Quantum Electrodynamics

Ling-Fong Li

Quantum Electrodynamics

Lagrangian density for QED ,

$$\mathcal{L} = \overline{\psi}(x)\gamma^{\mu}\left(i\partial_{\mu} - eA_{\mu}\right)\psi(x) - m\overline{\psi}(x)\psi(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Equations of motion are

$$\begin{aligned}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) &= eA_{\mu}\gamma^{\mu}\psi \qquad \text{non-linear coupled equations}\\ \partial_{v}F^{\mu\nu} &= e\overline{\psi}\gamma^{\mu}\psi\end{aligned}$$

Quantization

Write $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

$$\begin{aligned}\mathcal{L}_0 &= \overline{\psi}\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\\ \mathcal{L}_{int} &= -e\overline{\psi}\gamma^{\mu}\psi A_{\mu}\end{aligned}$$

where $\mathcal{L}_0$,free field Lagrangian, $\mathcal{L}_{int}$ is interaction part.

Conjugate momenta for fermion

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_0\psi_{\alpha}\right)} = i\psi^{\dagger}_{\alpha}(x)$$

For em fields choose the gauge

$$\overrightarrow{\nabla}\cdot\overrightarrow{A} = 0$$

Conjugate mometa

$$\pi^i = \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A^i\right)} = -F^{0i} = E^i$$

From equation of motion

$$\partial_\nu F^{0\nu} = e\psi^\dagger\psi \qquad \Longrightarrow \qquad -\nabla^2 A^0 = e\psi^\dagger\psi$$

A^0 is not an independent field ,

$$A^0 = e\int d^3x' \, \frac{\psi^\dagger\left(x',t\right)\psi\left(x',t\right)}{4\pi|\overrightarrow{x'}-\overrightarrow{x}|} = e\int \frac{d^3x'\rho\left(x',t\right)}{|\overrightarrow{x}-\overrightarrow{x'}|}$$

Commutation relations

$$\begin{aligned}\left\{\psi_\alpha\left(\overrightarrow{x},t\right),\psi_\beta^\dagger\left(\overrightarrow{x'},t\right)\right\} &= \delta_{\alpha\beta}\delta^3\left(\overrightarrow{x}-\overrightarrow{x'}\right) \qquad \left\{\psi_\alpha\left(\overrightarrow{x},t\right),\psi_\beta\left(\overrightarrow{x'},t\right)\right\} = \ldots = 0\\ \left[\dot{A}_i\left(\overrightarrow{x},t\right),A_j\left(\overrightarrow{x'},t\right)\right] &= i\delta_{ij}^{tr}\left(\overrightarrow{x}-\overrightarrow{x'}\right)\end{aligned}$$

where

$$\delta_{ij}^{tr}(\vec{x}-\vec{y}) = \int \frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}(\delta_{ij}-\frac{k_ik_j}{k^2})$$

Commutators involving A_0

$$\left[A_0\left(\overrightarrow{x},t\right),\psi_\alpha\left(\overrightarrow{x'},t\right)\right] = e\int\frac{d^3x''}{4\pi|\overrightarrow{x}-\overrightarrow{x''}|}\left[\psi^\dagger\left(\overrightarrow{x''},t\right)\psi\left(\overrightarrow{x''},t\right),\psi_\alpha\left(\overrightarrow{x'},t\right)\right] = -\frac{e}{4\pi}\frac{\psi_\alpha\left(\overrightarrow{x'},t\right)}{|\overrightarrow{x}-\overrightarrow{x'}|}$$

Hamiltonian density

$$
\begin{aligned}
\mathcal{H} &= \frac{\partial \mathcal{L}}{\partial\left(\partial_0 \psi_\alpha\right)} \dot{\psi}_\alpha + \frac{\partial \mathcal{L}}{\partial\left(\partial_0 A^k\right)} \dot{A}_k - \mathcal{L} \\
&= \psi^\dagger \left(-i \overrightarrow{\alpha} \cdot \overrightarrow{\nabla} + \beta m\right) \psi + \frac{1}{2}\left(\overrightarrow{E}^2 + \overrightarrow{B}^2\right) + \overrightarrow{E} \cdot \overrightarrow{\nabla} A_0 + e \overline{\psi} \gamma^\mu \psi A_\mu
\end{aligned}
$$

and

$$
H = \int d^3x \mathcal{H} = \int d^3x \{ \psi^\dagger \left[\overrightarrow{\alpha} \cdot (-i \overrightarrow{\nabla} - e \overrightarrow{A}) + \beta m\right] \psi + \frac{1}{2}\left(\overrightarrow{E}^2 + \overrightarrow{B}^2\right) \}
$$

A_0 does not appear in the interaction,
But if we write

$$
\overrightarrow{E} = \overrightarrow{E_l} + \overrightarrow{E_t} \qquad \text{where} \quad \overrightarrow{E_l} = -\overrightarrow{\nabla} A_0 \qquad , \overrightarrow{E_t} = -\frac{\partial \overrightarrow{A}}{\partial t}
$$

Then

$$
\frac{1}{2} \int d^3x \left(\overrightarrow{E}^2 + \overrightarrow{B}^2\right) = \frac{1}{2} \int d^3x \overrightarrow{E_l}^2 + \int d^3x \left(\overrightarrow{E_t}^2 + \overrightarrow{B}^2\right)
$$

longitudinal part is

$$
\frac{1}{2} \int d^3x \overrightarrow{E_l}^2 = \frac{e}{4\pi} \int d^3x d^3y \frac{\rho(\overrightarrow{x}, t) \rho(\overrightarrow{y}, t)}{|\overrightarrow{x} - \overrightarrow{y}|} \qquad \text{Coulomb interaction}
$$

Without classical solutions, can not do mode expansion to get creation and annihilation operators We can only do perturbation theory.

Recall that the free field part $\overrightarrow{A}_0$ satisfy massless Klein-Gordon equation

$$\Box \overrightarrow{A}^{(0)} = 0$$

The solution is

$$\vec{A}^{(0)}(\vec{x}, t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_\lambda \vec{\epsilon}(k,\lambda)[a(k,\lambda)e^{-ikx} + a^+(k,\lambda)e^{ikx}] \qquad w = k_0 = |\overrightarrow{k}|$$

$$\vec{\epsilon}(k,\lambda),\ \lambda = 1,2 \quad \text{with} \qquad \vec{k}\cdot\vec{\epsilon}(k,\lambda) = 0$$

Standard choice

$$\vec{\epsilon}(k,\lambda)\cdot\vec{\epsilon}(k,\lambda') = \delta_{\lambda\lambda'}, \quad \vec{\epsilon}(-k,1) = -\vec{\epsilon}(k,1), \quad \vec{\epsilon}(-k,2) = \vec{\epsilon}(-k,2)$$

It is convienent to write the mode expansion as,

$$A_\mu(\vec{x}, t) = \int \frac{d^3k}{\sqrt{2\omega(2\pi)^3}} \sum_\lambda \epsilon_\mu(k,\lambda)[a(k,\lambda)e^{-ikx} + a^+(k,\lambda)e^{ikx}]$$

where

$$\epsilon_\mu(k,\lambda) = (0, \vec{\epsilon}(k,\lambda))$$

Photon Propagator

Feynman propagatpr for photon is

$$\begin{aligned} iD_{\mu\nu}(x,x') &= \left\langle 0 \left| T\left(A_\mu(x)A_\nu(x')\right)\right| 0\right\rangle \\ &= \theta(t-t')\left\langle 0\left|A_\mu(x)A_\nu(x')\right|0\right\rangle + \theta(t'-t)\left\langle 0\left|A_\nu(x')A_\mu(x)\right|0\right\rangle \end{aligned}$$

From mode expansion,

$$\begin{aligned} \left\langle 0\left|A_\mu(x)A_\nu(x')\right|0\right\rangle &= \int \frac{d^3kd^3k'}{(2\pi)^3\sqrt{2\omega_k 2\omega_{k'}}}\sum_{\lambda,\lambda'}\epsilon_\mu(k,\lambda)\epsilon_\nu(k',\lambda')\left\langle 0\left|[a(k,\lambda)e^{-ikx}]a^+(k',\lambda')e^{ik'x'}\right|0\right\rangle \\ &= \int \frac{d^3kd^3k'}{(2\pi)^3 2\omega_k}\sum_{\lambda,\lambda'}\epsilon_\mu(k,\lambda)\epsilon_\nu(k',\lambda')\ \delta^3(k-k')\, e^{-ikx+ik'x'} \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\sum_{\lambda,\lambda'}\epsilon_\mu(k,\lambda)\epsilon_\nu(k,\lambda')e^{-ik(x-x')} \end{aligned}$$

Note that

$$\frac{1}{2\pi}\int \frac{dk_0}{k_0^2-\omega^2+i\varepsilon}e^{-ik_0(t-t')} = \begin{cases} -i\frac{1}{2\omega}e^{-i\omega(t-t')} & \text{for} \quad t>t' \\ -i\frac{1}{2\omega}e^{i\omega(t-t')} & \text{for} \quad t'>t \end{cases}$$

We then get

$$\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(x'-x)}}{k^2+i\varepsilon} = -i\int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\theta(t-t')e^{-ik(x-x')} + \theta(t-t')e^{ik(x-x')}\right]$$

and

$$
\begin{aligned}
\left\langle 0 \left| T\left(A_\mu(x) A_\nu(x')\right) \right| 0 \right\rangle &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{\lambda,\lambda'} \epsilon_\mu(k,\lambda)\epsilon_\nu(k,\lambda')[\theta(t-t')e^{-ik(x-x')} + \theta(t-t')e^{ik(x-x')}] \\
&= i\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x'-x)}}{k^2+i\varepsilon} \sum_{\lambda=1}^{2} \epsilon_\nu(k,\lambda)\epsilon_\mu(k,\lambda) = iD_{\mu\nu}(x,x')
\end{aligned}
$$

Or

$$
D_{\mu\nu}(x,x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x'-x)}}{k^2+i\varepsilon} \sum_{\lambda=1}^{2} \epsilon_\nu(k,\lambda)\epsilon_\mu(k,\lambda)
$$

polarization vectors $\epsilon_\mu(k,\lambda)$, $\lambda = 1,2$ are perpendicular to each other. Add 2 more unit vectors to form a complete set

$$
\eta^\mu = (1,0,0,0), \qquad \hat{k}^\mu = \frac{k^\mu - (k\cdot\eta)\eta^\mu}{\sqrt{(k\cdot\eta)^2 - k^2}}
$$

completeness relation is then,

$$
\begin{aligned}
\sum_{\lambda=1}^{2} \epsilon_\nu(k,\lambda)\epsilon_\mu(k,\lambda) &= -g_{\mu\nu} - \eta_\mu\eta_\nu - \hat{k}_\mu\hat{k}_\nu \\
&= -g_{\mu\nu} - \frac{k_\mu k_\nu}{(k\cdot\eta)^2 - k^2} + \frac{(k\cdot\eta)\left(k_\mu\eta_\nu + \eta_\mu k_\nu\right)}{(k\cdot\eta)^2 - k^2} - \frac{k^2\eta_\mu\eta_\nu}{(k\cdot\eta)^2 - k^2}
\end{aligned}
$$

If we define propagator in momentum space as

$$D_{\mu\nu}\left(x,x'\right)=\int\frac{d^4k}{\left(2\pi\right)^4}e^{-ik\cdot\left(x'-x\right)}D_{\mu\nu}\left(k\right)$$

then

$$D_{\mu\nu}\left(k\right)=\frac{1}{k^2+i\varepsilon}\left[-g_{\mu\nu}-\frac{k_\mu k_\nu}{\left(k\cdot\eta\right)^2-k^2}+\frac{\left(k\cdot\eta\right)\left(k_\mu\eta_\nu+\eta_\mu k_\nu\right)}{\left(k\cdot\eta\right)^2-k^2}-\frac{k^2\eta_\mu\eta_\nu}{\left(k\cdot\eta\right)^2-k^2}\right]$$

terms proportional to k_μ will not contribute to physical processes and the last term is of the form $\delta_{\mu0}\ \delta_{\nu0}$ will be cancelled by the Coulom interaction..

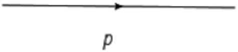
Feynman rule in QED

The interaction Hamiltonian is ,

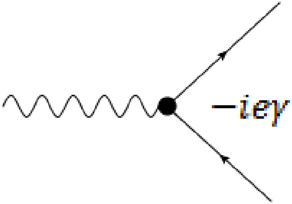
$$H_{int} = e \int d^3x \bar{\psi} \gamma^\mu \psi A_\mu$$

The Feynman propagators, vertices and external wave functions are given below.

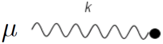
$$\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}$$



$$\frac{i}{\not{p} - m}$$

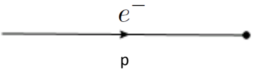


$$-ie\gamma^\mu$$

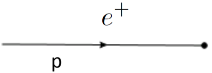


$$\varepsilon_\mu(k)$$

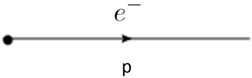
$$\varepsilon_\mu(k)$$



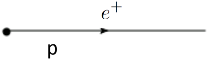
$u(p, s)$



$\overline{v}(p, s)$



$\overline{u}(p, s)$



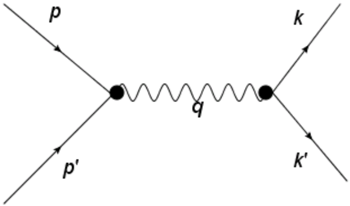
$v(p, s)$

$e^+e^- \to \mu^+\mu^-$

Total Cross Section

momenta for this reaction

$$e^+ (p') + e^- (p) \to \mu^+ (k') + \mu^- (k)$$



Use Feynman rule to write the matrix element as

$$\begin{aligned} M(e^+e^- \quad &\to \quad \mu^+\mu^-) = \bar{v}(p', s') \left(-ie\gamma^\mu\right) u(p,s) \left(\frac{-ig_{\mu\nu}}{q^2}\right) \bar{u}(k', r') \left(-ie\gamma^\nu\right) v(k,r) \\ &= \quad \frac{ie^2}{q^2} \bar{v}(p', s')\gamma^\mu u(p,s)\, \bar{u}(k', r')\gamma_\mu v(k,r) \end{aligned}$$

where $q = p + p'$. Notet that electron vertex have property,

$$q_\mu \bar{v}(p')\gamma^\mu u(p) = (p + p')_\mu \bar{v}(p')\gamma^\mu u(p) = \bar{v}(p') \left(\not{p} + \not{p}'\right) u(p) = 0$$

This shows the term proportional to photon momentum q^μ will not contribute in the physical processes. For cross section, we need M^* which contains factor $(\bar{v}\gamma^\mu u)^*$

$$(\bar{v}\gamma^\mu u)^* = u^\dagger (\gamma^\mu)^\dagger (\gamma_0)^\dagger v = u^\dagger \gamma_0 \gamma^\mu v = \bar{u}\gamma^\mu v$$

More generally,

$$(\bar{v}\Gamma u)^* = \bar{u}\bar{\Gamma} v, \qquad \text{with } \bar{\Gamma} = \gamma^0 \Gamma^\dagger \gamma^0$$

It is easy to see

$$\bar{\gamma}_\mu = \gamma_\mu$$

$$\overline{\gamma_\mu \gamma_5} = -\gamma_\mu \gamma_5$$

$$\overline{\not{a}\not{b}\cdots\not{p}} = \not{p}\cdots\not{b}\not{a}$$

unpolarized cross section which requires the spin sum,

$$\sum_s u_\alpha(p,s)\,\bar{u}_\beta(p,s) = (\not{p}+m)_{\alpha\beta}$$

$$\sum_s v_\alpha(p,s)\,\bar{v}_\beta(p,s) = (\not{p}-m)_{\alpha\beta}$$

This can be seen as follows.

$$\begin{aligned}
\sum_s u_\alpha(p,s)\,\bar{u}_\beta(p,s) &= (E+m)\begin{pmatrix} 1 \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} \sum_s \chi_s \chi_s^\dagger \begin{pmatrix} 1 & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \end{pmatrix} = (E+m)\begin{pmatrix} 1 & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & \frac{-\vec{p}^2}{(E+m)^2} \end{pmatrix} \\
&= \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E+m \end{pmatrix} = \not{p}+m
\end{aligned}$$

Similarly for the $v-$spinor,

$$\begin{aligned}
\sum_s v_\alpha(p,s)\,\bar{v}_\beta(p,s) &= (E+m)\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ 1 \end{pmatrix}\chi_s\chi_s^\dagger\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & -1 \end{pmatrix} = (E+m)\begin{pmatrix} \frac{\vec{p}^2}{(E+m)^2} & -\frac{\vec{\sigma}\cdot\vec{p}}{E+m} \\ \frac{\vec{\sigma}\cdot\vec{p}}{E+m} & -1 \end{pmatrix} \\
&= \begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E+m) \end{pmatrix} = \not{p} - m
\end{aligned}$$

A typical calculation is,

$$\begin{aligned}
& \sum_{s,s'} \bar{v}_\alpha(p',s') \left(\gamma^\mu\right)_{\alpha\beta} u_\beta(p,s)\, \bar{u}_\rho(p,s) \left(\gamma^\nu\right)_{\rho\sigma} v_\sigma(p,s) \\
= \quad & \sum_{s'} \bar{v}_\alpha(p',s') \left(\gamma^\mu\right)_{\alpha\beta} \left(\not{p}+m\right)_{\beta\rho} \left(\gamma^\nu\right)_{\rho\sigma} v_\sigma(p,s) \\
= \quad & \left(\gamma^\mu\right)_{\alpha\beta} \left(\not{p}+m\right)_{\beta\rho} \left(\gamma^\nu\right)_{\rho\sigma} \left(\not{p}-m\right)_{\sigma\alpha} \\
= \quad & Tr\left[\gamma^\mu \left(\not{p}+m\right) \gamma^\nu \left(\not{p}-m\right)\right]
\end{aligned}$$

trace of product of γ matrices.

$$Tr\left(\gamma^\mu\right) = 0$$

$$Tr\left(\gamma^\mu\gamma^\nu\right) = 4g^{\mu\nu}$$

$$Tr\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = 4\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha}\right)$$

$$\begin{aligned}
& Tr\left(\not{a}_1\not{a}_2\cdots\not{a}_n\right) \\
= \quad & \left(a_1\cdot a_2\right) Tr\left(\not{a}_3\cdots\not{a}_n\right) - \left(a_1\cdot a_3\right) Tr\left(\not{a}_2\cdots\not{a}_n\right) + \cdots + \left(a_1\cdot a_n\right) Tr\left(\not{a}_2\not{a}_3\cdots\not{a}_{n-1}\right), \qquad n \quad \text{even} \\
= \quad & 0 \qquad n \quad \text{odd}
\end{aligned}$$

With these tools

$$\frac{1}{4}\sum_{spin}\left|M(e^+e^-\to\mu^+\mu^-)\right|^2 = \frac{e^4}{q^4} Tr\left[\left(\not{p}'-m_e\right)\gamma^\mu\left(\not{p}+m_e\right)\gamma^\nu\right] Tr\left[\left(\not{k}'+m_\mu\right)\gamma_\mu\left(\not{k}+m_\mu\right)\gamma^\nu\right]$$

$$
\begin{aligned}
Tr\left[\left(\not{p}' - m_e\right)\gamma^\mu\left(\not{p} + m_e\right)\gamma^\nu\right] &= Tr\left[\not{p}'\gamma^\mu\not{p}\gamma^\nu\right] - m^2\, Tr\left[\gamma^\mu\gamma^\nu\right] \\
&= 4\left[p'^\mu p^\nu - g^{\mu\nu}\left(p\cdot p'\right) + p^\mu p'^\nu\right] - 4m_e^2 g^{\mu\nu}
\end{aligned}
$$

$$
\begin{aligned}
Tr\left[\left(\not{k}' + m_\mu\right)\gamma_\mu\left(\not{k} - m_\mu\right)\gamma^\nu\right] &= Tr\left[\not{k}'\gamma_\mu\not{k}\gamma^\nu\right] - m_\mu^2\, Tr\left[\gamma_\mu\gamma^\nu\right] \\
&= 4\left[k'^\mu k^\nu - g^{\mu\nu}\left(k\cdot k'\right) + k^\mu k'^\nu\right] - 4m_\mu^2 g^{\mu\nu}
\end{aligned}
$$

for energies $\gg m_\mu$.

$$
\frac{1}{4}\sum_{spin'}\left|M(e^+e^- \to \mu^+\mu^-)\right|^2 = 8\frac{e^4}{q^4}\left[(p\cdot k)\left(p'\cdot k'\right) + (p'\cdot k)\left(p\cdot k'\right)\right]
$$

In center of mass,

$$
p_\mu = (E, 0, 0, E)\,, \qquad p'_\mu = (E, 0, 0, -E)
$$

$$
k_\mu = \left(E, \vec{k}\right), \qquad k'_\mu = \left(E, -\vec{k}\right), \qquad \text{with } \vec{k}\cdot\hat{z} = E\cos\theta
$$

If we set $m_\mu = 0$, $E = \left|\vec{k}\right|$ and

$$
q^2 = (p + p')^2 = 4E^2, \qquad p\cdot k = p'\cdot k' = E^2\left(1 - \cos\theta\right),
$$

$$
p'\cdot k = p\cdot k' = E^2\left(1 + \cos\theta\right)
$$

Then

$$\frac{1}{4}\sum_{spin'}|M|^2 = \frac{8e^4}{16E^4}\left[E^4\left(1-\cos\theta\right)^2 + E^4\left(1+\cos\theta\right)^2\right]$$
$$= e^4\left(1+\cos^2\theta\right)$$

Note that under the parity $\theta \to \pi - \theta$. this matrix element conserves the parity
The cross section is

$$d\sigma = \frac{1}{I}\frac{1}{2E}\frac{1}{2E}(2\pi)^4\delta^4(p+p'-k-k')\frac{1}{4}\sum_{spin'}|M|^2\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}$$

use the $\delta-$function to carry out integrations . introduce the quantity ρ, called the **phase space**, given by

$$\rho = \int(2\pi)^4\delta^4(p+p'-k-k')\frac{d^3k}{(2\pi)^3 2\omega}\frac{d^3k'}{(2\pi)^3 2\omega'}$$
$$= \frac{1}{4\pi^2}\int\delta\left(2E-\omega-\omega'\right)\frac{d^3k}{4\omega\omega'} = \frac{1}{32\pi^2}\int\delta\left(E-\omega\right)\frac{k^2dkd\Omega}{\omega^2} = \frac{d\Omega}{32\pi^2}$$

The flux factor is

$$I = \frac{1}{E_1E_2}\sqrt{\left(p_1\cdot p_2\right)^2 - m_1^2m_2^2} = \frac{1}{E^2}2E^2 = 2$$

The differential crossection is then

$$d\sigma = \frac{1}{2}\frac{1}{4E^2}\left(\frac{1}{4}\sum_{spin'}|M|^2\right)\frac{d\Omega}{32\pi^2}$$

Or

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\left(1+\cos^2\theta\right)$$

where $\alpha = \dfrac{e^2}{4\pi}$ is the fine structure constant. The total cross section is

$$\sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{\alpha^2\pi}{3E^2}$$

Or

$$\sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{4\alpha^2\pi}{3s} \qquad \text{with} \qquad s = (p_1+p_2)^2 = 4E^2$$

$e^+e^- \longrightarrow hadrons$

One of the interesting processess in e^+e^- collider is the reaction

$$e^+e^- \longrightarrow hadrons$$

According to QCD, theory of strong interaciton, this processes will go through

$$e^+e^- \longrightarrow q\bar{q}$$

and then $q\bar{q}$ trun into hadrons. Since coupling of γ to $q\bar{q}$ differs from the coupling to $\mu^+\mu^-$ only in their charges cross section for $q\bar{q}$ as

$$\sigma\left(e^+e^- \longrightarrow q\bar{q}\right) = 3\left(Q_q^2\right)\frac{4\alpha^2\pi}{3s} = 3\left(Q_q^2\right)\sigma\left(e^+e^- \longrightarrow \mu^+\mu^-\right)$$

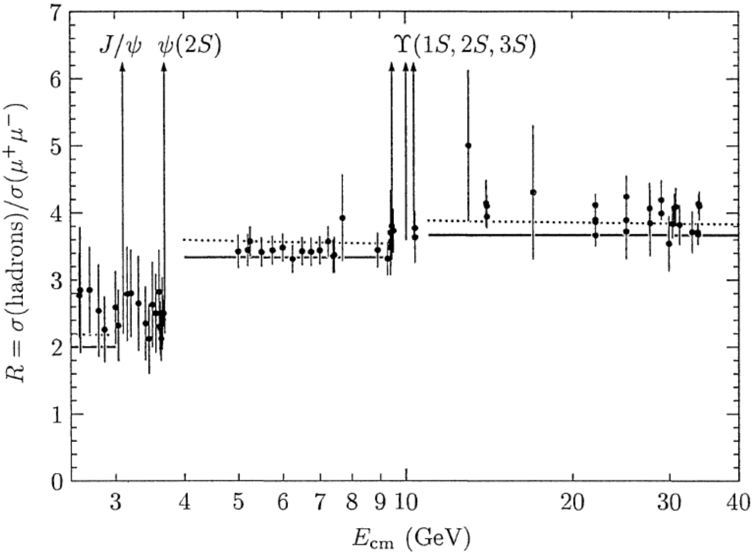
Q_q is electric charge of quark q. The factor of 3 because each quark has 3 colors. Then

$$\frac{\sigma\left(e^+e^- \longrightarrow hadrons\right)}{\sigma\left(e^+e^- \longrightarrow \mu^+\mu^-\right)} = 3\left(\sum_i Q_i^2\right)$$

Summation is over quarks which are allowed by the avaliable energies. e. g., for energy below the the charm quark only u, d, and s quarks should be included,

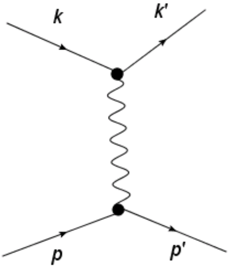
$$\frac{\sigma\left(e^+e^- \longrightarrow hadrons\right)}{\sigma\left(e^+e^- \longrightarrow \mu^+\mu^-\right)} = 3\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] = 2$$

which is not far from the reality.


J/ψ
$\psi(2S)$
$\Upsilon(1S, 2S, 3S)$
$R = \sigma(\text{hadrons})/\sigma(\mu^+\mu^-)$
E_{cm} (GeV)
0
1
2
3
4
5
6
7
3
4
5
6
7
8
9
10
20
30
40

$ep \to ep$,

$$e\,(k) + p\,(p) \longrightarrow e\,(k') + p\,(p')$$



Proton has strong interaction. First consider proton has no strong interaction and include strong interaction later. The lowest order contribution is ,

$$\begin{aligned} M(e+p \quad &\to \quad e+p) = \bar{u}(p',s')\,(-ie\gamma^{\mu})\,u\,(p,s)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}(k',r')\,(-ie\gamma^{\nu})\,u\,(k,r) \\ &= \quad \frac{ie^2}{q^2}\,\bar{u}(p',s')\gamma^{\mu}u\,(p,s)\,\bar{u}(k',r')\gamma_{\mu}u\,(k,r) \end{aligned}$$

where $q = k - k'$. For unploarized cross section, sum over the spins ,

$$\frac{1}{4}\sum_{spin}|M(e+p\to e+p)|^2 = \frac{e^4}{q^4} Tr\left[\left(\not{p}' + M\right)\gamma^\mu\left(\not{p}+M\right)\gamma^\nu\right] Tr\left[\left(\not{k}' + m_e\right)\gamma_\mu\left(\not{k}+m_e\right)\gamma^\nu\right]$$

Again neglect m_e. Compute the traces

$$Tr\left[\not{k}'\gamma_\mu\not{k}\gamma^\nu\right] = 4\left[k'^\mu k^\nu - g^{\mu\nu}\left(k\cdot k'\right) + k^\mu k'^\nu\right]$$

$$Tr\left[\left(\not{p}' + M\right)\gamma^\mu\left(\not{p}+M\right)\gamma^\nu\right] = 4\left[p'^\mu p^\nu - g^{\mu\nu}\left(p\cdot p'\right) + p^\mu p'^\nu\right] + 4M^2 g^{\mu\nu}$$

Then

$$\frac{1}{4}\sum_{spin}|M(e+p\to e+p)|^2 = \frac{e^4}{q^4}\left\{8\left[(p\cdot k)\left(p'\cdot k'\right) + (p'\cdot k)\left(p\cdot k'\right)\right] - 8M^2\left(k\cdot k'\right)\right\}$$

More useful to use the laboratoy frame

$$p_\mu = (M, 0, 0, 0)\,, \qquad k_\mu = \left(E, \vec{k}\right), \qquad k'_\mu = \left(E', \vec{k'}\right)$$

Then

$$p\cdot k = ME, \qquad p.k' = ME', \qquad k\cdot k' = EE'\,(1-\cos\theta)$$

$$p'\cdot k' = (p+k-k')\,.k' = p.k' + k\cdot k', \qquad p'\cdot k = (p+k-k')\,.k = p.k - k\cdot k'$$

$$q^2 = (k-k')^2 = -2k\cdot k' = -2EE'\,(1-\cos\theta)$$

Differential cross section is

$$d\sigma = \frac{1}{I}\frac{1}{2p_0}\frac{1}{2k_0}(2\pi)^4\delta^4(p+k-p'-k')\frac{1}{4}\sum_{spin'}|M|^2\frac{d^3p'}{(2\pi)^3 2p'_0}\frac{d^3k'}{(2\pi)^3 2k'_0}$$

The phase space is

$$\begin{aligned}\rho &= \int (2\pi)^4\delta^4(p+k-p'-k')\frac{d^3p'}{(2\pi)^3 2p'_0}\frac{d^3k'}{(2\pi)^3 2k'_0} \qquad (1)\\ &= \frac{1}{4\pi^2}\int \delta\left(p_0+k_0-p'_0-k'_0\right)\frac{d^3k'}{2p'_0 2k'_0}\end{aligned}$$

where

$$p'_0 = \sqrt{M^2 + \left(\vec{p}+\vec{k}-\vec{k'}\right)^2} = \sqrt{M^2 + \left(\vec{k}-\vec{k'}\right)^2}$$

Use the momenta in lab frame,

$$\begin{aligned}\rho &= \frac{1}{4\pi^2}\int \delta\left(M+E-p_0'-E'\right)\frac{k'^2 dk' d\Omega}{2p_0' 2E'} \\ &= \frac{1}{4\pi^2}\int \delta\left(M+E-p_0'-E'\right)\frac{d\Omega E' dE'}{p_0'}\end{aligned}$$

Let

$$x = -E + p_0' + E'$$

Then

$$dx = dE'\left(1+\frac{dp_0'}{dE'}\right) = dE'\left(\frac{p_0' + E' - E\cos\theta}{p_0'}\right)$$

and

$$\rho = \frac{1}{4\pi^2}\int \delta\left(x-M\right)\frac{d\Omega E' dx}{\left(p_0' + E' - E\cos\theta\right)} = \frac{1}{4\pi^2}\frac{d\Omega E'}{M + E\left(1-\cos\theta\right)}$$

From the argument of the $\delta-$function we get the relation, $M = x = -E + p_0' + E'$
From momentum conservation

$$p_0'^2 = M^2 + \left(\vec{k} - \vec{k}'\right)^2 = M^2 + E^2 + E'^2 - 2EE'\cos\theta$$

and from energy conservation

$$p_0'^2 = \left(M + E - E'\right)^2 = M^2 + E^2 + E'^2 - 2EE' + 2ME - 2ME'$$

Comparing these 2 equations we can solve for E',

$$E' = \frac{ME}{E\left(1-\cos\theta\right)+M} = \frac{E}{1+\left(\frac{2E}{M}\right)\sin^2\frac{\theta}{2}}$$

The phase space is then

$$\rho = \frac{d\Omega}{4\pi^2}\frac{ME}{\left(M+E\left(1-\cos\theta\right)\right)^2} = \frac{d\Omega}{4\pi^2}\frac{E'^2}{ME}$$

The flux factor is

$$I = \frac{1}{ME}p\cdot k = 1$$

The differential cross section is then

$$d\sigma = \frac{1}{I}\frac{1}{2p_0}\frac{1}{2k_0}(2\pi)^4\delta^4(p+k-p'-k')\frac{1}{4}\sum_{spin'}|M|^2\frac{d^3p'}{(2\pi)^3 2p_0'}\frac{d^3k'}{(2\pi)^3 2k_0'}$$

Or

$$\frac{d\sigma}{d\Omega} = \frac{1}{4ME}\frac{1}{4\pi^2}\frac{E'^2}{ME}\frac{1}{4}\sum_{spin'}|M|^2 = \left(\frac{E'}{E}\right)^2\frac{1}{16\pi^2M^2}\frac{e^4}{q^4}\left\{8\left[(p\cdot k)\left(p'\cdot k'\right)+(p'\cdot k)\left(p\cdot k'\right)\right]-8M^2\left(k\cdot k'\right)\right\}$$

It is staightforward to get

$$
\begin{aligned}
& \left[(p\cdot k)\left(p'\cdot k'\right)+(p'\cdot k)\left(p\cdot k'\right)\right]-M^2\,(k\cdot k') \\
= \quad & \left[(p\cdot k)\,(p+k-k')\cdot k'+(p\cdot k')\,(p+k-k')\cdot k-M^2\,(k\cdot k')\right] \\
= \quad & 2EE'M^2+(k\cdot k')\,(p\cdot q-M^2) \\
= \quad & 2EE'M^2+M^2EE'\,(1-\cos\theta)\left(-\frac{q^2}{2M^2}-1\right) \\
= \quad & 2EE'M^2\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]
\end{aligned}
$$

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left(\frac{E'}{E}\right)^2\frac{\alpha^2}{M^2}\frac{1}{\left(4EE'\sin^2\frac{\theta}{2}\right)^2}2EE'M^2\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right] \\
&= \frac{\alpha^2}{4}\frac{E'}{E^3}\frac{1}{\sin^4\frac{\theta}{2}}\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]
\end{aligned}
$$

Or

$$
\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{4E^2}\frac{1}{\sin^4\frac{\theta}{2}}\frac{\left[\cos^2\frac{\theta}{2}-\frac{q^2}{2M^2}\sin^2\frac{\theta}{2}\right]}{\left[1+\left(\frac{2E}{M}\right)\sin^2\frac{\theta}{2}\right]}
$$

Now include the strong interaction. Use the fact that the γpp interaction is local to parametrize the γpp matrix element as

$$\left\langle p' \left| J_\mu \right| p \right\rangle = \bar{u}(p', s') \left[\gamma^\mu F_1 \left(q^2\right) + \frac{i\sigma_{\mu\nu} q^\nu}{2M} F_2 \left(q^2\right) \right] u\left(p, s\right) \qquad \text{with} \qquad q = p - p' \tag{2}$$

Lorentz covariance and current conservation have been used. Another useful relation is the Gordon decomposition

$$\bar{u}(p')\gamma_\mu u\left(p\right) = \bar{u}(p') \left[\frac{\left(p + p'\right)^\mu}{2m} + \frac{i\sigma^{\mu\nu}\left(p' - p\right)_\nu}{2m} \right] u\left(p\right)$$

This can be derived as follows. From Dirac equation

$$(\not{p} - m)u(p) = 0, \qquad \bar{u}(p')(\not{p}' - m) = 0$$

and

$$\bar{u}(p')\gamma^\mu(\not{p} - m)u(p) = 0, \qquad \bar{u}(p')(\not{p}' - m)\gamma^\mu u(p) = 0$$

Adding these equations,

$$\begin{aligned} 2m\bar{u}(p')\gamma_\mu u\left(p\right) &= \bar{u}(p')\left(\gamma_\mu \not{p} + \not{p}'\gamma_\mu\right) u\left(p\right) = \bar{u}(p')\left(p^\nu \gamma_\mu\gamma_\nu + p'^\nu \gamma_\nu\gamma_\mu\right) u\left(p\right) \\ &= \bar{u}(p')\left(p^\nu\left(\frac{1}{2}\left\{\gamma_\mu, \gamma_\nu\right\} + \frac{1}{2}\left[\gamma_\mu, \gamma_\nu\right]\right) + p'^\nu\left(\frac{1}{2}\left\{\gamma_\mu, \gamma_\nu\right\} - \frac{1}{2}\left[\gamma_\mu, \gamma_\nu\right]\right)\right) u\left(p\right) \end{aligned}$$

From this we get

$$\bar{u}(p')\gamma_\mu u\left(p\right) = \bar{u}(p') \left[\frac{\left(p + p'\right)^\mu}{2m} + \frac{i\sigma^{\mu\nu}\left(p' - p\right)_\nu}{2m} \right] u\left(p\right)$$

$F_1\left(q^2\right)$, charge form factor

$F_2\left(q^2\right)$, magnetic form factor .
Note that $F_1\left(q^2\right)=1$ and $F_2\left(q^2\right)=0$ correspond to point particle.
The charge form factor satifies the condition $F_1(0)=1$. From

$$Q\,|p\rangle=|p\rangle$$

we get

$$\left\langle p'\,|Q|\,p\right\rangle=\left\langle p'|p\right\rangle=2E\,(2\pi)^3\,\delta^3\left(\vec{p}-\vec{p}'\right)$$

On the other hand from Eq(2) we see that

$$\begin{aligned}\left\langle p' \left| Q \right| p \right\rangle &= \int d^3x \left\langle p' \left| J_0\left(x\right) \right| p \right\rangle = \int d^3x \left\langle p' \left| J_0\left(0\right) \right| p \right\rangle e^{i\left(p'-p\right)\cdot x} \\ &= \left(2\pi\right)^3 \delta^3\left(\vec{p}-\vec{p}'\right) \bar{u}(p',s')\gamma_0 u\left(p,s\right) F_1\left(0\right) \\ &= 2E\left(2\pi\right)^3 \delta^3\left(\vec{p}-\vec{p}'\right) F_1\left(0\right)\end{aligned}$$

compare two equations $\Longrightarrow F_1\left(0\right) = 1$. To gain more insight, write Q in terms of charge density

$$Q = \int d^3x\rho\left(x\right) = \int d^3xJ_0\left(x\right)$$

Then

$$\left\langle p' \left| J_0\left(x\right) \right| p \right\rangle = e^{iq\cdot x} \left\langle p' \left| J_0\left(0\right) \right| p \right\rangle = e^{iq\cdot x} F_1\left(q^2\right) \bar{u}(p',s')\gamma_0 u\left(p,s\right)$$

$F_1\left(q^2\right)$ is the Fourier transform of charge density distribution i.e.

$$F_1\left(q^2\right) \sim \int d^3x\rho\left(x\right) e^{-i\vec{q}\cdot\vec{x}}$$

Expand $F_1\left(q^2\right)$ in powers of q^2,

$$F_1\left(q^2\right) = F_1\left(0\right) + q^2 F_1'\left(0\right) + \cdots$$

$F_1\left(0\right)$ is total charge and $F_1'\left(0\right)$ is related to the charge radius.
Calulate cross section as before,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2} \frac{\left[\cos^2\frac{\theta}{2}\left(\frac{1}{1-q^2/4M^2}\right)\left[G_E^2 - \left(q^2/4M^2\right)G_M^2\right] - \frac{q^2}{2M^2}\sin^2\frac{\theta}{2}G_M^2\right]}{\sin^4\frac{\theta}{2}\left[1+\left(\frac{2E}{M}\right)\sin^2\frac{\theta}{2}\right]}$$

where

$$G_E = F_1 + \frac{q^2}{4M^2} F_2$$

$$G_M = F_1 + F_2$$

Experimentally, G_E and G_M have the form,

$$G_E\left(q^2\right) \approx \frac{G_M\left(q^2\right)}{\varkappa_p} \approx \frac{1}{\left(1 - q^2/0.7Gev^2\right)^2} \tag{3}$$

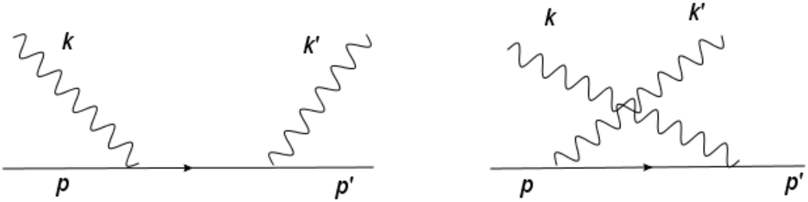
where $\varkappa_p = 2.79$ magnetic moment of the proton. If proton were point like, we would have $G_E\left(q^2\right)$ $= G_M\left(q^2\right) = 1$

Dependence of q^2 in Eq(3) $\Longrightarrow$ proton has a structure. For large q^2 the elastic cross section falls off rapidly as $G_E \approx G_M \sim q^{-4}$.

Compton Scattering

$$\gamma(k) + e(p) \longrightarrow \gamma(k') + e(p')$$

Two diagrams contribute,



The amplitude is given by

$$\begin{aligned} M(\gamma e \longrightarrow \gamma e) \quad &= \quad \bar{u}(p')(-ie\gamma^{\mu})\varepsilon'_{\mu}(k')\,\frac{i}{\not{p}+\not{k}-m}\,(-ie\gamma^{\nu})\,\varepsilon_{\nu}(k)\,u(p) \\ &\quad + \bar{u}(p')(-ie\gamma^{\mu})\varepsilon_{\mu}(k)\,\frac{i}{\not{p}-\not{k}'-m}\,(-ie\gamma^{\nu})\,\varepsilon'_{\nu}(k')\,u(p) \end{aligned}$$

Put the γ- matrices in the numerator,

$$M = -ie^2\varepsilon'_{\mu}\varepsilon_{\nu}\left[\bar{u}(p')\gamma^{\mu}\,\frac{\not{p}+\not{k}+m}{2p\cdot k}\gamma^{\nu}u(p) + \bar{u}(p')\gamma^{\nu}\,\frac{\not{p}-\not{k}'+m}{-2p\cdot k'}\gamma^{\mu}u(p)\right]$$

Using the relations,

$$\left(\not{p} + m\right)\gamma^{\nu} u\left(p\right) = 2p^{\nu} u\left(p\right),$$

we get

$$M = -ie^{2}\bar{u}(p')\left[\frac{\not{\varepsilon}'\not{k}\not{\varepsilon} + 2\left(p\cdot\varepsilon\right)\not{\varepsilon}'}{2p\cdot k} + \frac{-\not{\varepsilon}\not{k}'\not{\varepsilon}' + 2\left(p\cdot\varepsilon\right)\not{\varepsilon}'}{-2p\cdot k'}\right]u\left(p\right)$$

The photon polarizations are,

$$\varepsilon_\mu = \left(0, \vec{\varepsilon}\right), \qquad \text{with} \quad \vec{\varepsilon} \cdot \vec{k} = 0, \qquad \varepsilon'_\mu = \left(0, \vec{\varepsilon}'\right), \qquad \text{with} \quad \vec{\varepsilon}' \cdot \vec{k}' = 0,$$

Lab frame , $p_\mu = (m, 0, 0, 0)$, $\Longrightarrow$ $(p \cdot \varepsilon) = (p \cdot \varepsilon') = 0$ and

$$M = -ie^2 \bar{u}(p') \left[\frac{\not{\varepsilon}' \not{k} \not{\varepsilon}}{2p \cdot k} + \frac{\not{\varepsilon} \not{k}' \not{\varepsilon}'}{2p \cdot k'} \right] u(p)$$

Summing over spin of the electron

$$\frac{1}{2} \sum_{spin} |M|^2 = e^4 Tr \left\{ (\not{p}' + m) \left[\frac{\not{\varepsilon}' \not{k} \not{\varepsilon}}{2p \cdot k} + \frac{\not{\varepsilon} \not{k}' \not{\varepsilon}'}{2p \cdot k'} \right] (\not{p} + m) \left[\frac{\not{\varepsilon}' \not{k} \not{\varepsilon}}{2p \cdot k} + \frac{\not{\varepsilon} \not{k}' \not{\varepsilon}'}{2p \cdot k'} \right] \right\}$$

The cross section is given by

$$d\sigma = \frac{1}{I} \frac{1}{2p_0} \frac{1}{2k_0} (2\pi)^4 \delta^4(p + k - p' - k') \frac{1}{4} \sum_{spin'} |M|^2 \frac{d^3 p'}{(2\pi)^3 2p'_0} \frac{d^3 k'}{(2\pi)^3 2k'_0}$$

phase space

$$\rho = \int (2\pi)^4 \delta^4(p + k - p' - k') \frac{d^3 p'}{(2\pi)^3 2p'_0} \frac{d^3 k'}{(2\pi)^3 2k'_0}$$

is exactly the same as the case for *ep* scattering and the result is

$$\rho = \frac{d\Omega}{4\pi^2} \frac{\omega'^2}{m\omega}$$

It is straightforward to compute the trace with result,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4m^2} \left(\frac{\omega'}{\omega}\right)^2 \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 4\left(\varepsilon \cdot \varepsilon'\right)^2 - 2\right]$$

This is **Klein-Nishima** relation. In the limit $\omega \to 0$,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{m^2} \left(\varepsilon \cdot \varepsilon'\right)^2$$

here $\dfrac{\alpha}{m}$ is classical electron radius.

For unpolarized cross section, sum over polarization of photon,

$$\sum_{\lambda\lambda'} \left[\varepsilon(k,\lambda)\cdot\varepsilon'(k',\lambda')\right]^2 = \sum_{\lambda\lambda'} \left[\vec{\varepsilon}(k,\lambda)\cdot\vec{\varepsilon}'(k',\lambda')\right]^2$$

Since $\vec{\varepsilon}(k,1)$, $\vec{\varepsilon}(k,2)$ and $\vec{k}$ form basis in 3-dimension, completeness relation is

$$\sum_{\lambda} \varepsilon_i(k,\lambda)\,\varepsilon_j(k,\lambda) = \delta_{ij} - \hat{k}_i\hat{k}_j$$

Then

$$\sum_{\lambda\lambda'} \left[\vec{\varepsilon}(k,\lambda)\cdot\vec{\varepsilon}'(k',\lambda')\right]^2 = \left(\delta_{ij} - \hat{k}_i\hat{k}_j\right)\left(\delta_{ij} - \hat{k}'_i\hat{k}'_j\right) = 1 + \cos^2\theta$$

where $\hat{k}\cdot\hat{k}' = \cos\theta$. The cross section is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]$$

The total cross section,

$$\sigma = \frac{\pi\alpha^2}{m^2}\int_{-1}^{1} dz\{\frac{1}{\left[1+\frac{\omega}{m}(1-z)\right]^3} + \frac{1}{\left[1+\frac{\omega}{m}(1-z)\right]} - \frac{1-z^2}{\left[1+\frac{\omega}{m}(1-z)\right]^2}\}$$

At low energies, $\omega \to 0$, we

$$\sigma = \frac{8\pi\alpha^2}{3m^2}$$

and at high energies

$$\sigma = \frac{\pi\alpha^2}{\omega m}\left[\ln\frac{2\omega}{m} + \frac{1}{2} + O\left(\frac{m}{\omega}\ln\frac{m}{\omega}\right)\right]$$