Quantum Field Theory

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Path integral formalism has close relationship to classical dynamics, e.g. the transition amplitude

$$\langle f|i\rangle = \int [dx]\, e^{iS/\hbar}$$

as $\hbar \to 0$, the trajectory with smallest S dominates, the action principle. Here uses the ordinary functions not the operators. Later in non-Abelian gauge theory, to remove unphysical degrees of freedom can be accomodated in the path integral formalism by imposing constraints in the integral.

Quantum Mechanics in 1-dimension

In QM, transition from $|q,t\rangle$ to $\langle q',t'|$, can be written as,

$$\langle q' t'|qt\rangle = \langle q|^{\prime - iH(t-t')}|q\rangle$$

where $|q\rangle's$ are eigenstates of position operator Q in the Schrodinger picture,

$$Q|q\rangle = q|q\rangle$$

and $|q,t\rangle$ denotes corresponding state in Heisenberg picture,

$$|q,t\rangle = e^{iHt}|q\rangle$$

Ipath integral formalism,this can be written as

$$\langle q' t'|qt\rangle = N \int [dq] exp\{i \int_t^{t'} d\tau L(q,\dot{q})\}$$

To get this formula, divide the interval $(t\prime, t)$ into n intervals ,

$$\delta t = \frac{t' - t}{n}$$

and write ,

$$\langle q'|e^{-iH(t'-t)}|q\rangle = \int dq_1 ... dq_{n-1} \langle q'|e^{-iH\delta t}|q_{n-1}\rangle \langle q_{n-1}|e^{-iH\delta t}|q_{n-2}\rangle ... \langle q_1|e^{-iH\delta t}|q\rangle$$

If we take the Hamiltonian to be in the simple form,

$$H(P, Q) = \frac{p^2}{2m} + V(Q)$$

then

$$\begin{aligned}
\langle q_j|H|q_i\rangle &= \langle q_j|\frac{p^2}{2m}|q_i\rangle + V(\frac{q_i + q_j}{2})\delta(q_i - q_j) \\
&= \int \langle q_j|\frac{p^2}{2m}|p_k\rangle \langle p_k|q_i\rangle (\frac{dp_k}{2\pi}) + V(\frac{q_i + q_j}{2}) \int \frac{dp_k}{2\pi} e^{ip_k(q_j - q_i)} \\
&= \int \frac{dp_k}{2\pi} e^{ip_k(q_j - q_i)} [\frac{p_k^2}{2m} + V(\frac{q_i + q_j}{2})]
\end{aligned}$$

where we have used

$$\langle p|q\rangle = e^{-ipq}$$

which is the momentum eigenfunction in coordinate space. Exponentiation of this infinitesmal result gives

$$\langle q_j|e^{-iH\delta t}|q_i\rangle \simeq \int \frac{dp_k}{2\pi} e^{ip_k(q_j-q_i)}\{1-i\delta t[\frac{p_k^2}{2m}+V(\frac{q_i+q_j}{2})]\} \simeq \int \frac{dp_k}{2\pi} e^{ip_k(q_j-q_i)} \exp\{-i\delta t[\frac{p_k^2}{2m}+V(\frac{q_i+q_j}{2})]\}$$

The whole transition matrix element can then be written as

$$\langle q'|e^{-iH(t'-t)}|q\rangle \cong \int(\frac{dp_1}{2\pi})...(\frac{dp_n}{2\pi})\int dq_1...dq_{n-1} \exp\{i\left[\sum_{i=1}^{n} p_i(q_i-q_{i-1})-(\delta t)H(p_i,\frac{q_i+q_{i+1}}{2})\right]\}$$

This can be written formally as

$$\langle q'|e^{-iH(t'-t)}|q\rangle = \int[\frac{dpdq}{2\pi}]exp\{i\int_t^{t'} dt[p\dot{q}-H(p,q)]\}$$

$$\equiv \lim_{n\to\infty}\int(\frac{dp_1}{2\pi})...(\frac{dp_n}{2\pi})\int dq_1...dq_{n-1} exp\{i\sum_{i=1}\delta t[p_i(\frac{q_i-q_{i-1}}{\delta t})-H(p_i,\frac{q_i+q_{i+1}}{2})]\}$$

If Hamiltonian depends quadractically on p, use the formula

$$\int_{-\infty}^{+\infty}\frac{dx}{2\pi}e^{-ax^2+bx} = \frac{1}{\sqrt{4\pi a}}e^{\frac{b^2}{4a}}$$

to get

$$\int\frac{dp_i}{2\pi}\exp[\frac{-i\delta t}{2m}p_i^2+ip_i(q_i-q_{i-1})] = (\frac{m}{2\pi i\delta t})^{1/2}exp[\frac{im(q_i-q_{i-1})^2}{2\delta t}]$$

Then

$$\langle q'|e^{-iH(t'-t)}|q\rangle = \lim_{n\to\infty}(\frac{m}{2\pi i\delta t})^{n/2}\int\prod_{i=1}^{n-1}dq_i \exp\{i\sum_{i=1}^{n}\delta t[\frac{m}{2}(\frac{q_i-q_{i-1}}{\delta t})^2 - V]\}$$

or

$$\langle q't'|qt\rangle = \langle q'|e^{-iH(t'-t)}|q\rangle = N\int[dq]\exp\{i\int_t^{t'}d\tau[\frac{m}{2}\dot{q}^2 - V(q)]\}$$

This is the path integral representation for amplitude from initial state $|q,t\rangle$ to final state $\langle q',t'|$. Or

$$\langle q't'|qt\rangle == N\int[dq]\exp iS$$

Green's functions

To generalize this to field theory where we have vacuum expectation value of field operators, we consider

$$G(t_1, t_2) = \langle 0|T(Q^H(t_1)Q^H(t_2))|0\rangle$$

Inserting complete sets of states,

$$G(t_1, t_2) = \int dqdq'\langle 0|q', t'\rangle\langle q', t'|T(Q^H(t_1)Q^H(t_2))|q, t\rangle\langle q, t|0\rangle$$

The matrix element

$$\langle 0|q, t\rangle = \phi_0(q)e^{-iE_0 t} = \phi_0(q, t)$$

is the wavefunction for ground state. Consider the case

$$t' > t_1 > t_2 > t,$$

we can write

$$\langle q', t'|T(Q^H(t_1)Q^H(t_2))|q, t\rangle = \langle q'|e^{-iH(t'-t_1)}Q^s e^{-iH(t_1-t_2)}Q^s e^{-iH(t_2-t)}|q\rangle$$

$$= \int \langle q'|e^{-iH(t'-t_1)}|q_1\rangle q_1\langle q_1|e^{-iH(t_1-t_2)}|q_2\rangle q_2\langle q_2|e^{-iH(t_2-t)}|q\rangle dq_1 dq_2$$

$$= \int [\frac{dpdq}{2\pi}]q_1(t_1)q_2(t_2)exp\{i\int_t^{t'} d\tau[p\dot{q} - H(p, q)]\}$$

For the other time sequence

$$t' > t_2 > t_1 > t,$$

we get same formula, because path integral orders the time sequence automatically through the division of time interval into small pieces. The Green's function is then

$$G(t_1, t_2) = \int dqdq' \phi_0(q', t')\phi_0^*(q, t) \int [\frac{dpdq}{2\pi}] q_1(t_1) q_2(t_2) exp\{i \int_t^{t'} d\tau [p\dot{q} - H(p, q)]\} \quad (1)$$

We remove wavefunction $\phi_0(q, t)$ by the following procedure. Write

$$\langle q', t' | \theta(t_1, t_2) | q, t \rangle = \int dQdQ' \langle q', t' | Q', T' \rangle \langle Q', T' | \theta(t_1, t_2) | Q, T \rangle \langle Q, t | q, t \rangle$$

where

$$\theta(t_1, t_2) = T(Q^H(t_1) Q^H(t_2))$$

Let $|n>$ be eigenstate with energy E_n and wave function ϕ_n, i.e.,

$$H|n >= E_n|n >, \quad \langle q|n \rangle = \phi_n^*(q)$$

Then

$$\langle q', t' | Q', t' \rangle = \langle q' | e^{-iH(t'-T')} | Q' \rangle = \sum_n \langle q' | n \rangle e^{-iE_n(t'-T')} \langle n | Q' \rangle = \sum_n \phi_n^*(q') \phi_n(Q') e^{-iE_n(t'-T')}$$

To isolate the ground state wavefunction, take an "unusual limit",

$$\lim_{t' \to -i\infty} \langle q', t' | Q', T' \rangle = \phi_0^*(q') \phi_0(Q') e^{-E_0 |t'|} e^{iE_0 T'}$$

Similarly,

$$\lim_{t\to i\infty}\langle Q,T|q,t\rangle = \phi_0(q)\phi_0^*(Q)e^{-E_0|t|}e^{-iE_0T}$$

With these we write

$$\lim_{\substack{t'\to -i\infty \\ t\to i\infty}} \langle q',t'|\theta(t_1,t_2)|q,t\rangle = \int dQdQ'\phi_0^*(q')\phi_0(Q')\langle Q',T'|\theta(t_1,t_2)|Q,T\rangle\phi_0^*(Q)\phi_0(q)e^{-E_0|t'|}e^{iE_0T'}e^{-iE_0T}e^{-E_0|t|}$$

$$= \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|}G(t_1,t_2)$$

It is easy to see that

$$\lim_{\substack{t'\to -i\infty \\ t\to i\infty}} \langle q',t'|q,t\rangle = \phi_0^*(q')\phi_0(q)e^{-E_0|t'|}e^{-E_0|t|}$$

Finally, the Green function can be written as,

$$G(t_1,t_2) = \lim_{\substack{t'\to -i\infty \\ t\to i\infty}} \left[\frac{\langle q',t'|T(Q^H(t_1)Q^H(t_2))|q,t\rangle}{\langle q',t'|q,t\rangle}\right]$$

$$= \lim_{\substack{t'\to -i\infty \\ t\to i\infty}} \frac{1}{\langle q',t'|q,t\rangle}\int\left[\frac{dpdq}{2\pi}\right]q(t_1)q(t_2)\exp\{i\int_t^{t'}d\tau[p\dot{q}-H(p,q)]\}$$

This can generalized to n-point Green's function with the result,

$$G(t_1,t_2,...,t_n) = \langle 0|T(q(t_1)q(t_2)...q(t_n))|0\rangle$$

$$= \lim_{\substack{t' \to -i\infty \\ t \to i\infty}} \frac{1}{\langle q', t' | q, t \rangle} \int \left[\frac{dpdq}{2\pi} \right] q(t_1) q(t_2) ... q(t_n) \exp\{ i \int_t^{t'} d\tau [p\dot{q} - H(p,q)] \}$$

It is very useful to introduce generating functional for these n-point functions

$$W[J] = \lim_{\substack{t' \to -i\infty \\ t \to i\infty}} \frac{1}{\langle q', t' | q, t \rangle} \int \left[\frac{dpdq}{2\pi} \right] \exp\{ i \int_t^{t'} d\tau [p\dot{q} - H(p,q) + J(\tau) q(\tau)] \}$$

Then

$$G(t_1, t_2, ..., t_n) = (-i)^n \left. \frac{\delta^n}{\delta J(t_1) ... \delta J(t_n)} \right|_{J=0}$$

The unphysical limit, $t' \to -i\infty, t \to i\infty$, should be interpreted in term of Eudidean Green's functions defined by

$$S^{(n)}(\tau_1, \tau_2, ..., \tau_n) = i^n G^{(n)}(-i\tau_1, -i\tau_2, ..., -i\tau_n)$$

Generating functional for $S^{(n)}$ is then

$$W_E[J] = \lim_{\substack{\tau' \to \infty \\ \tau \to -\infty}} \int [dq] \frac{1}{\langle q', t' | q, t \rangle} \exp\{ \int_{\tau}^{\tau'} d\tau'' [-\frac{m}{2} \left(\frac{dq}{d\tau''} \right)^2 - V(q) + J(\tau'') q(\tau'')] \}$$

Since we can adjust the zero point of $V(q)$ such that

$$\frac{m}{2} \left(\frac{dq}{d\tau} \right)^2 + V(q) > 0$$

which provides the damping to give a converging Gaussian integral. In this form, we can see that any constant in the path integral which is independent of q will be canceled out in the generation functional.

Field Theory

From quantum mechanics to field theory of a scalar field $\phi(x)$ replace,

$$\prod_{i=1}^{\infty}[dq_i dp_i] \longrightarrow [d\phi(x)d\pi(x)]$$

$$L(q,\dot{q}) \longrightarrow \int \mathcal{L}(\phi,\partial_\mu\phi)d^3x \qquad H(p,q) \longrightarrow \int \mathcal{H}(\phi,\pi)d^3x$$

Generating functional is

$$W[J] \backsim \int [d\phi] \exp\{i\int d^4x[\mathcal{L}(\phi,\partial_\mu\phi)+J(x)\phi(x)]\}$$

functional derivative is defined by

$$\frac{\delta F[\phi(x)]}{\delta\phi(y)} = \lim_{\varepsilon\to 0}\frac{F[\phi(x)+\varepsilon\delta(x-y)]-F[\phi(x)]}{\varepsilon}$$

Then

$$\frac{\delta W[J]}{\delta J(y)} = i\int [d\phi]\,\phi(y)\exp\{i\int d^4x[\mathcal{L}(\phi,\partial_\mu\phi)+J(x)\phi(x)]\} \quad (2)$$

and

$$\frac{\delta^2 W[J]}{\delta J(y_1)\,\delta J(y_2)} = (i)^2\int [d\phi]\,\phi(y_1)\,\phi(y_2)\exp\{i\int d^4x[\mathcal{L}(\phi,\partial_\mu\phi)+J(x)\phi(x)]\}$$

Consider $\lambda\phi^4$ theory

$$\mathcal{L}(\phi) = \mathcal{L}_0(\phi)+\mathcal{L}_1(\phi)$$

$$\mathcal{L}_0(\phi) = \frac{1}{2}(\partial_\lambda \phi)^2 - \frac{\mu^2}{2}\phi^2, \qquad \mathcal{L}_1(\phi) = -\frac{\lambda}{4!}\phi^4$$

Use Euclidean time the generating functional

$$W[J] = \int [d\phi] \exp\{-\int d^4x [\frac{1}{2}(\frac{\partial\phi}{\partial\tau})^2 + \frac{1}{2}(\overrightarrow{\bigtriangledown}\phi)^2 + \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 - J\phi]\}$$

can be written as

$$W[J] = \left[\exp \int d^4x \mathcal{L}_I \left(\frac{\delta}{\delta J(x)}\right)\right] W_0[J]$$

We have used Eq(2) to write the interaction term in terms of function derivative with repect to the source $J(x)$ .Here $W_0[J]$ is the free field generating function

$$W_0[J] = \int [d\phi] \exp[-\frac{1}{2}\int d^4x d^4y \phi(x) K(x,y) \phi(y) + \int d^4z J(z)\phi(z)]$$

and

$$K(x,y) = \delta^4(x-y)\left(-\frac{\partial^2}{\partial\tau^2} - \overrightarrow{\bigtriangledown}^2 + \mu^2\right)$$

. The Gaussian integral for many variables is

$$\int d\phi_1 d\phi_2 ... d\phi_n \exp\left[-\frac{1}{2}\sum_{i,j}\phi_i K_{ij}\phi_j + \sum_k J_k\phi_k\right] \backsim \frac{1}{\sqrt{detK}} \exp\left[\frac{1}{2}\sum_{i,j} J_i (K^{-1})_{ij} J_j\right]$$

This can be derived as follows. For Gaussian integral in one variable, we have

$$I = \int_{-\infty}^{\infty} \exp\left(-ax^2 + bx\right) dx = \int_{-\infty}^{\infty} dx \exp[-a\left(x + \frac{b}{2a}\right)^2 + \frac{b^2}{4a}] = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a}\right)$$

Generalization to more than one varibles,

$$I_n = \int dx_1 \cdots dx_n \exp[-\frac{1}{2}\sum_{ij} A_{ij}x_i x_j + \sum_j b_j x_j] = \int dx_1 \cdots dx_n \exp[-\frac{1}{2}(x, Ax) + (B, x)]$$

where

$$(x, Ax) = \sum_{ij} A_{ij}x_i x_j, \qquad (B, x) = \sum_j b_j x_j$$

Since A is a real symmetric matrix, it can be diagonalized by a othorgonal matrix S,

$$SAS^{-1} = D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}, \qquad \text{or} \qquad A = S^{-1}DS = S^T DS$$

Then

$$(x, Ax) = (Sx, DSx) = (y, Dy), \qquad (B, x) = (B', y) \qquad \text{where} \quad y = Sx, \quad B' = SB$$

We can then write

$$\begin{aligned} I_n &= \int dy_1 \cdots dy_n \exp[-\frac{1}{2}(y, Dy) + (B', y)] = \prod_i \left(\int_{-\infty}^{\infty} \exp\left(-\frac{y_i^2}{2d_i} + b_i' y_i\right) dy_i\right) \\ &= \prod_i \left[\sqrt{\frac{2\pi}{d_i}} \exp\left(\frac{b_i^2}{2d_i}\right)\right] \end{aligned}$$

Note that

$$\prod_i \left[\exp\left(\frac{b_i^2}{2d_i}\right)\right] = \exp\left[\sum_i \left(\frac{b_i'^2}{2d_i}\right)\right], \qquad \prod_i \sqrt{\frac{2\pi}{d_i}} = \frac{(2\pi)^{n/2}}{\sqrt{\det D}} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}$$

We can write

$$\sum_i \left(\frac{b_i'^2}{2d_i} \right) = \frac{1}{2} \left(B', D^{-1} B' \right) = \frac{1}{2} \left(SB, D^{-1} SB \right) = \frac{1}{2} \left(B, A^{-1} B \right)$$

The result is then

$$I_n = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp \left[\frac{1}{2} \left(B, A^{-1} B \right) \right] = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp \left[\frac{1}{2} \left(b_i \left(A^{-1} \right)_{ij} b_j \right) \right]$$

Apply this to the case of scalar fields,

$$W_0 [J] = \exp \left[\frac{1}{2} \int d^4 x d^4 y J(x) \triangle (x, y) J(y) \right]$$

where

$$\int d^4 y K(x, y) \triangle (y, z) = \delta^4 (x - z)$$

$\triangle(x, y)$ can be calculated by Fourier transform to give,

$$\triangle(x, y) = \int \frac{d^4 k_E}{(2\pi)^4} \frac{e^{ik_E (x-y)}}{k_E^2 + \mu^2}$$

where $k_E = (ik_0, \overrightarrow{k})$,the Euclidean momentum.
We now give an alternative way to derive the same result. Define

$$\phi (x) = \bar{\phi} (x) + \phi_c (x) \qquad \text{where} \qquad \phi_c (x) = \int \Delta (x, z) J (z) d^4 z$$

then we can write

$$S = -\frac{1}{2}\int d^4xd^4y\phi(x)K(x,y)\phi(y) + \int d^4zJ(z)\phi(z) = -\frac{1}{2}\{\int d^4xd^4y\bar{\phi}(x)K(x,y)\bar{\phi}(y) + \int d^4xd^4y\bar{\phi}(x)K(x$$

$$\int d^4xd^4y\phi_c(x)K(x,y)\bar{\phi}(y) + \int d^4xd^4y\phi_c(x)K(x,y)\phi_c(y)\} + \int d^4zJ(z)\phi(z)$$

The first term is

$$\int d^4xd^4y\bar{\phi}(x)K(x,y)\phi_c(y) = \int d^4x\bar{\phi}(x)\int d^4yK(x,y)\int \Delta(y,z)\,J(z)\,d^4z = \int d^4x\bar{\phi}(x)J(x)$$

Similarly

$$\int d^4xd^4y\phi_c(x)K(x,y)\bar{\phi}(y) = \int d^4yJ(y)\,\bar{\phi}(y)$$

$$\int d^4xd^4y\phi_c(x)K(x,y)\phi_c(y) = \int d^4xd^4yJ(x)\,\Delta(x,y)\,J(y)$$

Put all these together,

$$S = -\frac{1}{2}\{\int d^4xd^4y\bar{\phi}(x)K(x,y)\bar{\phi}(y) + \int d^4x\bar{\phi}(x)J(x) + \int d^4yJ(y)\,\bar{\phi}(y) + \int d^4xd^4yJ(x)\,\Delta(x,y)\,J(y)\} +$$

$$= -\frac{1}{2}\int d^4xd^4y\bar{\phi}(x)K(x,y)\bar{\phi}(y) + \frac{1}{2}\int d^4xd^4yJ(x)\,\Delta(x,y)\,J(y)$$

The first term is independent of $J(x)$ and can be dropped. We then get the same result as given in $W_0[J]$. Perturbative expansion in power of λ gives

$$W[J] = W_0[J]\{1 + \lambda w_1[J] + \lambda^2 w_2[J] + ...\}$$

where

$$w_1 = -\frac{1}{4!} W_0^{-1}[J] \{ \int d^4x \left[\frac{\delta}{\delta J(x)} \right]^4 \} W_0[J]$$

$$w_2 = -\frac{1}{2\,(4!)^2} W_0^{-1}[J] \{ \int d^4x \left[\frac{\delta}{\delta J(x)} \right]^4 \}^2 W_0[J]$$

Use explicit form for $W_0[J]$,

$$\begin{aligned} W_0[J] \quad &= \quad 1 + \frac{1}{2} \int d^4x d^4y J(x) \triangle (x,y) J(y) + \\ & \left(\frac{1}{2}\right)^2 \frac{1}{2!} \int d^4y_1 d^4y_2 d^4y_3 d^4y_4 \left[J(y_1) \triangle (y_1,y_2) J(y_2) J(y_3) \triangle (y_3,y_4) J(y_4) \right] + \ldots \end{aligned}$$

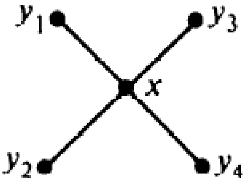
We get for w_1,

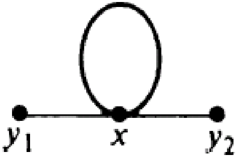
$$w_1 = -\frac{1}{4!} \left[\int \triangle(x,y_1) \triangle(x,y_2) \triangle(x,y_3) \triangle(x,y_4) J(y_1) J(y_2) J(y_3) J(y_4) + 3! \triangle(x,y_1) \triangle(x,y_2) J(y_1) J(y_2) \triangle(x,\right.$$

we dropped all J independent terms, and all (x_i, y_i) are integrated over. In this computation we have used the identity,

$$\frac{\delta}{\delta J(x)} \int d^4y_1 J(y_1) f(y_1) = \int \delta^4(x - y_1) d^4y_1 f(y_1) = f(x)$$

Graphical representation for w_1





The connected Green's function is

$$G^{(n)}(x_1, x_2, ...x_n) = \frac{\delta^n \ln W[J]}{\delta J(x_1)\delta J(x_2)...\delta J(x_n)}|_{J=0}$$

Thus replacing y_i by external x_i, we get contributions for 4-point, 2-point functions.

Grassmann algebra

For fermion fields in path integral, we need to use anti-commuting c-number functions.This can be realized as elements of Grassmann algebra.

In an n-dimensional Grassmann algebra,the n generators θ_1, θ_2, $\theta_3, ..., \theta_n$ satisfy the anti-commutation relations,

$$\{\theta_i, \theta_j\} = 0 \qquad i, j = 1, 2, ..., n$$

and every element can be expanded in a finite series,

$$P(\theta) = P_0 + P^{(1)}_{i_1}\theta_{i_1} + P^{(2)}_{i_1 i_2}\theta_{i_1}\theta_{i_2} + ... + P_{i_1 \dots i_n}\theta_{i_1}...\theta_{i_n}$$

Simplest case:n=1

$$\{\theta, \theta\} = 0 \qquad \text{or} \qquad \theta^2 = 0 \qquad P(\theta) = P_0 + \theta P_1$$

We can define the "differentiation" and "integration" as follows,

$$\frac{d}{d\theta}\theta = \theta\frac{\overleftarrow{d}}{d\theta} = 1 \quad \Longrightarrow \frac{d}{d\theta}P(\theta) = P_1$$

Integration is defined in such a way that it is invariant under translation,

$$\int d\theta P(\theta) = \int d\theta P(\theta + \alpha)$$

α is another Grassmann variable. This implies

$$\int d\theta = 0$$

We can normalize the integral

$$\int d\theta \theta = 1$$

Then

$$\int d\theta P(\theta) = P_1 = \frac{d}{d\theta} P(\theta)$$

Consider a change of variable

$$\theta \longrightarrow \widetilde{\theta} = a + b\theta$$

Since

$$\int d\widetilde{\theta} P\left(\widetilde{\theta}\right) = \frac{d}{d\widetilde{\theta}} P\left(\widetilde{\theta}\right) = P_1$$

$$\int d\theta P\left(\widetilde{\theta}\right) = \int d\theta \left[P_0 + \widetilde{\theta} P_1\right] = \int d\theta \left[P_0 + (a + b\theta) P_1\right] = bP_1$$

we get

$$\int d\widetilde{\theta} P\left(\widetilde{\theta}\right) = \int d\theta \left(\frac{d\widetilde{\theta}}{d\theta}\right)^{-1} P\left(\widetilde{\theta}(\theta)\right)$$

The "Jacobian" is the inverse of that for c-number integration.

Generalize to n-dimensional Grassmann algebra,

$$\frac{d}{d\theta_i}\left(\theta_1,\theta_2,\theta_3,...,\theta_n\right)=\delta_{i_1}\theta_2...\theta_n-\delta_{i_2}\theta_1\theta_3...\theta_n+...+\left(-1\right)^{n-1}\delta_{in}\theta_1\theta_2...\theta_{n-1}$$

$$\left\{d\theta_i,d\theta_j\right\}=0$$

$$\int d\theta_i=0 \qquad \int d\theta_i\theta_j=\delta_{ij}$$

For a change of variables of the form

$$\widetilde{\theta}_i=b_{ij}\theta_j$$

we have

$$\int d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1 P\left(\widetilde{\theta}\right)=\int d\theta_n...d\theta_1\left[det\frac{d\widetilde{\theta}}{d\theta}\right]^{-1}P\left(\widetilde{\theta}\left(\theta\right)\right)$$

Proof:

$$\widetilde{\theta}_1\widetilde{\theta}_2...\widetilde{\theta}_n=b_{1i_1}b_{2i_2}...b_{ni_n}\theta_{i_1}...\theta_{i_n}$$

RHS is non-zero only if $i_1, i_2..., i_n$ are all different and we can write

$$\begin{aligned}\widetilde{\theta}_1\widetilde{\theta}_2...\widetilde{\theta}_n &= b_{1i_1}b_{2i_2}...b_{ni_n}\epsilon_{i_1,i_2...,i_n}\theta_{i_1}...\theta_{i_n}\\ &= \left(\det b\right)\theta_1\theta_2\theta_3...\theta_n\end{aligned}$$

From the normalization condition,

$$1=\int d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1\left(\widetilde{\theta}_1\widetilde{\theta}_2...\widetilde{\theta}_n\right)=\left(\det b\right)\int d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1\left(\theta_1\theta_2\theta_3...\theta_n\right)$$

we see that

$$d\widetilde{\theta}_n d\widetilde{\theta}_{n-1}...d\widetilde{\theta}_1=\left(\det b\right)^{-1}d\theta_1...d\theta_n$$

In field theory, we need Gaussian integral of the form,

$$G(A) \equiv \int d\theta_n ... d\theta_1 \exp\left(\frac{1}{2}(\theta, A\theta)\right) \qquad \text{where } (\theta, A\theta) = \theta_i A_{ij} \theta_j$$

First consider n=2

$$A = \begin{pmatrix} 0 & A_{12} \\ -A_{12} & 0 \end{pmatrix}$$

Then

$$G(A) = \int d\theta_2 d\theta_1 \exp(\theta_1\theta_2 A_{12}) \simeq \int d\theta_2 d\theta_1 (1 + \theta_1\theta_2 A_{12}) = A_{12} = \sqrt{\det A}$$

For the general $n = even$, we first bring the matrix A into the standard form by a unitary transformation,

$$UAU^\dagger = A_s$$

$$A_s = \begin{bmatrix} a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & b\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \\ & & \ddots \end{bmatrix}$$

This can be seen as follows. Since iA is Hermitian, it cab diagonalized by a unitary transfomation,

$$V(iA)V^\dagger = A_d$$

where A_d is real and diagonal. The diagonal elements are solutions to the secular equation,

$$\det|iA - \lambda I| = 0$$

Since $A = -A^T$, we have

$$0 = \det|iA - \lambda I|^T = \det|-iA - \lambda I|$$

This means that if λ is a solution, -λ is also a solution and A_d is of the form,

$$A_d = \begin{pmatrix} a & & & & \\ & -a & & & \\ & & b & & \\ & & & -b & \\ & & & & \ddots \end{pmatrix}$$

To put this matrix into the standard we use the unitary matrix

$$S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$$

which has the property

$$S_2 (-i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} S_2^\dagger = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Thus we get

$$S(-iA_d)S^\dagger = A_s, \quad \text{where} \quad S = \begin{pmatrix} S_2 & & \\ & S_2 & \\ & & \ddots \end{pmatrix}$$

For arbitrary n, we get

$$G\left(A\right)=\int d\theta_n...d\theta_1 \exp\left(\frac{1}{2}\left(\theta,A\theta\right)\right)=\sqrt{\det A} \qquad \text{n even}$$

and for "complex" Grassmann variables

$$\int d\theta_n d\bar{\theta}_n d\theta_{n-1} d\bar{\theta}_{n-1}...d\theta_1 d\bar{\theta}_1 \exp\left(\overline{\theta},A\theta\right)=\det A$$

For the Fermion fields, the generating functional is of the form,

$$W\left[\eta,\overline{\eta}\right]=\int\left[d\psi\left(x\right)\right]\left[d\overline{\psi}\left(x\right)\right]\exp\{i\int d^4x\left[\mathcal{L}\left(\psi,\overline{\psi}\right)+\overline{\psi}\eta+\overline{\eta}\psi\right]\}$$

If $\mathcal{L}$ depends on $\psi,\overline{\psi}$ quadratically

$$\mathcal{L}=\left(\overline{\psi},A\psi\right)$$

then we have

$$W=\int\left[d\psi\left(x\right)\right]\left[d\overline{\psi}\left(x\right)\right]\exp\{\int d^4x\overline{\psi}A\psi\}=\det A$$