, each with energy ω. So the operator $N = a^\dagger a$ is the number operator.

Appendix 2–U(1) local symmetry

The free Maxwll's equations are

$$\overrightarrow{\nabla}\cdot\overrightarrow{B}=0,\quad \overrightarrow{\nabla}\times\overrightarrow{E}+\frac{\partial\overrightarrow{B}}{\partial t}=0,$$

$$\overrightarrow{B}=\nabla\times\overrightarrow{A},\qquad \overrightarrow{E}=-\nabla\phi-\frac{\partial\overrightarrow{A}}{\partial t}$$

Solve the first two equations by introducing $\overrightarrow{A},\phi$

$$\overrightarrow{B}=\nabla\times\overrightarrow{A},\qquad \overrightarrow{E}=-\nabla\phi-\frac{\partial\overrightarrow{A}}{\partial t}\tag{5}$$

Convenient to write

$$F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu\quad\text{with}\quad F^{0i}=\partial^0A^i-\partial^iA^0=-E^i,\quad F^{ij}=\partial^iA^j-\partial^jA^i=-\epsilon_{ijk}B_k$$

For a charged particle moving in electromagnetic field, the equation of motion is,

$$m\frac{d^2\overrightarrow{x}}{dt^2}=e\left(\overrightarrow{E}+\overrightarrow{v}\times\overrightarrow{B}\right)$$

The Lagrangian for these equations is

$$L = \frac{1}{2} m \left(\vec{v}\right)^2 + e\vec{A} \cdot \vec{v} - eA_0$$

To see this, we compute the derivatives with respect to $\vec{x}$ and $\vec{v}$,

$$\frac{\partial L}{\partial v_i} = mv_i + eA_i, \quad \frac{\partial L}{\partial x_i} = e\frac{\partial A_j}{\partial x_i} v_j - e\frac{\partial A_0}{\partial x_i}$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = m\frac{dv_i}{dt} + e\frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt} + e\frac{\partial A_i}{\partial t}$$

Euler-Lagrange equation gives

$$m\frac{dv_i}{dt} + e\frac{\partial A_i}{\partial x_j}\frac{dx_j}{dt} + e\frac{\partial A_i}{\partial t} = e\frac{\partial A_j}{\partial x_i} v_j - e\frac{\partial A_0}{\partial x_i}$$

On the other hand,

$$\left(\vec{v} \times \vec{B}\right)_i = \varepsilon_{ijk} v_j B_k = \varepsilon_{ijk} v_j \varepsilon_{klm} \partial_l A_m = v_j \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) \partial_l A_m = v_j \left(\partial_i A_j - \partial_j A_i\right)$$

Then we get

$$m\frac{dv_i}{dt} = -e\frac{\partial A_i}{\partial x_j} v_j - e\frac{\partial A_i}{\partial t} + e\frac{\partial A_j}{\partial x_i} v_j - e\frac{\partial A_0}{\partial x_i}$$

Or

$$m\frac{dv_i}{dt} = e\left(\partial_i A_j - \partial_j A_i\right) v_j + e\left(-\partial_i A_0 - \partial_0 A_i\right) = e\left(\overrightarrow{E} + \overrightarrow{v} \times \overrightarrow{B}\right)_i$$

which is the correct equation of motion.
From Lagrangian define the conjugate momentum,

$$p_i = \frac{\partial L}{\partial v_i} = mv_i + eA_i, \qquad \Longrightarrow \qquad v_i = \frac{1}{m}\left(p_i - eA_i\right)$$

The Hamiltonian is then

$$\begin{aligned} H &= p_i v_i - L = p_i v_i - \frac{1}{2}m\left(\overrightarrow{v}\right)^2 - e\overrightarrow{A}\cdot\overrightarrow{v} + eA_0 \\ &= \frac{1}{2m}\left(\overrightarrow{p} - e\overrightarrow{A}\right)^2 + eA_0 \end{aligned}$$

The Schrodinger equation for a charged particle moving in the electromagnetic field is,

$$[-\frac{1}{2m}\left(\overrightarrow{\nabla} - ie\overrightarrow{A}\right)^2 + eA_0]\psi = i\frac{\partial\psi}{\partial t}$$

This shows that it is the potentials $\overrightarrow{A}, A_0$,not the $\overrightarrow{E}$, $\overrightarrow{B}$ fields show up in the Schrodinger equation. However, Schrodinger equation is not invariant under the gauge transformation,

$$A^\mu \longrightarrow A^\mu + \partial^\mu\alpha, \qquad \text{or} \qquad \overrightarrow{A} \longrightarrow \overrightarrow{A} - \overrightarrow{\nabla}\alpha, \qquad A_0 \longrightarrow A_0 + \partial_0\alpha$$

But it turns out that we can recover the Schrodinger equation if we also change the wave function ψ by a phase,

$$\psi \longrightarrow \psi' = e^{-ie\alpha}\psi$$

This can be seen as follows. Define the covariant derivative as

$$\vec{D}\psi = \left(\vec{\partial} - ie\vec{A}\right)\psi$$

The covariant derivative for the new fields is then,

$$\begin{aligned}\vec{D}\psi' &= \left(\vec{\partial} - ie\vec{A}'\right)\psi' = e^{-ie\alpha}[\vec{\partial} - ie\vec{\nabla}\alpha - ie\left(\vec{A} - \vec{\nabla}\alpha\right)]\psi \\ &= e^{-ie\alpha}\left(\vec{D}\psi\right)\end{aligned}$$

So the covariant derivative $\vec{D}\psi$ transforms by a phase in the same way as the field ψ.In other words, the covariant derivative $\vec{D} = \left(\vec{\partial} - ie\vec{A}\right)$ does not change the transformation property of the object it acts on. It is then easy to see that

$$\vec{D}^2\psi' = e^{-ie\alpha}\left(\vec{D}^2\psi\right)$$

For the time derivative, we have

$$D_0\psi = (\partial_0 + ieA_0)\,\psi$$

and

$$D_0\psi' = e^{-ie\alpha}\left(\partial_0 + ie\partial_0\alpha - ieA_0 - ie\partial_0\alpha\right)\psi = e^{-ie\alpha}D_0\psi$$

With this phase transformation, the Schrodinger equation

$$[-\frac{1}{2m}\left(\vec{\nabla} - ie\vec{A}'\right)^2 + eA_0']\psi' = i\frac{\partial\psi'}{\partial t}$$

becomes

$$e^{-ie\alpha}[-\frac{1}{2m}\left(\vec{\nabla} - ie\vec{A}\right)^2 + eA_0]\psi = e^{-ie\alpha}i\frac{\partial\psi}{\partial t}$$

After cancelling out the phase $e^{-ie\alpha}$, we get back the original Schrodinger equation.
The phase transformation of the wave function is a symmetry transformation and is a local symmetry because $\alpha = \alpha\left(\vec{x}, t\right)$. The phase transformation in usually referred to as $U(1)$ transformation and we call the elecromagnetic possesses $U(1)$ local symmetry.