QFT Perturbation Theory

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Interaction Theory

Theories with only quardratic terms or lower contain only free fields. $\Longrightarrow$ terms cubic or higher have non-trivial interactions.
To introduce interaction terms we need guidence from symmetry principles.
As an illustration, take electromagnetic interaction. Lagrangian density is

$$\mathcal{L} = \overline{\psi}\left(x\right)\gamma^{\mu}\left(i\partial_{\mu} - eA_{\mu}\right)\psi\left(x\right) - m\overline{\psi}\left(x\right)\psi\left(x\right) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

The term $\bar{\psi}\gamma^{\mu}eA_{\mu}\psi$ describes the interaction between electrons and photons.
Equations of motion

$$\begin{aligned} \left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi\left(x\right) &= eA_{\mu}\gamma^{\mu}\psi \qquad \text{non-linear coupled equations} \\ \partial_{\upsilon}F^{\mu\nu} &= e\overline{\psi}\gamma^{\mu}\psi \end{aligned}$$

No exact solutions are known. If we expand the fields in terms of Fourier components,

$$\psi\left(\vec{x}, t\right) = \sum_{s}\int \frac{d^{3}p}{\sqrt{\left(2\pi\right)^{3}2E_{p}}}\left[b\left(p, s, t\right)u\left(p, s\right)e^{-i\vec{p}\cdot\vec{x}} + d^{\dagger}\left(p, s, t\right)\upsilon\left(p, s\right)e^{i\vec{p}\cdot\vec{x}}\right]$$

Then the operators b and $d^{\dagger}$ will be time dependent controlled by the interaction terms.
Quantization
Write $\mathcal{L} = \mathcal{L}_{0} + \mathcal{L}_{int}$

$$\begin{aligned}\mathcal{L}_0 &= \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ \mathcal{L}_{int} &= -e\overline{\psi}\gamma^\mu\psi A_\mu\end{aligned}$$

where $\mathcal{L}_0$,free field part, while $\mathcal{L}_{int}$ interaction.
Conjugate momenta

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_0\psi_\alpha\right)} = i\psi^\dagger_\alpha\left(x\right)$$

For electromagnetic fields, choose the gauge

$$\overrightarrow{\nabla}\cdot\overrightarrow{A} = 0\ , \qquad \text{Radiaiton Gauge}$$

then

$$\pi^i = \frac{\partial\mathcal{L}}{\partial\left(\partial_0 A^i\right)} = -F^{0i} = E^i$$

From equation of motion

$$\partial_\nu F^{0\nu} = e\psi^\dagger\psi \qquad \Longrightarrow \qquad -\nabla^2 A^0 = e\psi^\dagger\psi$$

We can't take A_0 to be zero but A^0 can be expressed in terms of other field,

$$A^0 = e\int d^3x'\,\frac{\psi^\dagger\left(x',t\right)\psi\left(x',t\right)}{4\pi|\overrightarrow{x'}-\overrightarrow{x}|} = e\int\frac{d^3x'\rho\left(x',t\right)}{4\pi|\overrightarrow{x}-\overrightarrow{x'}|}$$

Commutation relations

$$\left\{\psi_\alpha\left(\overrightarrow{x},t\right),\psi_\beta^\dagger\left(\overrightarrow{x'},t\right)\right\} = \delta_{\alpha\beta}\delta^3\left(\overrightarrow{x}-\overrightarrow{x'}\right) \qquad \left\{\psi_\alpha\left(\overrightarrow{x},t\right),\psi_\beta\left(\overrightarrow{x'},t\right)\right\} = \ldots = 0$$

$$\left[\dot{A}_i\left(\overrightarrow{x},t\right),A_j\left(\overrightarrow{x'},t\right)\right] = i\delta_{ij}^{tr}\left(\overrightarrow{x}-\overrightarrow{x'}\right)$$

Commutators with A_0 ,

$$\begin{aligned}\left[A_0\left(\overrightarrow{x},t\right),\psi_\alpha(\overrightarrow{x'},t)\right] &= e\int\frac{d^3x''}{4\pi|\overrightarrow{x}-\overrightarrow{x''}|}\left[\psi^\dagger(\overrightarrow{x''},t)\psi(\overrightarrow{x''},t),\psi_\alpha\left(\overrightarrow{x'},t\right)\right] \\ &= -\frac{e}{4\pi}\frac{\psi_\alpha\left(\overrightarrow{x'},t\right)}{|\overrightarrow{x}-\overrightarrow{x'}|}\end{aligned}$$

We have used the relation

$$\nabla^2\frac{1}{4\pi|\overrightarrow{x}-\overrightarrow{x'}|} = -\delta^3\left(\overrightarrow{x}-\overrightarrow{x'}\right)$$

This can be seen as follows. In the spherical coordinates the Laplacian is of the form,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

From this we get

$$\nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{(-1)}{r^2}\right) = 0, \qquad \text{if } \; r\neq 0$$

To deal with the singularity at $r \to 0$, consider the integral

$$\int d^3x\ \nabla^2 \left(\frac{1}{r}\right) = \int d^3x\ \vec{\nabla} \cdot [\vec{\nabla} \left(\frac{1}{r}\right)] = \oint d\vec{S} \cdot\ [\vec{\nabla} \left(\frac{1}{r}\right)]$$

We can take the surface to be the surface of a sphere with radius a centered around the origin so that $d\vec{S} = a^2 \hat{r} d\Omega$. Using $\vec{\nabla} \left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$, we get

$$\int d^3x\ \nabla^2 \left(\frac{1}{r}\right) = \int d\Omega a^2 \hat{r} (-\frac{\hat{r}}{a^2}) = -4\pi$$

From these we see that

$$\nabla^2 (\frac{1}{4\pi r}) = -\delta^3 \left(\vec{r}\right)$$

Hamiltonian

$$H = \int d^3x \mathcal{H} = \int d^3x \{\psi^\dagger \left[\vec{\alpha} \cdot (-i\vec{\nabla} - e\vec{A}) + \beta m\right] \psi + \frac{1}{2} \left(\vec{E}^2 + \vec{B}^2\right)\}$$

No A_0 in the interaction. If we write

$$\vec{E} = \vec{E_l} + \vec{E_t} \qquad \text{where} \quad \vec{E_l} = -\vec{\nabla} A_0 \qquad , \vec{E_t} = -\frac{\partial \vec{A}}{\partial t}$$

Then

$$\frac{1}{2} \int d^3x \left(\vec{E}^2 + \vec{B}^2\right) = \frac{1}{2} \int d^3x \vec{E_l}^2 + \int d^3x \left(\vec{E_t}^2 + \vec{B}^2\right)$$

The longitudinal part is

$$\frac{1}{2}\int d^3x \overrightarrow{E_l}^2 = -\frac{1}{2}\int d^3x A_0 \nabla^2 A_0 = \frac{e^2}{4\pi}\int d^3x d^3y \frac{\rho\left(\overrightarrow{x}, t\right)\rho\left(\overrightarrow{y}, t\right)}{\left|\overrightarrow{x} - \overrightarrow{y}\right|}$$

Perturbation Theory

Can't solve the classical equations of motion. Can't do mode expansion to introduce a and $a^\dagger$.The only approximation we can do in field theory is the perturbation theory.
We will now set up the framework for the perturbation.

Physical states

In high energy physics, we study the scattering processes.
Assume interactions all short-range, far away from interaction region, particles propagate as free particles.
Choose the physical states to be eigensates of energy momentum operators,

$$P_\mu|\Psi\rangle = p_\mu|\Psi\rangle$$

Satisfy requirements;

1. eigenvalues p_μ all in forward light cone,

$$p^2 = p_\mu p^\mu \geqslant 0, \qquad p_0 \geqslant 0$$

2. non-degenerate Lorentz invariant ground state $|0>$ with zero energy ,

$$p^0|0\rangle = 0, \qquad \Longrightarrow \overrightarrow{p}|0\rangle = 0$$

3. There exists stable single particle states $|\overrightarrow{p_i}\rangle$ with $p_i^2 = m_i^2$ for each stable particle.
4. vaccum and one particle states form discrete spectra in p^μ

assume interactions do not violently the spectrum of states. there is no room to describe bound states

In-fields and in-states—asymptotic conditions

Consider

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{\mu_0^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

Equation of motion

$$\left(\Box + \mu_0^2\right)\phi = j\left(x\right) = \frac{\lambda}{3!}\phi^3$$

conjugate momenta

$$\pi\left(x\right) = \frac{\partial \mathcal{L}}{\partial_0 \phi} = \partial_0 \phi$$

Commutation relations

$$\left[\pi\left(x,t\right), \phi\left(y,t\right)\right] = -i\delta^3\left(x-y\right) \qquad \left[\pi\left(x,t\right), \pi\left(y,t\right)\right] = \left[\phi\left(x,t\right), \phi\left(y,t\right)\right] = 0$$

At $t = -\infty$, $\phi_{in}\left(x\right)$ creates free particle propagating with physical mass μ.

$$\left(\Box + \mu^2\right)\phi_{in}\left(x\right) = 0$$

we allow physical mass μ to be different from μ_0 .
Assume that $\phi_{in}\left(x\right)$ transforms same way as $\phi\left(x\right)$. In particular,

$$\left[p_\mu, \phi_{in}\left(x\right)\right] = -i\partial_\mu \phi_{in}\left(x\right)$$

$\phi_{in}\left(x\right)$ creates one particle state from vacuum.
Expand $\phi_{in}\left(x\right)$ in terms of free solution of Klein-Gordon equation,

$$\phi_{in}\left(x\right) = \int d^3k\left[a_{in}\left(k\right) f_k\left(x\right) + a_{in}^\dagger\left(k\right) f_k^*\left(x\right)\right] \qquad f_k\left(x\right) = \frac{1}{\sqrt{\left(2\pi\right)^3 2w_k}} e^{-ik\cdot x}$$

Invert this expansion

$$a_{in}(k) = i\int d^3x f_k^*(x)\overleftrightarrow{\partial_0}\phi_{in}(x)$$

We also have

$$\left[p^\mu, a_{in}(k)\right] = -k^\mu a_{in}(k), \qquad \left[p^\mu, a_{in}^\dagger(k)\right] = k^\mu a_{in}^\dagger(k)$$

States are defined by

$$|k_1, in\rangle = \sqrt{(2\pi)^3 2w_k} a_{in}^\dagger(k)|0\rangle$$

$$|k_1, k_2, ...k_n|in\rangle = \left[\prod_i \sqrt{(2\pi)^3 2w_{k_i}} a_{in}^\dagger(k_i)\right]|0\rangle$$

With normalization

$$\langle k_2, in|k_1, in\rangle = (2\pi)^3 2w_1\delta^3\left(\overrightarrow{k_1} - \overrightarrow{k_2}\right)$$

$$\langle p_1, p_2, ..., p_m, in|k_1, k_2, ...k_n|in\rangle = 0$$

unless m=n and $(p_1, p_2, ..., p_m)$ conicides with $(k_1, k_2, ...k_n)$

Relation between $\phi_{in}(x)$ and $\phi(x)$

The field equations for these fields

$$\left(\Box + \mu_0^2\right)\phi(x) = j(x)$$

Or

$$\left(\Box + \mu^2\right)\phi(x) = j(x) + \delta\mu^2\phi(x) = \widetilde{j(x)} \qquad \delta\mu^2 = \mu^2 - \mu_0^2$$

$$\left(\Box + \mu^2\right)\phi_{in}(x) = 0$$

Formally relates $\phi(x)$ to $\phi_{in}(x)$ by Green's function,

$$\sqrt{z}\phi_{in}(x) = \phi(x) - \int d^4y \, \triangle_{ret}\left(x-y,\mu^2\right)\widetilde{j(y)}$$

where

$$\left(\Box_x + \mu^2\right)\triangle_{ret}\left(x-y,\mu^2\right) = \delta^4(x-y) \ , \ \triangle_{ret}\left(x-y,\mu^2\right) = 0 \quad \text{for} \quad x_0 < y_0$$

This suggests that as $x_0 \to -\infty$, $\phi(x) \to \sqrt{z}\phi_{in}(x)$. It turns out that this is not correct.

Correct asymptotic condition (Lehmann, Symanzik, and Zimmermann**)**

Let $|\alpha\rangle, |\beta\rangle$ be any two normalizable states, $\phi^f(t)$ is defined

$$\phi^f(t) \equiv i\int d^3x f_k^*\left(\overrightarrow{x},t\right)\overleftrightarrow{\partial_0}\phi\left(\overrightarrow{x},t\right) \quad \text{with} \quad \left(\Box + \mu^2\right)f = 0$$

$f_k\left(\overrightarrow{x},t\right)$ is an arbitrary ***normalizable*** solution to Klein-Gordon equation. Then

The correct asymptotic condition is

$$\lim_{x_0\to-\infty}\langle\alpha|\phi^f(t)|\beta\rangle = \sqrt{z}\langle\alpha|\phi_{in}^f(t)|\beta\rangle \quad \text{with} \quad \phi_{in}^f(t) = i\int d^3x f^*\left(\overrightarrow{x},t\right)\overleftrightarrow{\partial_0}\phi_{in}\left(\overrightarrow{x},t\right)$$

this is a weak convergence relation.

Out fields and out states

Similar procedure applies to ϕ_{out}

$$\left(\Box + \mu_0^2\right) \phi_{out}(x) = 0$$

$$\phi_{out}(x) = \int d^3k \left[a_{out}(k) f_k(x) + a^{\dagger}_{out}(k) f_k^*(x)\right],$$

$$\left[p^{\mu}, a^{\dagger}_{out}(k)\right] = -k^{\mu} a^{\dagger}_{out}(k)$$

Asymptotic condition

$$\lim_{t\to\infty} \langle\alpha|\phi^f(t)|\beta\rangle = \sqrt{z}\langle\alpha|\phi^f_{out}(t)|\beta\rangle$$

S-matrix

Scattering processes: start n non-interacting particles.They interact when close to each other. After interaction, m particles seperate

Initial state

$$|p_1, p_2, ..., p_n, in\rangle = |\alpha, in\rangle$$

Final state

$$|p_1', p_2', ..., p_m', out\rangle = |\beta, out\rangle$$

S-matrix

$$S_{\beta\alpha} \equiv \langle\beta, out|\alpha, in\rangle$$

Introduce S-operator which will take an $in - state$ and turn it into $out - state$,

$$\langle\beta, out| \equiv \langle\beta, in|S \qquad \langle\beta, out|S^{-1} = \langle\beta, in|$$

then the $S-matrix$ element can be written as a matrix element of $S-$operator between 2 $in-states$,

$$S_{\beta\alpha} = \langle\beta, out|\alpha, in\rangle = \langle\beta, in|S|\alpha, in\rangle$$

Properties of S-matrix

1. stability of vacuum $\Longrightarrow$ $|S_{00}| = 1$

$$\langle 0, in|S = \langle 0, out| = e^{-i\varphi}\langle 0, in|$$

2. Stability of one-particle state

$$\langle p, in|S|p, in\rangle = \langle p, out|p, in\rangle = 1 \qquad \because |p, in\rangle = |p, out\rangle$$

3.

$$\phi_{in}(x) = S\phi_{out}(x)\, S^{-1}$$

4. Unitarity $SS^{\dagger} = S^{\dagger}S = 1$

$$\langle\alpha, in|S = \langle\alpha, out|\ , \qquad \Longrightarrow \ S^{\dagger}|\alpha, in\rangle = |\alpha, out\rangle$$

5. S is translational and Lorentz invariance

$$U(\Lambda, b)\, SU^{-1}(\Lambda, b) = S$$

LSZ reduction

set up the framework to compute $S_{\beta\alpha}$.

Consider

$$S_{\beta,\alpha p} = \langle \beta, out | \alpha, p, in \rangle$$

Using creation operator for the $in - state$,

$$\begin{aligned} S_{\beta,\alpha p} &= \langle \beta, out|\alpha, p, in\rangle = \sqrt{(2\pi)^3 2w_p}\langle \beta, out| a^\dagger_{in}(p)|\alpha, in\rangle \\ &= \sqrt{(2\pi)^3 2w_p}\langle \beta, out| a^\dagger_{out}(p)|\alpha, in\rangle + \langle \beta, out|\left[a^\dagger_{in}(p) - a^\dagger_{out}(p)\right]|\alpha, in\rangle] \\ &= N\left[\langle \beta - p, out|\alpha, in\rangle - i\langle \beta, out| \int d^3x f_p(x) \overleftrightarrow{\partial_0} \left[\phi_{in}(x) - \phi_{out}(x)\right]\alpha, in\rangle\right] \end{aligned}$$

Here $\langle \beta - p, out|$ is state $\langle \beta, out|$ by removing a particle with momentum $\overrightarrow{p}$ and $N = \sqrt{(2\pi)^3 2w_p}$

Use the symptotic conditions

$$\langle \alpha|\phi_{in}(x)|\beta\rangle = \frac{1}{\sqrt{z}}\lim_{t\to-\infty}\langle \alpha|\phi(x)|\beta\rangle, \qquad \langle \alpha|\phi_{out}(x)|\beta\rangle = \frac{1}{\sqrt{z}}\lim_{t\to\infty}\langle \alpha|\phi(x)|\beta\rangle$$

and the identity

$$\left(\lim_{x_0\to\infty} - \lim_{x_0\to-\infty}\right)\int d^3x g_1(x)\overleftrightarrow{\partial_0} g_2(x) = \int_{-\infty}^{\infty} d^4x\left[g_1(x)\partial_0^2 g_2(x) - \partial_0^2 g_1(x) g_2(x)\right]$$

we get

$$\int d^3x f_p(x)\overleftrightarrow{\partial_0}\left[\phi_{in}(x)-\phi_{out}(x)\right] = \int d^4x\left[\partial_0^2 f_p(x)\phi(x)-f_p(x)\partial_0^2\phi(x)\right]$$
$$= -\int d^4x f_p(x)\ \left(\Box+\mu^2\right)\phi(x)$$

we get the **reduction formula**,

$$\langle\beta,out|\alpha,p,in\rangle = N\langle\beta-p,out|\alpha,in\rangle+\frac{i}{\sqrt{z}}\int e^{-ip\cdot x}d^4x\left(\Box+\mu^2\right)\langle\beta,out|\phi(x)|\alpha,in\rangle$$

To remove a particle with momentum p' from β

$$\langle\beta,out|\phi(x)|\alpha,in\rangle = \langle\gamma p',out|\phi(x)|\alpha,in\rangle = N\langle\gamma,out|a_{out}(p')\phi(x)|\alpha,in\rangle$$

$$= \left[\langle\gamma,out|\phi(x)a_{in}(p')|\alpha,in\rangle-\langle\gamma,out|(a_{out}(p')\phi(x)-\phi(x)a_{in}(p')|\alpha,in\rangle\right]$$

$$= \langle\gamma,out|\phi(x)|\alpha-p',in\rangle - i\int d^3y\langle\gamma,out|(\phi_{out}(y)\phi(x)-\phi(x)\phi_{in}(y))|\alpha,in\rangle\overleftrightarrow{\partial_0}f^*_{p'}(y)$$

$$= \langle\gamma,out|\phi(x)|\alpha-p',in\rangle$$
$$-\frac{i}{\sqrt{z}}\int d^3y\left(\lim_{y_0\to\infty}-\lim_{y_0\to-\infty}\right)\langle\gamma,out|\left(T(\phi(y)\phi(x))\right)|\alpha,in\rangle\overleftrightarrow{\partial_0}f^*_{p'}(y)$$

same procedure as before

$$\begin{aligned}\langle \beta, out|\phi(x)|\alpha, in\rangle &= \{\langle \gamma, out|\phi(x)|\alpha - p', in\rangle\} \\ &+ \frac{i}{\sqrt{z}}\int d^4y\langle \gamma, out|T(\phi(y)\phi(x))|\alpha, in\rangle\left(\overleftarrow{\Box}_y + \mu^2\right)e^{ip\cdot x}\end{aligned}$$

remove all particles from "in" and "out" state

$$\begin{aligned}\langle p_1, ..., p_n, out|q_1, ..., q_m, in\rangle &= \left(\frac{i}{\sqrt{z}}\right)^{m+n}\prod_{i=1}^{m}\prod_{j=1}^{n}\int d^4x_i d^4y_j e^{-iq_i x_i}\left(\overrightarrow{\Box}_x + \mu^2\right) \\ &\langle 0|T(\phi(y_1)...\phi(y_m)\phi(x_1)...\phi(x_m))|0\rangle\left(\overleftarrow{\Box}_{y_j} + \mu^2\right)e^{ip_j\cdot x_j}\end{aligned}$$

for all $p_j \neq q_i$

In and Out fields for Fermions

generalization to fermions.
in-field

$$\psi_{in}(x) = \int d^3p \sum_s \left[b_{in}(p,s)\, U_{p,s}(x) - d_{in}^{\dagger}(p,s)\, V_{p,s}(x) \right]$$

where

$$U_{p,s}(x) = \frac{1}{\sqrt{(2\pi)^3 2E_p}} u(p,s)\, e^{-ip\cdot x} \qquad V_{p,s}(x) = \frac{1}{\sqrt{(2\pi)^3 2E_p}} \upsilon(p,s)\, e^{ip\cdot x}$$

Inversion

$$b_{in}(p,s) = \int d^3x U_{p,s}^{\dagger}(x)\, \psi_{in}(x) \qquad d_{in}(p,s) = \int d^3x \psi_{in}^{\dagger}(x)\, V_{p,s}(x)$$

$$b_{in}^{\dagger}(p,s) = \int d^3x \psi_{in}^{\dagger}(x)\, U_{p,s}(x) \qquad d_{in}^{\dagger}(p,s) = \int d^3x V_{p,s}^{\dagger}(x)\, \psi_{in}(x)$$

Reduction formula for fermions

1. remove electron from in-state

$$\langle \beta, out | \alpha; ps, in \rangle = -\frac{i}{\sqrt{z_2}} \int d^4x \langle \beta, out | \overline{\psi_\alpha}(x) | \alpha, in \rangle \overleftarrow{\left(-i\gamma^\mu \partial_\mu - m\right)}_{\alpha\beta} u(p,s)\, e^{-ip\cdot x}$$

2. remove positron(anti-particle) from in-state

$$\langle \beta, out|\alpha; \overline{ps}, in\rangle = \frac{i}{\sqrt{z_2}} \int d^4x e^{-i\overline{p}\cdot x} \overline{v_\alpha} (\overline{p}, \overline{s}) \overrightarrow{(i\gamma^\mu \partial_\mu - m)}_{\alpha\beta} \langle \beta, out|\psi_\beta (x) |\alpha, in\rangle$$

3. remove electron from out-state

$$\langle \beta; p's', out|\alpha, in\rangle = -\frac{i}{\sqrt{z_2}} \int d^4x \overline{u_\alpha} (p', s') e^{ip'\cdot x} \overrightarrow{(i\gamma^\mu \partial_\mu - m)}_{\alpha\beta} \langle \beta, out|\psi_\beta (x) |\alpha, in\rangle$$

4. remove positron from out-state

$$\langle \beta; \overline{p}'\overline{s}', out|\alpha, in\rangle = \frac{i}{\sqrt{z_2}} \int d^4x \langle \beta, out|\overline{\psi_\alpha} (x) |\alpha, in\rangle \overleftarrow{(-i\gamma^\mu \partial_\mu - m)}_{\alpha\beta} v (\overline{p},' \overline{s}') e^{-i\overline{p}'\cdot x}$$

U matrix

In perturbation theory want to find the relation between interacting fields $\phi(x)$, $\pi(x)$ and the free fields $\phi_{in}(x)$, $\pi_{in}(x)$. Assume

$$\phi(\overrightarrow{x},t) = U^{-1}(t)\,\phi_{in}(\overrightarrow{x},t)\,U(t)\,, \qquad \pi(\overrightarrow{x},t) = U^{-1}(t)\,\pi_{in}(\overrightarrow{x},t)\,U(t)$$

In-fields satisfy ,

$$\partial_0\phi_{in}(x) = i\left[H_{in}(\phi_{in},\pi_{in}),\phi_{in}\right], \qquad \partial_0\pi_{in}(x) = i\left[H_{in}(\phi_{in},\pi_{in}),\pi_{in}\right] \tag{1}$$

where $H_{in}(\phi_{in},\pi_{in})$ is free field Hamiltonian with mass μ.
Time evolution of $\phi(x)$, $\pi(x)$ is governed by full Hamiltonian,

$$\partial_0\phi(x) = i\left[H(\phi,\pi),\phi\right], \qquad \partial_0\pi(x) = i\left[H(\phi,\pi),\pi\right]$$

Then we find

$$\phi_{in} = U\phi U^{-1}, \qquad \Longrightarrow \qquad \partial_0\phi_{in} = \left(\frac{\partial U}{\partial t}\phi U^{-1} + U\partial_0\phi U^{-1} + U\phi\partial_0 U^{-1}\right)$$

Or

$$\partial_0\phi_{in} = \left(\frac{\partial U}{\partial t}U^{-1}\right)\phi_{in} + i\left[H(\phi_{in},\pi_{in}),\phi_{in}\right] - \phi_{in}\frac{\partial U}{\partial t}U^{-1}$$

Using Eq(1), we simplify

$$\left[\frac{\partial U}{\partial t}U^{-1} + iH_I(\phi_{in},\pi_{in}),\ \phi_{in}\right] = 0$$

where $H_I(\phi_{in},\pi_{in}) = H(\phi_{in},\pi_{in}) - H_{in}(\phi_{in},\pi_{in})$ contains interaction. Similarly,

$$\left[\frac{\partial U}{\partial t}U^{-1} + iH_I(\phi_{in},\pi_{in}),\ \pi_{in}\right] = 0$$

This means $\frac{\partial U}{\partial t}U^{-1}+iH_I$ commutes with all operators, take this to be zero. Thus

$$\boxed{i\frac{\partial U(t)}{\partial t}=H_I(t)\,U(t)} \tag{2}$$

For convenience, define

$$U(t,t')\equiv U(t)\,U^{-1}(t') \qquad \text{time evolution operator}$$

Eq(2) becomes,

$$i\frac{\partial U(t,t')}{\partial t}=H_I(t)\,U(t,t') \qquad \text{with} \quad U(t,t)=1$$

convert this to integral equation

$$U(t,t')=1-i\int_{t'}^{t}dt_1 H_I(t_1)\,U(t_1,t')$$

which includes the initial condition. Iterate this equation assuming H_I is "small",

$$\begin{aligned} U(t,t') \quad &= \quad 1-i\int_{t'}^{t}dt_1 H_I(t_1)+(-i)^2\int_{t'}^{t}dt_1 H_I(t_1)\int_{t'}^{t_1}dt_2 H_I(t_2)+... \\ &\quad +(-i)^n\int_{t'}^{t}dt_1\int_{t'}^{t_1}dt_2...\int_{t'}^{t_{n-1}}dt_n H_I(t_1)\,H_I(t_2)\,...H_I(t_n)+... \end{aligned}$$

The second term can be written as

$$\begin{aligned} U^{(2)} &= (-i)^2 \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2 H_I(t_1) H_I(t_2) \\ &= (-i)^2 \int_{t'}^{t} dt_2 \int_{t_2}^{t} dt_1 H_I(t_1) H_I(t_2) \\ &= (-i)^2 \int_{t'}^{t} dt_1 \int_{t_2}^{t} dt_2 H_I(t_2) H_I(t_1) \end{aligned}$$

where we have interchange the order of integration. Renaming t_1 and t_2 ,we get

$$U^{(2)} = (-i)^2 \int_{t'}^{t} dt_1 \int_{t_2}^{t} dt_2 H_I(t_2) H_I(t_1)$$

We can use time-ordered product to combine these two equivalent expression so that the t_2 integration goes from t' to t

$$U^{(2)} = \frac{(-i)^2}{2} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \, T(H_I(t_2) H_I(t_1))$$

We can generalize these steps to higher terms in U so that
This can be written as

$$\begin{aligned} U(t, t') &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2 \ldots \int_{t'}^{t} dt_n T(H_I(t_1) H_I(t_2) \ldots H_I(t_n)) \\ &= T\left(\exp\left[-i \int_{t'}^{t} d^4x \mathcal{H}_I(\phi_{in}, \pi_{in})\right]\right) \end{aligned}$$

Perturbation Expansion of Vaccum expectation value

From LSZ reduction, $S-matrix$ is of the form,

$$\tau\left(x_1,x_2,...,x_n\right)=\langle 0|T(\phi\left(x_1\right)\phi\left(x_2\right)...\phi\left(x_n\right))|0\rangle$$

Using U matrix, write this in terms of ϕ_{in}

$$\begin{aligned}\tau &= \langle 0|T\left(U^{-1}\left(t_1\right)\phi_{in}\left(x_1\right)U\left(t_1,t_2\right)\phi_{in}\left(x_2\right)U\left(t_2,t_3\right)...U\left(t_{n-1},t_n\right)\phi_{in}\left(x_n\right)U\left(t_n\right)\right)|0\rangle \\ &= \langle 0|T\left(U^{-1}\left(t\right)U\left(t,t_1\right)\phi_{in}\left(x_1\right)...\phi_{in}\left(x_n\right)U\left(t_n,t'\right)U\left(t'\right)\right)|0\rangle\end{aligned}$$

Let $t>t_1...t_n>t'$,then we can pull $U^{-1}\left(t\right)$ and $U\left(t'\right)$ out of the time-ordered product, and combine $U's$ and ϕ_{in}

$$\begin{aligned}\tau &= \langle 0|U^{-1}\left(t\right)TU\left(t,t_1\right)\phi_{in}\left(x_1\right)...\phi_{in}\left(x_n\right)U\left(t_n,t'\right))U\left(t'\right)|0\rangle \\ &= \langle 0|U^{-1}\left(t\right)T(\phi_{in}\left(x_1\right)...\phi_{in}\left(x_n\right)\exp\left[-i\int_{t'}^{t}H_I\left(t"\right)dt"\right])U\left(t'\right)|0\rangle\end{aligned}$$

Theorem: $|0\rangle$ is an eigenstate of$U(-t)$ as $t\rightarrow\infty$.
Proof :Consider a matrix element $\langle p,\alpha,in|U(-t)|0\rangle$. Use the method the same as reduction formula, we

$$\begin{aligned}\langle p,\alpha,in|U(-t)|0\rangle &= \sqrt{\left(2\pi\right)^3 2w_p}\langle\alpha,in|a_{in}\left(p\right)U(-t)|0\rangle \\ &= i\sqrt{\left(2\pi\right)^3 2w_p}\int d^3x f_p^*\left(\overrightarrow{x},-t'\right)\overleftrightarrow{\partial_0'}\langle\alpha,in|\phi_{in}\left(\overrightarrow{x},-t'\right)U(-t)|0\rangle \\ &= i\sqrt{\left(2\pi\right)^3 2w_p}\int d^3x f_p^*\left(\overrightarrow{x},-t'\right)\overleftrightarrow{\partial_0'}\langle\alpha,in|U(-t')\phi\left(\overrightarrow{x},-t'\right)U(-t')U(-t)|0\rangle\end{aligned}$$

Last term

$$
\begin{aligned}
& f_p^* \left(\overrightarrow{x}, -t'\right) \overleftrightarrow{\partial_0'} U(-t')\phi\left(\overrightarrow{x}, -t'\right) U(-t') U(-t) \\
= \quad & \partial_0' f_p^* \left(\overrightarrow{x}, -t'\right) U(-t)\phi(-t) - f_p^* \left(\overrightarrow{x}, -t'\right) \left[\dot{U}(-t)\phi(-t) + U(-t)\dot{\phi}(-t) + U(-t)\phi(-t)\dot{U}^{-1}(-t)U^{-1}(-t)\right]
\end{aligned}
$$

Then

$$
\langle p, \alpha, in|U(-t)|0\rangle = \sqrt{(2\pi)^3 2w_p}\{\langle \alpha, in|U(-t)a_{in}(p)|0\rangle + i\int d^3x f_p^* \left(\overrightarrow{x}, -t'\right) \langle \alpha, in|\dot{U}\phi + U\phi\dot{U}^{-1}U|0\rangle\}
$$

In the last term

$$
\begin{aligned}
\dot{U}\phi + U\phi\dot{U}^{-1}U \quad & = \quad \dot{U}\left(U^{-1}\phi_{in}U\right) + U\left(U^{-1}\phi_{in}U\right)\left(-U^{-1}\dot{U}U^{-1}\right)U \\
& = \quad \dot{U}U^{-1}\phi_{in}U - \phi_{in}\dot{U}U^{-1}U = \left[\dot{U}U^{-1}, \phi_{in}\right]U \\
& = \quad -i\left[H_I\left(\phi_{in}, \pi_{in}\right), \phi_{in}\right]U = 0
\end{aligned}
$$

Then we get the result $\langle p, \alpha, in|U(-t)|0\rangle = 0$ as $t \to \infty$ for all in-states.This means

$$
U(-t)|0\rangle = \lambda_-|0\rangle \qquad \lambda_- \quad \text{some phase as } t \to \infty
$$

This completes the proof.
Similarly we can show that

$$
U(t)|0\rangle = \lambda_+|0\rangle \qquad \lambda_+ \quad \text{some phase as } t \to \infty
$$

These phases can be written as

$$\lambda_- \lambda_+^* = [\langle 0| T \exp(-i \int_{-t}^{t} H_I(t')\, dt') |0\rangle]^{-1}$$

Now we have vacuum expectation value $\tau(x_1, x_2, \ldots, x_n)$ completely in terms of ϕ_{in},

$$\begin{aligned}
\tau(x_1, x_2, \ldots, x_n) &= \langle 0| U^{-1}(-t)\, T\left(\phi_{in}(x_1)\, \phi_{in}(x_2) \ldots \phi_{in}(x_n) \exp(-i \int_{-t}^{t} H_I(t')\, dt')\right) U(t) |0\rangle \\
&= \lambda_- \lambda_+^* \langle 0| T\left(\phi_{in}(x_1)\, \phi_{in}(x_2) \ldots \phi_{in}(x_n) \exp(-i \int_{-t}^{t} H_I(t')\, dt')\right) |0\rangle
\end{aligned}$$

or

$$\boxed{\tau(x_1, x_2, \ldots, x_n) = \frac{\langle 0| T\left(\phi_{in}(x_1)\, \phi_{in}(x_2) \ldots \phi_{in}(x_n) \exp(-i \int_{-\infty}^{\infty} H_I(t')\, dt')\right) |0\rangle}{\langle 0| T\left(\exp(-i \int_{-\infty}^{\infty} H_I(t')\, dt')\right) |0\rangle}}$$

For computation we need to expand the exponential of H_I , to write

$$\tau\left(x_1, x_2, \ldots, x_n\right)=\frac{\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} dy_1 \ldots dy_m \langle 0| T\left(\phi_{in}\left(x_1\right) \ldots \phi_{in}\left(x_n\right) \mathcal{H}_I\left(y_1\right) \mathcal{H}_I\left(y_2\right) \ldots \mathcal{H}_I\left(y_m\right)\right)|0\rangle}{\sum_{m=0}^{\infty} \frac{(-i)^m}{m!} \int_{-\infty}^{\infty} dy_1 \ldots dy_m \langle 0| T\left(\mathcal{H}_I\left(y_1\right) \mathcal{H}_I\left(y_2\right) \ldots \mathcal{H}_I\left(y_m\right)\right)|0\rangle}$$

Wick's theorem

To compute product of free fields ϕ_{in} between vacuum, convert to normal ordering . Results are summarized below;

$$\begin{aligned} T\left(\phi_{in}\left(x_1\right) \ldots \phi_{in}\left(x_n\right)\right) \quad = \quad & : \phi_{in}\left(x_1\right) \ldots \phi_{in}\left(x_n\right) : \\ & + \left[\langle 0|\phi_{in}\left(x_1\right) \phi_{in}\left(x_2\right)|0\rangle : \phi_{in}\left(x_3\right) \phi_{in}\left(x_4\right) \ldots \phi_{in}\left(x_n\right) : + permutations\right] \end{aligned}$$

$$+\left[\langle 0|\phi_{in}\left(x_1\right) \phi_{in}\left(x_2\right)|0\rangle\langle 0|\phi_{in}\left(x_3\right) \phi_{in}\left(x_4\right)|0\rangle : \phi_{in}\left(x_5\right) \ldots \phi_{in}\left(x_n\right) : + permutations\right] \ldots$$

$$+\begin{cases} \left[\langle 0|\phi_{in}\left(x_1\right) \phi_{in}\left(x_2\right)|0\rangle\langle 0|\phi_{in}\left(x_3\right) \phi_{in}\left(x_4\right)|0\rangle \ldots\langle 0|\phi_{in}\left(x_{n-1}\right) \phi_{in}\left(x_n\right)|0\rangle + permutations\right] & neven \\ \left[\langle 0|\phi_{in}\left(x_1\right) \phi_{in}\left(x_2\right)|0\rangle \ldots\langle 0|\phi_{in}\left(x_{n-2}\right) \phi_{in}\left(x_{n-1}\right)|0\rangle \phi_{in}\left(x_n\right) + permutations\right] & \text{n odd} \end{cases}$$

This can be proved by induction.

Illustrate this for n=2. Difference between $T()$ and :() :is a c-number,

$$T\left(\phi_{in}\left(x_1\right) \phi_{in}\left(x_2\right)\right) =: \phi_{in}\left(x_1\right) \phi_{in}\left(x_2\right) : + (c - number)$$

take matrix element between vacuum state,

$$\langle 0|\, T(\phi_{in}(x_1)\,\phi_{in}(x_2))|0\rangle = (c - number)$$

Then

$$T(\phi_{in}(x_1)\,\phi_{in}(x_2)) =: \phi_{in}(x_1)\,\phi_{in}(x_2) : + \langle 0|\, T(\phi_{in}(x_1)\,\phi_{in}(x_2))|0\rangle$$

Most useful application of Wick's theorem

$$\begin{aligned} \langle 0|\, T(\phi_{in}(x_1)\ldots\phi_{in}(x_n))|0\rangle &= 0 \qquad \text{n odd} \\ \langle 0|\, T(\phi_{in}(x_1)\ldots\phi_{in}(x_n))|0\rangle &= \sum_{permu} [\langle 0|\, T(\phi_{in}(x_1)\,\phi_{in}(x_2))|0\rangle \langle 0|\, T(\phi_{in}(x_3)\,\phi_{in}(x_4))|0\rangle \ldots] \qquad \text{n even} \end{aligned}$$

<u>Notation</u>

$$\underbrace{\phi_{in}(x_1)\,\phi_{in}(x_2)} = \langle 0|\, T(\phi_{in}(x_1)\,\phi_{in}(x_2))|0\rangle \qquad \text{Contraction}$$

Example:

$$\begin{aligned} \langle 0|T(\phi_{in}(x_1)\,\phi_{in}(x_2) \quad &: \quad \phi_{in}^3(y_1) :: \phi_{in}^3(y_2) :)|0\rangle \\ &= \langle 0|T(\phi_{in}(x_1)\,\phi_{in}(x_2) : \phi_{in}(y_1)\,\phi_{in}(y_1)\,\phi_{in}(y_1) :: \phi_{in}(y_2)\,\phi_{in}(y_2)\,\phi_{in}(y_2) :)|0\rangle \\ &\quad + \ldots \end{aligned}$$

Feynman Propagators

From Wick's theorem most important quantity is vacuum expecation of two free fields, called **Feynman propagator**.

$$\begin{aligned}
\langle 0|\, T(\phi_{in}(x)\,\phi_{in}(y))|0\rangle &= i\triangle_F\left(x-y,\mu^2\right) = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2-\mu^2+i\varepsilon} \\
&= i\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x-y)}\, i\triangle_F(k) \\
\text{with} \quad i\triangle_F(k) &= \frac{i}{k^2-\mu^2+i\varepsilon}
\end{aligned}$$

For complex scalar field

$$\langle 0|\, T(\phi_{in}(x)\,\phi^*_{in}(y))|0\rangle = i\triangle_F\left(x-y,\mu^2\right) = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2-\mu^2+i\varepsilon}$$

Fermion field

$$\begin{aligned}
\langle 0|\, T(\psi^{in}_\alpha(x)\,\overline{\psi}^{in}_\beta(y))|0\rangle &= iS_F(x-y,m)_{\alpha\beta} \\
&= i\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)}\frac{\left(\gamma^\mu p_\mu + m\right)_{\alpha\beta}}{p^2-m^2+i\varepsilon} = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)}\, iS_F(p)_{\alpha\beta}
\end{aligned}$$

photon field

$$\langle 0| T(A_\mu^{in}(x)\, A_\upsilon^{in}(y))|0\rangle = iD_F^{tr}(x-y) = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x-y)}}{k^2+i\varepsilon}\times$$
$$\left[-g_{\mu\nu} - \frac{k_\mu k_\nu}{(k\cdot\eta)^2-k^2} + \frac{(k\cdot\eta)\left(k_\mu\eta_\nu + k_\nu\eta_\mu\right)}{(k\cdot\eta)^2-k^2} - \frac{k^2\eta_\mu\eta_\nu}{(k\cdot\eta)^2-k^2}\right]$$

where $\eta_\mu = (1,0,0,0)$
It can be shown that in QED only term contributes is "$-g_{\mu\nu}$" as a consequence of the gauge invariance.

Graphical representation
Each line (propagator) represents a contraction in Wick's expansion
e.q.

$$\underset{y}{\circ}\cdots\cdots\underset{x}{\circ} \qquad i\triangle_F\left(x-y,\mu^2\right)$$

$$\overset{\beta}{\underset{y}{}}\!\!-\!\!\!-\!\!>\!\!-\!\!\!-\overset{\alpha}{\underset{x}{}} \qquad iS_F\left(x-y,m\right)_{\alpha\beta}$$

$$\sim^{\nu}\sim\sim\sim\sim\sim\sim^{\mu} \qquad iD_F^{tr}\left(x-y\right)$$

Vaccum Amplitude
In the denominator of $\tau-$function, there are no external lines

$$\sum_{m=0}^{\infty}\frac{(-i)^m}{m!}\int_{-\infty}^{\infty}d^4y_1...d^4y_m\langle 0|T\left(\mathcal{H}_I\left(\phi_{in}\left(y_1\right)\right)...\mathcal{H}_I\left(\phi_{in}\left(y_m\right)\right)\right)|0\rangle$$

e.q. 2nd order term for the case $\mathcal{H}_I=\frac{\lambda}{3!}:\phi_{in}^3:$

$$\langle 0|T\left(\mathcal{H}_I\left(\phi_{in}\left(y_1\right)\right)\mathcal{H}_I\left(\phi_{in}\left(y_m\right)\right)\right)|0\rangle = \left(\frac{\lambda}{3!}\right)^2 \langle 0|T\left(:\phi_{in}^3\left(y_1\right)::\phi_{in}^3\left(y_2\right):\right)|0\rangle$$

$$= \left(\frac{\lambda}{3!}\right)^2 :\phi_{in}\left(y_1\right)\phi_{in}\left(y_1\right)\phi_{in}\left(y_1\right)::\phi_{in}\left(y_2\right)\phi_{in}\left(y_2\right)\phi_{in}\left(y_2\right): 3\times 2$$

closed loop diagram :graphs with no external lines(lines with open end)
disconnected diagram :a subgraph not connected to any external lines
connected diagram :graph not disconnected

All graphs appearing in the numerator of the $\tau-$function can be seperated uniquely into connected and disconnected parts. It turns out that disconnected part is cancelle by those in denominator.
Example :$\mathcal{H}_I = \frac{\lambda}{3!}\phi_{in}^3$

$$\phi(q_1) + \phi(q_2) \longrightarrow \phi(p_1) + \phi(p_2)$$

$$\begin{aligned}
S_{\beta\alpha} &= \langle \beta, out|\alpha, in\rangle = \langle p_1, p_2, out|q_1, q_2, in\rangle \\
&= \left(\frac{-i}{\sqrt{z}}\right)^4 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{ip_1y_1} e^{ip_2y_2} \left(\Box_{y_1} + \mu^2\right)\left(\Box_{y_2} + \mu^2\right) \langle 0| T\left(\phi(y_1)\phi(y_2)\phi(x_1)\phi(x_2)\right)|0> \\
&\quad \left(\overleftarrow{\Box}_{x_1} + \mu^2\right)\left(\overleftarrow{\Box}_{x2} + \mu^2\right) e^{-iq_1x_1} e^{-iq_2x_2} \\
&= \left(\frac{-i}{\sqrt{z}}\right)^4 \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 \left(\mu^2 - p_1^2\right)\left(\mu^2 - p_2^2\right)\left(\mu^2 - q_1^2\right)\left(\mu^2 - q_2^2\right) \\
&\quad \times \tau(y_1, y_2, x_1, x_2) e^{i(p_1y_1 + p_2y_2)} e^{-i(q_1x_1 + q_2x_2)}
\end{aligned}$$

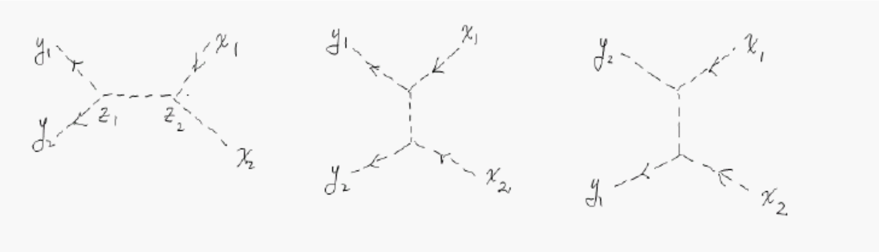
Perturbation expansion of $\tau-$function

$$\tau(y_1, y_2, x_1, x_2) = \sum_n \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} d^4z_1 \ldots d^4z_n \langle 0| T\left(\phi_{in}(y_1)\phi_{in}(y_2)\phi_{in}(x_1)\phi_{in}(x_2)\mathcal{H}_I(\phi_{in}(z_1)) \ldots \mathcal{H}_I(\phi_{in}(z_n))\right)|0\rangle$$

Lowest order contribution

$$\tau^{(2)}(y_1, y_2, x_1, x_2) = \frac{(-i)^2}{2!} \int_{-\infty}^{\infty} d^4z_1 d^4z_2 \langle 0| T\left(\phi_{in}(y_1)\phi_{in}(y_2)\phi_{in}(x_1)\phi_{in}(x_2)\left(\frac{\lambda}{3!}\phi_{in}^3(z_1)\right)\left(\frac{\lambda}{3!}\phi_{in}^3(z_2)\right)\right)|0\rangle$$

Using Wick's theorem, the connected diagrams are,



Their contribution to $\tau(y_1, y_2, x_1, x_2)$ is

$$\tau^{(2)}(y_1, y_2, x_1, x_2) = \frac{(-i\lambda)^2}{2!} \int_{-\infty}^{\infty} d^4 z_1 d^4 z_2 i \triangle_F (y_1 - z_1) i \triangle_F (y_2 - z_1) \\ i \triangle_F (z_2 - x_1) i \triangle_F (z_2 - x_2) i \triangle_F (z_1 - z_2) + \ldots$$

use the propagator in momentum space

$$i \triangle_F (x) = \int \frac{d^4 k}{(2\pi)^4} \frac{i}{k^2 - \mu^2 + i\varepsilon} e^{-ik \cdot x}$$

Then

$$\tau^{(2)}(y_1, y_2, x_1, x_2) = \frac{(-i\lambda)^2}{2!} \int_{-\infty}^{\infty} d^4 z_1 d^4 z_2 \int \frac{d^4 k_1}{(2\pi)^4} \int \frac{d^4 k_2}{(2\pi)^4} \cdots \int \frac{d^4 k_5}{(2\pi)^4}$$
$$e^{-ik_1\cdot(y_1-z_1)} i \triangle_F(k_1)\, e^{-ik_2\cdot(z_1-x_1)} i \triangle_F(k_2)$$
$$e^{-ik_3\cdot(z_1-z_2)} i \triangle_F(k_3)\, e^{-ik_4\cdot(y_2-z_2)} i \triangle_F(k_4)\, e^{-ik_5\cdot(z_2-x_2)} i \triangle_F(k_5)$$

$$z_1 \text{ integration} \quad \int d^4 z_1 e^{i(k_1-k_2-k_3)\cdot z_1} = (2\pi)^4 \delta^4(k_1 - k_2 - k_3)$$
$$z_2 \text{ integration} \quad \int d^4 z_2 e^{i(k_3+k_4-k_5)\cdot z_2} = (2\pi)^4 \delta^4(k_3 + k_4 - k_5)$$

energy-momentum conservation at each vertex
Then

$$\tau^{(2)}(y_1, y_2, x_1, x_2) = \frac{(-i\lambda)^2}{2!} \int \frac{d^4 k_1}{(2\pi)^4} \cdots \frac{d^4 k_4}{(2\pi)^4} (2\pi)^4 \delta^4(k_1 - k_2 + k_4 - k_5)$$
$$i \triangle_F(k_1)\, i \triangle_F(k_2)\, i \triangle_F(k_4)\, i \triangle_F(k_5)\, i \triangle_F(k_1 - k_2)\, e^{-ik_1\cdot y_1} e^{ik_2\cdot x_1} e^{-ik_4\cdot y_2} e^{ik_5\cdot x_2}$$

$$\int \tau^{(2)}(y_1, y_2, x_1, x_2)\, d^4x_1 d^4x_2 d^4y_1 d^4y_2\, e^{i(p_1 y_1 + p_2 y_2)} e^{-i(q_1 x_1 + q_2 x_2)} e^{-ik_1\cdot y_1} e^{ik_2\cdot x_1} e^{-ik_4\cdot y_2} e^{ik_5\cdot x_2}$$
$$= (2\pi)^4 \delta^4(k_2 - q_1)(2\pi)^4 \delta^4(k_5 - q_2)(2\pi)^4 \delta^4(p_1 - k)(2\pi)^4 \delta^4(p_2 - k_4)$$

We see that the external line propagators cancell out and

$$S_{\beta\alpha} = \frac{(-i\lambda)^2}{2!} \left(\frac{1}{\sqrt{z}} \right)^4 (2\pi)^4 \delta^4 (p_1 + p_2 - q_1 - q_2) + \ldots$$

This is rather simple answer in momentum space.

Cross section and Decay rate

Write the S-matrix elements as

$$S_{fi} = \delta_{fi} + i(2\pi)^4 \delta^4(p_f - p_i) T_{fi}$$

T_{fi} : invariant amplitude for $i \rightarrow f$.
For $i \neq f$, the transistion probability is

$$|S_{fi}|^2 = (2\pi)^4 \delta^4(0)[(2\pi)^4 \delta^4(p_f - p_i) \, |T_{fi}|^2]$$

To interprete $\delta^4(0)$, we write

$$(2\pi)^4 \delta^4(p_f - p_i) = \int d^4 x e^{-i(p_f - p_i)x}$$

The integration is over some large but finite volume V and time interval T.
Then we can interprete $\delta^4(0)$ as

$$(2\pi)^4 \delta^4(0) = VT$$

and write

$$|S_{fi}|^2 = VT[(2\pi)^4 \delta^4(p_f - p_i) \, |T_{fi}|^2$$

The transistion rate (transistion probability per unit time) is then

$$\omega_{fi} = (2\pi)^4 \delta^4(p_f - p_i) \, |T_{fi}|^2 \, V$$

Decay rates

For a general decay processes with kinematics,

$$a(p) \rightarrow c_1(k_1) + c_2(k_2) + ... + c_n(k_n) \quad p_f = \sum_{l=1}^{n} k_i \quad p_i = p$$

number of states in the volume elements $d^3k_1 \ldots d^3k_n$ in momentum space is

$$\prod_{l=1}^{n} \frac{d^3k_l}{(2\pi)^3 2\omega_{kl}}$$

The transition rate, summing over final states is

$$d\omega' = (2\pi)^4 \delta^4(p - \Sigma_{j=1}^{n} k_j)\,|T_{fi}|^2\, V \prod_{l=1}^{n} \frac{d^3k_l}{(2\pi)^3 2\omega_{kl}}$$

For the invariant normalization of the physical states

$$< p|p' >= (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,')2\omega_p$$

For $p = p'$,

$$< p|p >= (2\pi)^3 \delta^3(0) 2\omega_p = 2V\omega_p$$

which is the number of particle in the initial state.
The decay rate per particle is then

$$d\omega = \frac{d\omega'}{2V\omega_p} = (2\pi)^4 \delta^4(p - \Sigma_{j=1}^{n} k_j)\,|T_{fi}|^2\, \frac{1}{2\omega_p} \prod_{l=1}^{n} \frac{d^3k_l}{(2\pi)^3 2\omega_{kl}}$$

If there are "m" identical particles in the final state, divide this by m!

$$d\omega = \frac{1}{2\omega_p} \left| T_{fi} \right|^2 \frac{d^3 k_1}{(2\pi)^3 2\omega_1} \cdots \frac{d^3 k_n}{(2\pi)^3 2\omega_n} (2\pi)^4 \delta^4 (p - \Sigma_{j=1}^{n} k_j) S \qquad S = \prod_j \frac{1}{(m_j)!}$$

Cross section

For a scattering processes ,

$$a(p_1) + b(p_2) \to c_1(k_1) + c_2(k_2) + ... + c_n(k_n)$$

the transition rate is, after summing over final states,

$$d\omega' = (2\pi)^4 \delta^4(p_1 + p_2 - \Sigma_{j=1}^n k_j)\, |T_{fi}|^2\, V \prod_{l=1}^{n} \frac{d^3 k_l}{(2\pi)^3 2\omega_{kl}}$$

Normalize this to 1 particle in the beam and 1 particle in the target and divide this by the flux∼relative velocity divided by the volume, to get differential cross section

$$d\sigma = \frac{1}{2\omega_{p_1} V} \frac{1}{2\omega_{p_2} V} (2\pi)^4 \delta^4(p_1 + p_2 - \Sigma_{j=1}^n k_j)\, |T_{fi}|^2\, V \prod_{l=1}^{n} \frac{d^3 k_l}{(2\pi)^3 2\omega_{kl}} \frac{V}{|\vec{v_1} - \vec{v_2}|}$$

Velocity factor can be written as

$$I = |\vec{v_1} - \vec{v_2}| = \left| \frac{\vec{p_1}}{E_1} - \frac{\vec{p_2}}{E_2} \right|$$

In the C.M. frame $\vec{p_1} = -\vec{p_2} = \vec{p} \qquad p_1 = (E_1, \vec{p}), p_2 = (E_2, -\vec{p})$

$$I = \frac{|\vec{p}|}{E_1 E_2}(E_1 + E_2)$$

$$(p_1 \cdot p_2)^2 = (E_1 E_2 + \vec{p}^{\,2})^2 = E_1^2 E_2^2 + 2E_1 E_2 \vec{p}^{\,2} + \vec{p}^{\,4}$$

$$
\begin{aligned}
(p_1 \cdot p_2)^2 - m_1^2 m_2^2 &= (\vec{p}^{\,2} + m_1^2)(\vec{p}^{\,2} + m_2^2) + 2E_1 E_2 \vec{p}^{\,2} + \vec{p}^{\,4} - m_1^2 m_2^2 \\
&= \vec{p}^{\,2}[2\vec{p}^{\,2} + (m_1^2 + m_2^2) + 2E_1 E_2] \\
&= \vec{p}^{\,2}(E_1 + E_2)^2
\end{aligned}
$$

$$
\Rightarrow I = \frac{1}{E_1 E_2}\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}
$$

$$
d\sigma = \frac{1}{I}\frac{1}{2\omega_{p_1}}\frac{1}{2\omega_{p_2}}(2\pi)^4 \delta^4(p_1 + p_2 - \Sigma_{j=1}^{n} k_j)\,|T_{fi}|^2 \prod_{l=1}^{n} \frac{d^3 k_l}{(2\pi)^3 2\omega_l}
$$

Feynman Rules

Since the final forms for transition matrix elements T_{fi} are quite simple,we can use simple rules to sidestep all those tedious intermediate steps.

Draw all connected Feynman graphs with appropriate external lines.Label each with momenta and impose momentum conservation for each vertex.

1. For each internal fermion line with momentum p,enter the propagator

$$iS_F(p) = \frac{i}{p^\mu \gamma_\mu - m + i\varepsilon}$$

2.For each internal boson line of spin 0,with momentum q,enter the propagator

$$i\triangle_F(q) = \frac{i}{q^2 - \mu^2 + i\varepsilon}$$

3. For each internal photon line with momentum k,enter the propagator

$$iD_F(k)_{\mu v} = \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon}$$

4. For each internal momentum l not fixed by momentum conservation,enter

$$\int \frac{d^4 l}{(2\pi)^4}$$

5. For each closed fermion loop,enter (-1) .Also they should be factor of (-1) between graphs which differ only by an interchange of two external identical fermion lines.

6. At each vertex,the factors depend on the explicit form of interactions.

a. $\frac{1}{3!} \quad \lambda\phi^3 \qquad (-i\lambda)$

b. $\frac{1}{4!} \quad \lambda\phi^4 \qquad (-i\lambda)$

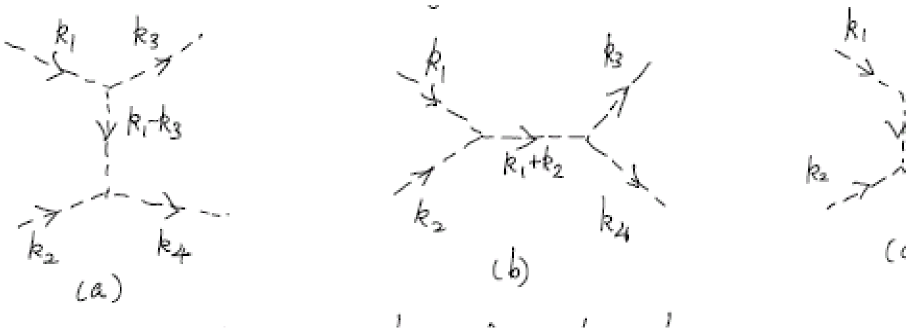
c. $e\overline{\psi}\gamma_\mu\psi A^\mu \qquad \left(-ie\gamma_\mu\right)$

d. $f \ \overline{\psi}\gamma_5\psi\phi \qquad (-if\gamma_5)$

Example in $\lambda\phi^3$ theory

In $\lambda\phi^3$ theory, consider scattering processes $\phi(k_1) + \phi(k_2) \longrightarrow \phi(k_3) + \phi(k_4)$
To second order in λ , we have following 3 Feynman diagrams for this reaction

We can write down the matrix element for each graph,



$$T^{(a)} = (-i\lambda)^2 \frac{i}{(k_1 - k_3)^2 - \mu^2} \qquad T^{(b)} = (-i\lambda)^2 \frac{i}{(k_1 + k_2)^2 - \mu^2} \qquad T^{(c)} = (-i\lambda)^2 \frac{i}{(k_1 - k_4)^2 - \mu^2}$$

Total amplitude $T = T^{(a)} + T^{(b)} + T^{(c)}$
Mandelstam variables

$$\begin{aligned} s &= (k_1 + k_2)^2 && \text{total energy in c.m. frame} \\ t &= (k_1 - k_3)^2 && \text{momentum transfer(scattering angle)} \\ u &= (k_1 - k_4)^2 && \end{aligned}$$

$$s + t + u = 4\mu^2$$

Usually these amplitudes are written as

$$T^{(a)} = (\lambda)^2 \frac{i}{t - \mu^2} \qquad T^{(b)} = (\lambda)^2 \frac{i}{s - \mu^2} \qquad T^{(c)} = (\lambda)^2 \frac{i}{u - \mu^2}$$