Renormalization

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Renormalization

Renomalization is a general physical phenomena Consider an electron moving inside a solid. Due to interaction of electorn with ions on the lattice, the effective mass of the elelctron $m^* \neq m$. Electron mass is changed (renormalized) from m to m^*. Clearly both m and m^* are finite and measurable.
In relativistic field theory the concept of renormalization is the same.
Two important distinctions.

1. Modification due to interaction is infinite.
2. Can't turn off interaction to measure bare mass

Technically, theory of renormalization is quite complicated. We will explain the main ideas

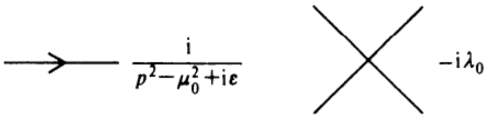
Renormalization in $\lambda\phi^4$ Theory

Consider $\lambda\phi^4$ theory

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

$$\mathcal{L}_0 = \frac{1}{2}[(\partial_\mu \phi_0)^2 - \mu_0^2 \phi_0^2] \quad , \quad \mathcal{L}_I = -\frac{\lambda_0}{4!}\phi_0^4$$

Feynman rule
vertex and propagator are

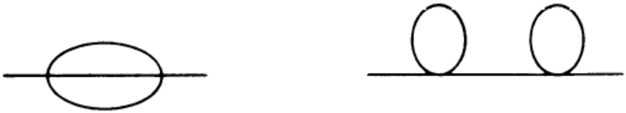


4-momentum conservation at each vertex.

1. Integrate over internal momenta not fixed by momentum conservation
2. no propagator for external line

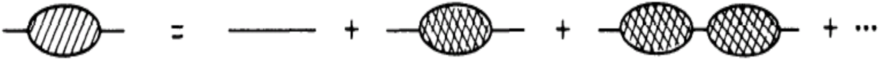
Simple example

2-point function has contribution from following graphs



(a) (b)

Define **1PI**: one-particle irreducible graphs— graphs which can not be disconnected by cutting any one line.
Complete 2 point function in terms of 1PI graphs

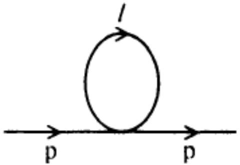


$$\frac{i}{(p^2-\mu_0^2+i\varepsilon)}+\frac{i}{(p^2-\mu_0^2+i\varepsilon)}(-i\Sigma(p^2))\frac{i}{(p^2-\mu_0^2+i\varepsilon)}+\ldots\ldots$$

$$=\frac{i}{(p^2-\mu_0^2+i\varepsilon)}[\frac{1}{1+i\Sigma(p^2)\frac{i}{p^2-\mu_0^2+i\varepsilon}}]=\frac{i}{p^2-\mu_0^2-\Sigma(p^2)+i\varepsilon}$$

1-loop diagrams

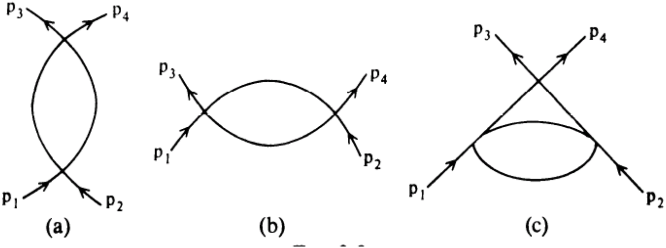
In one-loop we have



The self energy

$$-i\Sigma(p) = -\frac{i\lambda_0}{2}\int \frac{d^4 l}{(2\pi)^4}\frac{i}{l^2-\mu_0^2+i\varepsilon}$$

is quadratically divergent.
The 4-point function are



Graph (a) gives

$$\Gamma(p^2) = \frac{(i\lambda_0)^2}{2} \int \frac{d^4 l}{(2\pi)^4} \frac{i}{(l-p)^2 - \mu_0^2 + i\varepsilon} \frac{i}{l^2 - \mu_0^2 + i\varepsilon} , \qquad p = p_1 + p_2$$

and is logarithmically divergent. If we differentiate $\Gamma\left(p^2\right)$ with respect to p, power of l will increase in denominator and make the integral more convergent,

$$\frac{\partial}{\partial p^2}\Gamma(p^2) = \frac{1}{2p^2} p_\mu \frac{\partial}{\partial p_\mu}\Gamma(p^2) = \frac{\lambda_0^2}{p^2} \int \frac{d^4 l}{(2\pi)^4} \frac{(l-p)\cdot p}{[(l-p)^2 - \mu_0^2 + i\varepsilon]^2} \frac{1}{l^2 - \mu_0^2 + i\varepsilon} \rightarrow \text{ convergent}$$

where we have used

$$\frac{\partial}{\partial p^\mu} = \frac{\partial}{\partial p^2}\frac{\partial p^2}{\partial p^\mu} = 2p_\mu \frac{\partial}{\partial p^2}, \qquad \text{or} \qquad 2p^2\frac{\partial}{\partial p^2} = p^\mu \frac{\partial}{\partial p^\mu}$$

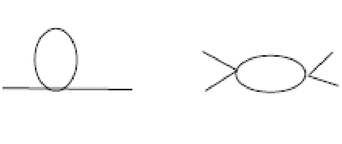
If expand $\Gamma(p^2)$ in Taylor series,

$$\Gamma(p^2) = a_0 + a_1 p^2 + \ldots$$

divergences are contained in first few terms. In our simple case,

$$\Gamma(p^2) = \Gamma(0) + \tilde{\Gamma}(p^2)$$

$\tilde{\Gamma}(p^2)$ is finite.
In 1-loop, the divergent graphs are (1PI)



Other 1-loop graphs are either finite or contain the above graphs as subgraphs

Mass and wavefunction renormalization

Taylor expansionof 1PI self energy, has 2 divergent terms,

$$\Sigma(p^2) = \Sigma(\mu^2) + (p^2 - \mu^2)\Sigma'(p^2) + \tilde{\Sigma}(p^2) \qquad \mu^2 : \text{arbitrary}$$

$\Sigma(\mu^2)$ is quadratically, $\Sigma'^2(\mu^2)$ logarithmically divergnet, 3rd term $\tilde{\Sigma}(p^2)$ is finite and ,

$$\tilde{\Sigma}(\mu^2) = 0, \qquad \tilde{\Sigma}'(\mu^2) = 0$$

Complete propagator is

$$i\Delta(p^2) = \frac{i}{p^2 - \mu_0^2 - \Sigma(\mu^2) - (p^2 - \mu^2)\Sigma' - \tilde{\Sigma}(p^2)}$$

Choose μ^2 such that

$$\mu_0^2 - \Sigma(\mu^2) = \mu^2 \qquad \text{mass renormalization}$$

then $\Delta(p^2)$ has a pole at $p^2 = \mu^2$. $\Longrightarrow \mu^2$ physical mass and μ_0^2 bare mass.
Full propagator is

$$i\Delta(p^2) = \frac{i}{(p^2 - \mu^2)[1 - \Sigma'\,(\mu^2)] - \tilde{\Sigma}(p^2)}$$

$\Sigma'(\mu^2)$ and $\tilde{\Sigma}(p^2)$ are both of order λ_0 or higher, we can approximate

$$\tilde{\Sigma}(p^2) \rightarrow (1 - \Sigma'\,(\mu^2))\tilde{\Sigma}(p^2)$$

Then

$$i\Delta(p^2) = \frac{iZ_\phi}{p^2 - \mu^2 - \tilde{\Sigma}(p^2) + i\varepsilon} \quad \text{with} \quad Z_\phi = \frac{1}{1 - \Sigma'(p^2)} \approx 1 + \Sigma'(p^2)$$

Get rid of Z_ϕ by defining renormalized field ϕ by

$$\phi = \frac{1}{\sqrt{Z_\phi}}\phi_0$$

propagator for ϕ is

$$i\Delta_R(p) = \int d^4 x e^{-px} \ \langle 0|\, T(\phi(x)\phi(0))|0\rangle = \frac{i}{P^2 - \mu^2 - \tilde{\Sigma}(p^2) + i\varepsilon}$$

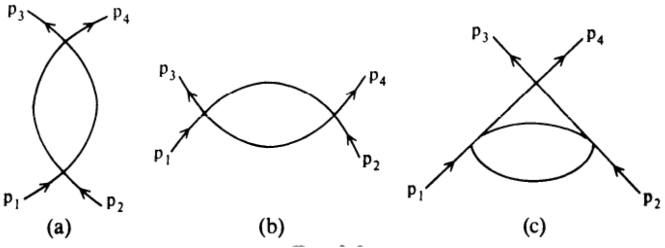
which is completely finite. Z_ϕ is called the *wave function renormalization constant*.
For general Green's functions of renomalized fields,

$$G_R^{(n)}(x_1 \ldots x_n) = \langle 0|\, T(\phi(x_1)\cdots\phi(x_n))|0\rangle$$

$$= Z_\phi^{-n/2} \langle 0|\, T(\phi_0(x_1)\cdots\phi_0(x_n))|0\rangle = Z_\phi^{-n/2} G_0^{(n)}(x_1 \ldots x_n)$$

Coupling constant renormalization

1PI 4-point functions $\Gamma^{(4)}(p_1 \cdots p_4)$,



Include tree diagram,

$$\Gamma_0^{(4)}(s,t,u) = -i\lambda_0 + \Gamma(s) + \Gamma(t) + \Gamma(u)$$

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2, \qquad s + t + u = 4\mu^2$$

These are logarithmically divergent, one substraction to make them finite.

To remove this divergence from the 4-point function, we need to make the substraction of 4-point function at some kineatical point. For conveience we can choose $s_0 = t_0 = u_0 = \dfrac{4\mu^2}{3}$,

$$\Gamma_0^{(4)}(s,t,u) = -i\lambda_0 + 3\Gamma(s_0) + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)$$

where

$$\tilde{\Gamma}(s) = \Gamma(s) - \Gamma(s_0),$$

is finite. Define Z_λ by

$$-i\lambda_0 + 3\Gamma(s_0) = -iZ_\lambda^{-1}\lambda_0$$

Thus

$$\Gamma_0^{(4)}(s,t,u) = -iZ_\lambda^{-1}\lambda_0 + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u)$$

At the symmetric point

$$\Gamma_0^{(4)}(s_0,t_0,u_0) = -iZ_\lambda^{-1}\lambda_0$$

with $\tilde{\Gamma}(s_0) = \tilde{\Gamma}(t_0) = \tilde{\Gamma}(u_0) = 0$. Renormalized 1PI 4 point function $\Gamma^{(4)}$ is related to Green's function by

$$\Gamma_R^{(4)} = \prod_{j=1}^{4}[i\Delta_R(p_j)]^{-1} G_R^{(4)}$$

which implies

$$\Gamma_R^{(4)}(s,t,u) = -Z_\phi^2 \Gamma_0^{(4)}(s,t,u)$$

Define renormalized coupling constant λ by

$$\lambda = Z_\phi^2 Z_\lambda^{-1}\lambda_0$$

then

$$\Gamma_R^{(4)}(p_1,\cdots,p_4\ ) = Z_\phi^2\Gamma_0^{(4)} = -iZ_\lambda^{-1}Z_\varphi^2\lambda_0 + Z_\varphi^2[\tilde{\Gamma}(s)+\tilde{\Gamma}(t)+\tilde{\Gamma}(u)] = -i\lambda + Z_\varphi^2[\tilde{\Gamma}(s)+\tilde{\Gamma}(t)+\tilde{\Gamma}(u)]$$

Since $Z_\varphi = 1 + O(\lambda_0)$, $\tilde{\Gamma} = O(\lambda_0^2)$ $\lambda = \lambda_0 + O(\lambda_0^2)$, we can approximate

$$\Gamma_R^{(4)}(\ p_1,\cdots,p_4) = -i\lambda + \tilde{\Gamma}(s) + \tilde{\Gamma}(t) + \tilde{\Gamma}(u) + 0(\lambda^3)$$

which is completely finite. From the original Lagrangian (unrenormalized Lagrangian)

$$\mathcal{L}_0 = \frac{1}{2}[(\partial_\mu \phi_0)^2 - \mu_0^2 \phi_0^2] - \frac{\lambda_0}{4!}\phi^4$$

we can write

$$\mathcal{L}_0 = \mathcal{L} + \Delta\mathcal{L}$$

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu \phi)^2 - \mu^2 \phi^2] - \frac{\lambda}{4!}\phi^4$$

$$\Delta\mathcal{L} = \mathcal{L}_0 - \mathcal{L} = \frac{1}{2}(Z_\phi - 1)[(\partial_\mu \phi)^2 - \mu^2 \phi^2] + \frac{\delta\mu^2}{2}\phi^2 - \frac{-\lambda(Z_\lambda - 1)}{4!}\phi^4$$

where

$$\mu^2 = \delta\mu^2 + \mu_0^2 \quad , \quad \phi = Z_\phi^{-\frac{1}{2}}\phi_0 \quad , \quad \lambda = Z_\lambda^{-1} Z_\phi^2 \lambda_0$$

Here $\mathcal{L}$ is usually called renormalized Lagrangian and $\Delta\mathcal{L}$ the counterterms.

BPH renormalization

An equivalent scheme is BPH (Bogoliubov, Parasiuk and Hepp) renormalization scheme.
Essential idea: use counter terms Lagrangian $\Delta\mathcal{L}$ as a device to cancel the divergences.

1. Starts with renormalized Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

   Generate free propagator and vertices from this Lagrangian.

2. The divergent parts of one-loop 1PI diagrams are isolated by Taylor expansion. Construct a set of counter terms $\Delta\mathcal{L}^{(1)}$ to cancel these divergences.
3. A new Lagrangian $\mathcal{L}^{(1)} = \mathcal{L} + \Delta\mathcal{L}^{(1)}$ is used to generate 2-loop diagrams and to counter terms $\Delta\mathcal{L}^{(2)}$ to cancel 2-loops divergences. This sequence of operation is iteratively applied.

To illustrate the usefulness of BPH scheme, we need to make use of the power counting method.

Power counting
Superficial degree of divergence D is defined as

$$D = (\sharp \ \ of \ \ loop \ \ momenta \ \ in \ \ numerator) - (\sharp \ \ of \ \ loop \ \ momenta \ \ indenominator)$$

We define the following quantities,
B= number of external lines
IB= number of internal lines
n= number of vertices
Counting the lines in the graph, we get

$$4n = 2(IB) + B$$

4-momentum conseravation at each vertex and overall 4-momentum conseravation which do not depend on the internal momenta.
number of loops L is

$$L = IB - n + 1$$

The superficial degree of divergence is

$$D = 4L - 2(IB)$$

Eliminating n, L and (IB),

$$D = 4 - B$$

Thus $D < 0$ for $B > 4$. Note that in this case D is independent of n, the order of pertubation, and depends only on B, the number of external lines.The $\lambda\phi^4$ theory has the symmetry $\phi \rightarrow -\phi$. which implies that $B = even$ and only $B = 2, 4$ are superficially divergent.
1) $B = 2, \Rightarrow \ D = 2$
Being quadratically divergent, the necessary Taylor expansion for the 2-point function is of the form,

$$\Sigma\left(p^2\right) = \Sigma\left(0\right) + p^2\Sigma'\left(0\right) + \widetilde{\Sigma}\left(p^2\right)$$

where $\Sigma(0)$ and $\Sigma'(0)$ are divergent and $\widetilde{\Sigma}(p^2)$. To cancel these divergences we need to add two counterterms,

$$\frac{1}{2}\Sigma(0)\phi^2 + \frac{1}{2}\Sigma'(0)\left(\partial_\mu\phi\right)^2$$

which give the following contributions,

Fig 5 Counter term for 2 point function

2) $B = 4, \Rightarrow \quad D = 0$
The Taylor expansion is

$$\Gamma^{(4)}(p_i) = \Gamma^{(4)}(0) + \widetilde{\Gamma}^{(4)}(p_i)$$

where $\Gamma^{(4)}(0)$ is logarithmically divergent will be cancelled by conunterterm of the form

$$\frac{i}{4!}\Gamma^{(4)}(0)\phi^4$$

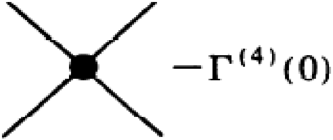


Fig 6 counter term for 4-point function

The general counterterrm Lagrangian is then

$$\Delta\mathcal{L} = \frac{1}{2}\Sigma(0)\,\phi^2 + \frac{1}{2}\Sigma'(0)\left(\partial_\mu\phi\right)^2 + \frac{i}{4!}\Gamma^{(4)}(0)\,\phi^4$$

which is clearly the same as Eq(**??**) with the identification

$$\begin{aligned}\Sigma'(0) &= \left(Z_\varphi - 1\right)\\ \Sigma(0) &= -\left(Z_\varphi - 1\right)\mu^2 + \delta\mu^2\\ \Gamma^{(4)}(0) &= -i\lambda\left(1 - Z_\lambda\right)\end{aligned}$$

This illustrates the equivalence of BPH renormalization and conventional renormalization.

More on BPH renormalization

The BPH renormalization looks very simple. There are many interesting and useful features in BPH:

1. **Convergence of Feynman diagrams**
    1. We have used the superficial degree of divergences D. To 1-loop D is the same as the real degree of divergence. Beyond 1-loop we can have an overall $D < 0$ while there are real divergences in the subgraphs.
    Weinberg's theorem: The general Feynman integral converges if D of the graph together with D of all subgraphs are negative. For example, consider a Feynman graph with n external lines and l loops. Introduce a cutoff Λ to estimate the order of divergence,

$$\Gamma^{(n)}\left(p_1,\cdots,p_{n-1}\right) = \int_0^\Lambda d^4q_1\cdots d^4q_l\, I\left(p_1,\cdots,p_{n-1};q_1,\cdots,q_l\right)$$

Take a subset $S = \{q'_1, q'_2, \cdots q'_m\}$ of the loop momenta $\{q_1, \cdots, q_i\}$ and scale them to infinity and all other momenta fixed. Let $D(S)$ be the superficial degree of divergence associated with integration over this set, i. e.,

$$\left| \int_0^{\Lambda} d^4 q'_1 \cdots d^4 q'_m I \right| \leq \Lambda^{D(s)} \{\ln \Lambda\}$$

where $\{\ln \Lambda\}$ is some function of $\ln \Lambda$. Then the convergent theorem states that the integral over $\{q_1, \cdots, q_i\}$ converges if the $D(S)' s$ for all possible choice of S are negative. For example the graph in the following figure

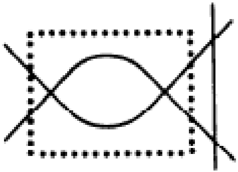


Fig 7 Divergent 6-point function

is a 6-point function with $D = -2$. But the integration inside the box with $D = 0$ is logarithmically divergent. However, in the BPH procedure this subdivergence is removed by lower order counter terms as shown below.

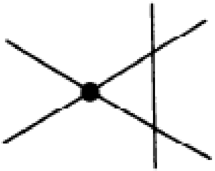


Fig 8 Counterterm for 6-point function

2. Primitively divergent graphs
   A primitively divergent graph has $D \geq 0$ but is convergent for all subintegrations. In these diagrams the only divergences is caused by all of the loop momenta growing large together. So when we differentiate with respect to external momenta will improve the convergence of the diagram. Then all the divergences can be isolated in the first few terms of the Taylor expansion.
3. Disjointed divergent graphs
   Here the divergent subgraphs are disjointed. For example, consider the 2-loop graph,

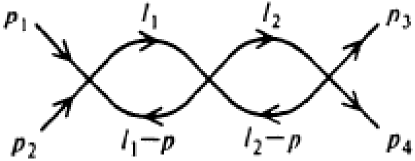


Fig 9 2-loop disjoint divergence

Differentiating with respect to the external momentum will improve only one of the loop integration but not both. Then not all divergences can be removed by subtracting out the first few terms in the Taylor expansion around external momenta. However, the lower order counter terms in the BPH scheme will come in to save the day. The Feynman integral is of the form,

$$\Gamma_a^{(4)}(p) \propto \lambda^3 [\Gamma(p)]^2$$

with

$$\Gamma(p) = \frac{1}{2}\int d^4 l \frac{1}{l^2-\mu^2+i\varepsilon}\frac{1}{\left[(l-p)^2-\mu^2+i\varepsilon\right]}$$

and $p = p_1 + p_2$. Since $\Gamma(p)$ is logarithmic divergent, $\Gamma_a^{(4)}(p)$ cannot be made convergent no matter how many derivatives act on it, even though the overall superficial degree of divergence is zero. However, the lower order counterm $-\lambda^2\Gamma(0)$ generates the additional contributions given in the following diagrams,

Fig 10 2-loop graphs with counterms

which are proportional to $-\lambda^3\Gamma(0)\Gamma(p)$. Adding these 3 contributions, we get

$$\lambda^3[\Gamma(p)]^2 - 2\lambda^3\Gamma(0)\Gamma(p)$$
$$= \lambda^3[\Gamma(p) - \Gamma(0)]^2 - \lambda^3[\Gamma(0)]^2$$

Since the combination in the first $[\cdots]$ is finite, the divergence in the last term can be removed by one differentiation. Thus the inclusion of lower order counterterms, the divergences take the form of polynomials in external momenta. Thus for graphs with disjointed divergences we need to include the lower order counter terms to remove the divergences by substractions in Taylor expansion.

4. Nested divergent graphs

In this case one of a pair of divergent 1PI is entirely contained within the other as shown in the following diagram,

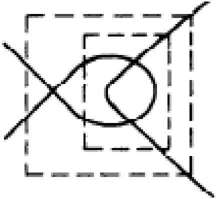


Fig 11 Nested divergence

After the subgraph divergence is removed by diagrams with lower order counterterms, the overall divergences is then renormalized by a λ^3 counter terms as shown below,

Fig 12 counterterm for nested divergence

Again diagrams with lower-order counterterm insertions must be included in order to aggregate the divergences into the form of polynomial in external momenta.

5. Overlapping divergent graphs

   These diagrams are those divergences which are neither nested nor disjointed. These are most difficult to analyze. An example of this is shown below,

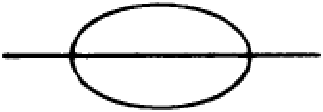


Fig 13 Overlapping divergence

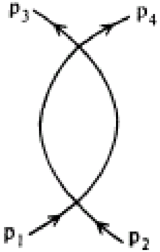
   The study of how to disentangle these overlapping divergences is beyond the scope of this simple introduction and we refer interested readers to the literature ([**?**].[**?**]).

It is clear that BPH renormalization scheme is quite useful in organizing the higher order divergences in a more systematic way for the removing of divergences by constructing the counterterms.
The general analysis of the renormalization program has been carried out by Bogoliubov, Parasiuk, Hepp ([**?**]).
The result is known as BPH theorem, which states that for a general renormalizable field theory, to any order in perturbation theory, all divergences are removed by the counterterms corresponding to superficially divergent amplitudes.

Regularization
Need first to make divergent integral finite before we can do any manipulation.
2 different schemes: Pauli-Villars regularization and dimensional regularization.
Pauli-Villars Regularization
Repalce the propagator by

$$\frac{1}{k^2-\mu_0^2} \to \left(\frac{1}{k^2-\mu_0^2} - \frac{1}{k^2-\Lambda^2}\right) = \frac{(\mu_0^2-\Lambda^2)}{(k^2-\mu_0^2)(k^2-\Lambda^2)} \to \frac{1}{k^4} \quad \text{for large } k$$

will make the integral more convergent.
4-point function from the following graph,



$$\Gamma\left(p^2\right) = \Gamma\left(s\right) = \frac{\left(-i\lambda\right)^2}{2}\int \frac{d^4 l}{\left(2\pi\right)^4}\frac{i}{\left(l-p\right)^2-\mu^2}\frac{i}{l^2-\mu^2}$$

With Pauli-Villars regularization this becomes,

$$\Gamma\left(p^2\right) = \frac{-\lambda^2\Lambda^2}{2}\int \frac{d^4 l}{\left(2\pi\right)^4}\frac{1}{\left[\left(l-p\right)^2-\mu^2\right]\left(l^2-\mu^2\right)\left(l^2-\Lambda^2\right)}$$

Taylor expansion around $p^2 = 0$,

$$\Gamma\left(p^2\right) = \Gamma\left(0\right) + \widetilde{\Gamma}\left(p^2\right)$$

with

$$\Gamma\left(0\right) = \frac{-\lambda^2\Lambda^2}{2}\int \frac{d^4 l}{\left(2\pi\right)^4}\frac{1}{\left(l^2-\mu^2\right)^2\left(l^2-\Lambda^2\right)}$$

$$\widetilde{\Gamma}\left(p^2\right) = \frac{-\lambda^2\Lambda^2}{2}\int \frac{d^4 l}{\left(2\pi\right)^4}\frac{2l\cdot p-p^2}{\left[\left(l-p\right)^2-\mu^2\right]\left(l^2-\mu^2\right)^2\left(l^2-\Lambda^2\right)}$$

Since the integral in $\widetilde{\Gamma}\left(p^2\right)$ is convergent enough, we can take the limit $\Lambda^2 \to \infty$ inside the integral to get,

$$\widetilde{\Gamma}\left(p^2\right) = \frac{\lambda^2}{2}\int \frac{d^4 l}{\left(2\pi\right)^4}\frac{2l\cdot p-p^2}{\left[\left(l-p\right)^2-\mu^2\right]\left(l^2-\mu^2\right)^2}$$

take the limit $\Lambda^2 \to \infty$ inside in $\widetilde{\Gamma}\left(p^2\right)$.

Combine the denominators by using the identities,

$$\frac{1}{a_1 a_2 \cdots a_n} = (n-1)! \int_0^1 \frac{dz_1 dz_2 \cdots dz_n}{(a_1 z_1 + \cdots + a_n z_n)^n} \delta\left(1 - \sum_{i=1}^{n} z_i\right)$$

$$\frac{1}{a_1^2 a_2 \cdots a_n} = n! \int_0^1 \frac{z_1 dz_1 dz_2 \cdots dz_n}{(a_1 z_1 + \cdots + a_n z_n)^{n+1}} \delta\left(1 - \sum_{i=1}^{n} z_i\right)$$

Here $\alpha_1, \cdots \alpha_n$ are called the Feynman parameters. Then

$$\frac{1}{\left[(l-p)^2 - \mu^2\right](l^2-\mu^2)^2} = 2\int \frac{(1-\alpha)\, d\alpha}{A^3}$$

where

$$A = (1-\alpha)\left(l^2 - \mu^2\right) + \alpha\left[(l-p)^2 - \mu^2\right] = (1-\alpha p)^2 - a^2$$

with

$$a^2 = \mu^2 - \alpha(1-\alpha)\, p^2$$

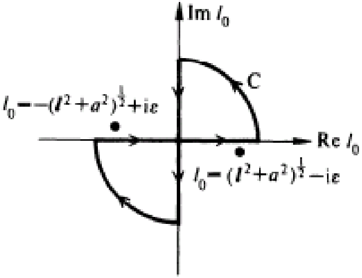
Thus

$$\begin{aligned}
\widetilde{\Gamma}\left(p^2\right) &= \lambda^2 \int_0^1 (1-\alpha)\, d\alpha \int \frac{d^4 l}{(2\pi)^4} \frac{2l\cdot p - p^2}{\left[(l-\alpha p)^2 - a^2\right]^3} \\
&= \lambda^2 \int_0^1 (1-\alpha)\, d\alpha \int \frac{d^4 l}{(2\pi)^4} \frac{(2\alpha - 1)\, p^2}{(l^2 - a^2 + i\varepsilon)^3}
\end{aligned}$$

we have changed the variable $l \to l + \alpha p$ and drop terms linear in l. In the complex l_0 plane, poles at

$$l_0 = \pm\left[\sqrt{l^2 + a^2} - i\varepsilon\right]$$

Do the integration by *Wick* rotation,



From Cauchy's theorem we have

$$\oint_C dl_0 f(l_0) = 0$$

where

$$f(l_0) = \frac{1}{\left[l_0^2 - \left(\sqrt{\vec{l}^{\,2} + a^2} - i\varepsilon\right)^2\right]^3}$$

Since $f(l_0) \to l_0^{-6}$ as $l_0 \to \infty$, circular part of contour C with very large radius vanishes and

$$\int_{-\infty}^{\infty} dl_0 f(l_0) = \int_{-i\infty}^{i\infty} dl_0 f(l_0)$$

Integration path has been rotated from along real axis to imaginary axis (Wick rotation). Changing the variable $l_0 = il_4$,

$$\int_{-i\infty}^{i\infty} dl_0 f\left(l_0\right) = i\int_{-\infty}^{\infty} dl_4 f\left(l_4\right) = -i\int_{-\infty}^{\infty} \frac{dl_4}{\left(l_1^2 + l_2^2 + l_3^2 + l_4^2 + a^2 - i\varepsilon\right)^3}$$

Define Euclidean momentum $k_i = (l_1, l_2, l_3, l_4)$ with $k^2 = l_1^2 + l_2^2 + l_3^2 + l_4^2$. The integral is then

$$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{\left(l^2 - a^2 + i\varepsilon\right)^3} = -i\int \frac{d^4 k}{(2\pi)^4} \frac{1}{\left(k^2 + a^2 - i\varepsilon\right)^3}$$

Many intergrals in loop integration can be worked out using Gamma and Beta functions

———————————————————

Gamma function is defined by

$$\Gamma\left(n\right) = \int_0^{\infty} u^{n-1} e^{-u} du \tag{1}$$

Then

$$\Gamma\left(m\right)\Gamma\left(n\right) = \int_0^{\infty} u^{n-1} e^{-u} du \int_0^{\infty} v^{m-1} e^{-v} dv$$

Let $u = x^2, v = y^2$

$$\Gamma\left(m\right)\Gamma\left(n\right) = 4\int dxdye^{-\left(x^2+y^2\right)} x^{2n-1} y^{2m-1}$$

Now use polar coordinates, $x = r\cos\theta, y = \sin\theta$,

$$\begin{aligned}
\Gamma\left(m\right)\Gamma\left(n\right) &= 4\int dre^{-r^2} r^{2(n+m)-1} \int_0^{\pi/2} d\theta \left(\cos\theta\right)^{2n-1} \left(\sin\theta\right)^{2m-1} \\
&= 2\Gamma\left(m+n\right) \int_0^{\pi/2} d\theta \left(\cos\theta\right)^{2n-1} \left(\sin\theta\right)^{2m-1}
\end{aligned}$$

Thus we get the relation

$$\int_0^{\pi/2} d\theta \left(\cos\theta\right)^{2n-1} \left(\sin\theta\right)^{2m-1} = \frac{1}{2}\frac{\Gamma\left(m\right)\Gamma\left(n\right)}{\Gamma\left(m+n\right)} = \frac{1}{2}B\left(n,m\right)$$

Or

$$\int_0^{\pi/2} d\theta \left(\cos\theta\right)^{n} \left(\sin\theta\right)^{m} = \frac{1}{2}B\left(\frac{n+1}{2},\frac{m+1}{2}\right) \tag{2}$$

Let $u=\cos^2\theta$,

$$B\left(n,m\right) = \int_0^1 u^{m-1}\left(1-u\right)^{n-1} du$$

$u=x^2$,

$$B\left(n,m\right) = 2\int_0^1 x^{2m-1}\left(1-x^2\right)^{n-1} du$$

Let $t=\dfrac{x^2}{1-x^2}$

$$B\left(n,m\right) = \int_0^\infty \frac{t^{m-1}dt}{\left(1+t\right)^{m+n}} = \frac{\Gamma\left(m\right)\Gamma\left(n\right)}{\Gamma\left(m+n\right)} \tag{3}$$

Using polar coordinates in 4-dim

$$\int d^4k = \int_0^\infty k^3 dk \int_0^{2\pi} d\phi \int_0^\pi \sin\theta d\theta \int_0^\pi \sin^2\chi d\chi$$

and integrating over angles

$$\begin{aligned}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+a^2-i\varepsilon)^3} &= 2\pi^2\int_0^\infty \frac{k^3dk}{(2\pi)^4}\frac{1}{(k^2+a^2-i\varepsilon)^3}\\ &= \frac{1}{16\pi^2}\int_0^\infty \frac{k^2dk^2}{(k^2+a^2-i\varepsilon)^3}\end{aligned}$$

Using the formula

$$\int \frac{t^{m-1}dt}{(t+a^2)^n} = \frac{1}{(a^2)^{n-m}}\frac{\Gamma(m)\Gamma(n-m)}{\Gamma(n)}$$

we get

$$\int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+a^2-i\varepsilon)^3} = \frac{1}{32\pi^2(a^2-i\varepsilon)}$$

and

$$\widetilde{\Gamma}(p^2) = \frac{-i\lambda^2}{32\pi^2}\int_0^1 \frac{d\alpha\,(1-\alpha)(2\alpha-1)\,p^2}{[\mu^2-\alpha(1-\alpha)p^2-i\varepsilon]}$$

It is straightforward to carry out the integration to compute $\widetilde{\Gamma}(p^2)$ to get

$$\begin{aligned}\tilde{\Gamma}(p^2) &= \tilde{\Gamma}(s) = \frac{i\lambda^2}{32\pi^2}\left\{2+\left(\frac{4\mu^2-s}{|s|}\right)^{\frac{1}{2}}\ln\left[\frac{(4\mu^2-s)^{\frac{1}{2}}-(|s|)^{\frac{1}{2}}\}}{\{(4\mu^2-s)^{\frac{1}{2}}+(|s|)^{\frac{1}{2}}\}}\right]\right\} && \text{for } s<0\\ &= \frac{i\lambda^2}{32\pi^2}\left\{2-2\left(\frac{4\mu^2-s}{s}\right)^{\frac{1}{2}}\tan^{-1}\left(\frac{s}{4\mu^2-s}\right)^{\frac{1}{2}}\right\} && \text{for } 0<s<4\mu^2\\ &= \frac{i\lambda^2}{32\pi^2}\left\{2+(\frac{s-4\mu^2}{s})^{\frac{1}{2}}ln\left[\frac{s^{\frac{1}{2}}-(s-4\mu^2)^{\frac{1}{2}}}{s^{\frac{1}{2}}+(s-4\mu^2)^{\frac{1}{2}}}\right]+i\pi\right\} && \text{for } s>4\mu^2\end{aligned}$$

Using the same procedure, we can calculate the divergent term to give

$$\Gamma(0) = \frac{i\lambda^2\Lambda^2}{32\pi^2}\int d\alpha \frac{\alpha}{\alpha(\mu^2 - \Lambda^2) + \Lambda^2} \simeq \frac{i\lambda^2}{32\pi^2} \ln \frac{\Lambda^2}{\mu^2}, \qquad \text{for large } \Lambda$$

Dimensional regularization

The basic idea : since the divergences come from integration of momentum in 4-dim space, the integral can be made finite in lower dimensional space. We can define integrals as functions of space-time n and carry out the renormalization for lower values of n before taking the limit $n \to 4$.
Consider the integral

$$I = \int \frac{d^4 k}{(2\pi)^4} \left(\frac{1}{k^2 - \mu^2} \right) \left[\frac{1}{(k-p)^2 - \mu^2} \right]$$

which is divergent in 4-dimension. If we define this as integration over $n-$dimension

$$I\left(n\right) = \int \frac{d^n k}{(2\pi)^4} \frac{1}{(k^2 - \mu^2)} \left[\frac{1}{(k-p)^2 - \mu^2} \right]$$

then it is convergent for $n < 4$.To define this integral for non-integer values of n, we first combine the denominators using Feynman parameters and make the Wick rotation,

$$\begin{aligned} I\left(n\right) &= \int_0^1 d\alpha \int \frac{d^n k}{\left[\left(k - \alpha p\right)^2 - a^2 + i\varepsilon \right]^2} \\ &= i \int_0^1 d\alpha \int \frac{d^n k}{\left[k^2 + a^2 - i\varepsilon\right]^2} \qquad \text{with} \quad a^2 = \mu^2 - \alpha\left(1-\alpha\right) p^2 \end{aligned}$$

Now introduce the spherical coordinates

$$\begin{aligned} \int d^n k &= \int_0^\infty k^{n-1} dk \int_0^{2\pi} d\theta_1 \int_0^\pi \sin\theta_2 d\theta_2 \int \cdots \int_0^\pi \sin^{n-2}\theta_{n-1} d\theta_{n-1} \\ &= \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty k^{n-1} dk \end{aligned}$$

where we have used the formula,

$$\int_0^{\pi} \sin^m \theta d\theta = \frac{\sqrt{\pi}\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)}$$

Then the $n-$dimensional integral is

$$I(n) = \frac{2i\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^1 d\alpha \int_0^{\infty} \frac{k^{n-1}dk}{\left[k^2 + a^2 - i\varepsilon\right]^2}$$

The dependence on n is now explicit and the integral is well-defined for $0 < \mathrm{Re}(n) < 4$. We can extend this domain of analyticity by integration by parts

$$\frac{1}{\Gamma(\frac{n}{2})} \int_0^{\infty} \frac{k^{n-1}dk}{[k^2 + a^2 - i\varepsilon]^2} = \frac{-2}{\Gamma(\frac{n}{2}+1)} \int_0^{\infty} k^n dk \frac{d}{dk}\left(\frac{1}{[k^2 + a^2 - i\varepsilon]^2}\right)$$

where we have used

$$z\Gamma(z) = \Gamma(z+1)$$

The integral is now well defined for $2 < \mathrm{Re}(n) < 4$. Repeat this procedure m times, the analyticity domain is extended to $-2m < \mathrm{Re}(n) < 4$ and eventually to $\mathrm{Re}(n) \to -\infty$.

To see what happens as $n \to 4$, we can integrate over k to get

$$I(n) = i\pi^{n/2}\Gamma\left(2-\frac{n}{2}\right)\int_0^1 \frac{d\alpha}{[a^2 - i\varepsilon]^{2-n/2}}$$

Using the formula,

$$\Gamma\left(2-\frac{n}{2}\right) = \frac{\Gamma\left(3-\frac{n}{2}\right)}{2-\frac{n}{2}} \to \frac{2}{4-n} \quad as\ n \to 4$$

we see that the singularity at $n = 4$ is a simple pole. Expand everything around $n = 4$,

$$\Gamma\left(2-\frac{n}{2}\right) = \frac{2}{4-n} + A + (n-4)B + \cdots$$

$$a^{n-4} = 1 + (n-4)\ln a + \cdots$$

where A and B are some constants, we obtain the limit, as $n \longrightarrow 4$

$$I(n) \longrightarrow \frac{2i\pi^2}{4-n} - i\pi^2\int_0^1 d\alpha \ln[\mu^2 - \alpha(1-\alpha)p^2] + i\pi^2 A$$

and the 1-loop contribution to 4-point function is,

$$\Gamma(p^2) = \frac{\lambda^2}{32\pi^2}\left\{\frac{2i}{4-n} - i\int_0^1 d\alpha \ln[\mu^2 - \alpha(1-\alpha)p^2] + iA\right\}$$

Taylor expansion around $p^2 = 0$ gives

$$\Gamma(p^2) = \Gamma(0) - \tilde{\Gamma}(p^2)$$

$$\Gamma(0) = \frac{\lambda^2}{32\pi^2}\left(\frac{2i}{4-\pi} - i\ln\mu^2 + iA\right) \simeq \frac{i\lambda^2}{16\pi^2(4-n)}$$

and

$$\begin{aligned}\tilde{\Gamma}(p^2) &= \frac{-i\lambda^2}{32\pi^2}\int_0^1 d\alpha \ln\left[\frac{\mu^2-\alpha(1-\alpha)p^2}{\mu^2}\right] \\ &= \frac{-i\lambda^2}{32\pi^2}\int_0^1 \frac{d\alpha(1-\alpha)(2\alpha-1)p^2}{[\mu^2-\alpha(1-\alpha)p^2]}\end{aligned}$$

Clearly this finite part is exactly the same as that given by the method of covariant regulariztion. The 1-loop self energy in dimensional-regularization scheme becomes

$$-i\Sigma(p^2) = \frac{\lambda}{2}\int \frac{d^n k}{(2\pi)^4}\frac{1}{k^2-\mu^2+i\varepsilon} = \frac{-i\lambda\pi^{n/2}\Gamma\left(1-\frac{n}{2}\right)}{32\pi^4(\mu^2)^{1-n/2}}$$

From the relation,

$$\Gamma\left(1-\frac{n}{2}\right) = \frac{\Gamma\left(3-\frac{n}{2}\right)}{\left(1-\frac{n}{2}\right)\left(2-\frac{n}{2}\right)}$$

we see that the quadratic divergnece has pole at $n=4$ and also at $n=2$. For $n\to 4$ we have,

$$-i\Sigma(0) = \frac{i\lambda\mu^2}{16\pi^2}\left(\frac{1}{4-n}\right)$$

Power counting and Renormalizability

We first illustrate some simple examples.

1. 4-fermion interaction where the interaction Lagrangian is given by

$$\mathcal{L}_I = g\left(\bar{\psi}\psi\right)^2$$

Let

$$F = \text{number of external fermion lines}$$
$$IF = \text{number of internal fermion lines}$$

Then we have

$$F + 2\,(IF) = 4n$$

and

$$L = (IF) - n + 1$$

where n is the number of vertices. The superficial degree of divergence is

$$D = 4L - (IF)$$

and can simplied to give

$$D = 4 - \frac{3}{2}F + 2n$$

Thus for n large enough $D > 0$ for any value of F .The counter terms needed are

$$F = 2: \qquad \bar{\psi}\psi, \quad \bar{\psi}\partial\!\!\!/\psi, \quad \bar{\psi}\partial\!\!\!/\partial\!\!\!/\psi, \quad \bar{\psi}\partial\!\!\!/\partial\!\!\!/\partial\!\!\!/\psi, \quad \cdots$$

$$F = 4: \qquad \bar{\psi}\psi\bar{\psi}\psi, \quad \bar{\psi}\partial\!\!\!/\psi\bar{\psi}\psi, \quad \bar{\psi}\partial\!\!\!/\psi\bar{\psi}\partial\!\!\!/\psi, \quad \bar{\psi}\psi\bar{\psi}\partial\!\!\!/\partial\!\!\!/\psi, \quad \bar{\psi}\partial\!\!\!/\partial\!\!\!/\partial\!\!\!/\psi\psi\psi, \cdots$$

$$\cdots$$

and there are infinite different type of counterterms. Thus we will not be able to aborb these infinities by redefining the parameters in the Lagrangian.

2. Yukawa interaction with interaction,

$$\mathcal{L}_I = f\left(\bar{\psi}\psi\right)\phi$$

Then we have

$$F + 2\,(IF) = 2n, \qquad B + 2\,(IB) = n,$$

and

$$L = (IF) + (IB) - n + 1$$

where n is the number of vertices. The superficial degree of divergence is

$$D = 4L - (IF) - 2(IB)$$

and can simplied to give

$$D = 4 - B - \frac{3}{2}F$$

Now D is independent of n. Thus $D \geq 0$, only for small number of cases and the counterterms needed are

$$B = 2, \qquad \phi^2, \qquad \left(\partial_\mu \phi\right)^2, \qquad B = 4, \qquad \phi^4$$

$$F = 2; \qquad \bar{\psi}\psi, \quad \bar{\psi}\partial\!\!\!/\psi, \qquad F = 2,\ B = 1; \qquad \phi\bar{\psi}\psi,$$

So counter terms can be absorbed in redefinition of the parameters in the Lagrangin and theory is renormalizable. Note that we need to add a $\lambda\phi^4$ interaction in order to absorb $B = 4$ coounterterm.

Now discuss renormalization for more general interactions. It is clear that it is advantageous to use the BPH scheme.

Theories with fermions and scalar fields

We first study the case with fermion ψ and scalar field ϕ. Write the Lagrangian density as

$$\mathcal{L} = \mathcal{L}_0 + \sum_i \mathcal{L}_i$$

where $\mathcal{L}_0$ is the free Lagrangian and $\mathcal{L}_i$ are the interaction terms e.g.

$$\mathcal{L}_i = g_1 \bar{\psi}\gamma^\mu \psi \partial_\mu \phi, \qquad g_2 \left(\bar{\psi}\psi\right)^2, \qquad g_3 \left(\bar{\psi}\psi\right)\phi, \qquad \cdots$$

Define

n_i = number of $i-th$ type vertices
b_i = number of scalar lines in $i-th$ type vertex
f_i = number of fermion lines in $i-th$ type vertex
d_i = number of derivatives in $i-th$ type of vertex
B = number of external scalar lines
F = number of external fermion lines
IB = number of internal scalar lines
IF = number of internal fermion lines

Counting the scalar and fermion lines,

$$B + 2\,(IB) = \sum_i n_i b_i \tag{4}$$

$$F + 2\,(IF) = \sum_i n_i f_i \tag{5}$$

Using momentum conservation at each vertex, number of loop integration L is

$$L = (IB) + (IF) - n + 1, \qquad n = \sum_i n_i$$

where the last term is due to the overall momentum conservation. The superficial degree of divergence is

$$D = 4L - 2\,(IB) - (IF) + \sum_i n_i d_i$$

Using the relations given in Eqs(4,5) we get

$$D = 4 - B - \frac{3}{2}F + \sum_i n_i \delta_i \tag{6}$$

where

$$\delta_i = b_i + \frac{3}{2}f_i + d_i - 4$$

is called the *index of divergence* of the interaction. Since $\mathcal{L}$ has dimension 4 and scalar field, fermion field and the derivative have dimensions, 1, $\frac{3}{2}$, and 1 respectively, we get for the dimension of the coupling constant g_i as

$$\dim(g_i) = 4 - b_i - \frac{3}{2}f_i - d_i = -\delta_i$$

We distinguish 3 different cases;

1. $\delta_i < 0$
   Here D decreases with $n_{i\kappa}$ and the interaction is called $super - renormalizable\ interaction$. The divergences occur only in some lower order diagrams. There is only one type of theory in this category, namely ϕ^3 interaction.

2. $\delta_i = 0$
D is independent of n_i are called *renormalizable interactions*. The divergence are present in a finite number of Green's functions. Interactions in this category are of the form, $g\phi^4$, $f\left(\bar{\psi}\psi\right)\phi$.

3. $\delta_i > 0$
Then D increasesn_i and all Green's functions are divergent for large n_i. These are called *non − renormalizable interactions*. There are plenty of examples in this category, $g_1\bar{\psi}\gamma^\mu\psi\partial_\mu\phi$, $g_2\left(\bar{\psi}\psi\right)^2$, $g_3\phi^5, \cdots$

The index of divergence δ_i can be related to the operator's *canonical dimension* defined by the high energy behavior in the free field theory. More specifically, for any operator A, write the 2-point function as

$$D_A\left(p^2\right) = \int d^4xe^{-ip\cdot x}\left\langle 0\left|T\left(A\left(x\right)A\left(0\right)\right)\right|0\right\rangle$$

If the asymptotic behavior is,

$$D_A\left(p^2\right) \longrightarrow \left(p^2\right)^{-\omega_A/2}, \qquad as \qquad p^2 \longrightarrow \infty$$

then the canonical dimension is

$$d\left(A\right) = \left(4 - \omega_A\right)/2$$

Thus for fermion and scalar fields,

$$d\left(\phi\right) = 1, \qquad d\left(\partial^n\phi\right) = 1 + n$$

$$d\left(\psi\right) = \frac{3}{2}, \qquad d\left(\partial^n\psi\right) = \frac{3}{2} + n$$

Note that in these cases, these values are the same as those in the dimensional analysis in the classical theory are called the **naive dimensions**. As we will see for the vector field, the canonical dimension $\neq$ naive dimension.

For composite operators it is difficult to know their asymptotic behavior. So we define their canonical dimensions as the algebraic sum of their constituent fields. For example,

$$d\left(\phi^2\right) = 2, \qquad d\left(\bar{\psi}\psi\right) = 3$$

For general composite operators that show up in the those interaction described before, we have,

$$d\left(\mathcal{L}_i\right) = b_i + \frac{3}{2} f_i + d_i$$

and it is related to the index of divergence as

$$\delta_i = d\left(\mathcal{L}_i\right) - 4$$

So dimension 4 interaction is renormalizable and greater than 4 is non-renormalizable.

Counter terms

Recall that we add counterterms to cancel all the divergences in Green's functions with $D \geq 0$. For convenience we use the Taylor expansion around zero external momenta $p_i = 0$. Then general diagram with $D \geq 0$, counter terms are of the form

$$O_{ct} = (\partial_\mu)^\alpha (\psi)^F (\phi)^B, \qquad \alpha = 1, 2, \cdots D$$

and the canonical dimension is

$$d_{ct} = \frac{3}{2}F + B + \alpha$$

The index of divergence of the counterterms is

$$\delta_{ct} = d_{ct} - 4$$

Using the relation in Eq (6) we can write this as

$$\delta_{ct} = (\alpha - D) + \sum_i n_i \delta_i$$

Since $\alpha \leq D$, we have

$$\delta_{ct} \leq \sum_i n_i \delta_i$$

Thus, the counterterms induced by a Feynman diagrams have indices of divergences less or equal to the sum of the indices of divergences of all interactions δ_i in the diagram.

- renormalizable interactions with $\delta_i = 0$ will generate counterterms with $\delta_{ct} \leq 0$.
- if all the $\delta_i \leq 0$ terms are present in $\mathcal{L}$, the counter terms may be considered as redefining parameters like masses and coupling constants in the theory.

- non-renormalizable interactions which have $\delta_i > 0$ will generate counterterms with arbitrary large δ_{ct} and cannot be absorbed into the original Lagrangian by a redefinition of parameters δ_{ct}.
- more restricted definition of renormalizability: a Lagrangian is renormalizable if all the counterterms can be absorbed by redefinitions of parameters in the Lagrangian. Then Yukawa interaction $\bar{\psi}\gamma_5\psi\phi$ by itself, is not renormalizable even though the coupling constant is dimensionless. This is because the 1-loop diagram shown below

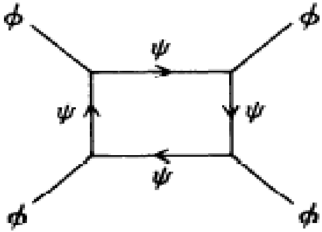


Fig 14 Box diagram for Yukawa

is logarithmically divergent and needs a counter term of the form ϕ^4 which is not present in the original Lagrangain.

Theories with vector fields

Here we distinguish massless from massive vector fields because their asymptotic behaviors for the free field propagators are very different.

1. Massless vector field

Massless vector field is associated with local gauge invariance as in QED. The asymptotic behavior of free field propagator is very similar to that of scalar field. For example, in the Feynman gauge

$$\Delta_{\mu\nu}(k) = \frac{-ig_{\mu\nu}}{k^2 + i\varepsilon} \longrightarrow O\left(k^{-2}\right), \qquad \text{for large } k^2$$

has same behavior as scalar field. Thus the power counting for theories with massless vector field interacting with fermions and scalar fields is the same as before. The renomalizable interactions in this category are of the type,

$$\bar{\psi}\gamma_\mu \psi A^\mu, \qquad \phi^2 A_\mu A^\mu, \qquad \left(\partial_\mu \phi\right)\phi A^\mu$$

2. Massive vector field

Here free Lagrangian is of the form,

$$\mathcal{L}_0 = -\frac{1}{4}\left(\partial_\mu V_\nu - \partial_\nu V_\mu\right)^2 + \frac{1}{2}M_V^2 V_\mu^2$$

where V_μ is a massive vector field and M_V is the mass of the vector field. The propagator in momentum space is of the form,

$$D_{\mu\nu}(k) = \frac{-i\left(g_{\mu\nu} - k_\mu k_\nu / M_V^2\right)}{k^2 - M_V^2 + i\varepsilon} \longrightarrow O(1), \quad \text{as } \; k \to \infty \tag{7}$$

This implies that canonical dimension of massive vector field is two rather than one. To see this we write

$$\int d^4x \mathcal{L}_0 = \int d^4x \frac{1}{2}\left[V_\mu \partial^2 V^\mu - V_\mu \partial^\mu \partial^\nu V_\nu + M_V^2 V_\mu V^\mu\right] = \int d^4x \frac{1}{2} V_\mu \left[\left(\partial^2 + M_V^2\right) g^{\mu\nu} - \partial^\mu \partial^\nu\right] V_\nu$$

The proppagator is deifined by

$$\left[\left(\partial^2 + M_V^2\right) g^{\mu\nu} - \partial^\mu \partial^\nu\right] G_{\nu\beta}(x-y) = g_\beta^\mu \delta^4 (x-y)$$

We use the Fourier transform to solve this equation. Write

$$G_{\nu\beta}(x-y) = \int d^4 x e^{-ikx} D_{\nu\beta}(k)$$

we get

$$\left[\left(-k^2 + M_V^2\right) g^{\mu\nu} + k^\mu k^\nu\right] D_{\nu\beta} = g_\beta^\mu$$

Decompose $D_{\nu\beta}$ as

$$D_{\nu\beta} = a g_{\nu\beta} + b k_\beta k_\nu$$

Then

$$a\left[\left(-k^2 + M_V^2\right) g_\beta^\mu + k^\mu k_\beta\right] + b\left[\left(-k^2 + M_V^2\right) k^\mu k_\beta + k^2 k^\mu k_\beta\right] = g_\beta^\mu$$

From these we get

$$a = -\frac{1}{k^2 - M_V^2}, \qquad b = \frac{1}{M_V^2}\frac{1}{k^2 - M_V^2}$$

Or

$$D_{\mu\nu}(k) = \frac{-i\left(g_{\mu\nu} - k_\mu k_\nu / M_V^2\right)}{k^2 - M_V^2 + i\varepsilon}$$

The superficial degree of divergence given by

$$D = 4 - B - \frac{3}{2} F - V + \sum_i n_i \left(\Delta_i - 4\right)$$

with

$$\Delta_i = b_i + \frac{3}{2} f_i + 2v_i + d_i$$

Here V is the number of external vector lines, v_i is the number of vector fields in the ith type of vertex and Δ_i is the canonical dimension of the interaction term in $\mathcal{L}$. The only renormalizable interaction with, $\Delta_i \leq 4$, is $\phi^2 A_\mu$ and is not Lorentz invariant. Thus there is no nontrivial interaction of the massive vector field which is renormalizabel. However, two important exceptions should be noted;

1. In a gauge theory with spontaneous symmetry breaking, the gauge boson will acquire mass in such a way to preserve the renormalizability of the theory ([?]).
2. A theory with a neutral massive vector boson coupled to a conserved current is also renormalizable. Heuristically, we can understand this as follows. The propagator in Eq(7) always appears between conserved currents $J^\mu(k)$ and $J^\nu(k)$ and the $k_\mu k_\nu / M_V^2$ term will not contribute because of current conservation, $k^\mu J_\mu(k) = 0$ or in coordinate space $\partial^\mu J_\mu(x) = 0$.

Composite operator

In some cases, we need to consider Green's function of composite operator, an operator with more than one fields at same space time.

Consider a simple composite operator of $\Omega(x) = \frac{1}{2}\phi^2(x)$ in $\lambda\phi^4$ theory. Green's function with one insertion of Ω is of the form,

$$G_\Omega^{(n)}(x; x_1, x_2, x_3, ..., x_n) = \left\langle 0 \right| T(\frac{1}{2}\phi^2(x)\phi(x_1)\phi(x_2)...\phi(x_n)) \left| 0 \right\rangle$$

In momentum space we have

$$(2\pi)^4 \delta^4(p + p_1 + p_2 + ... + p_n) G_{\phi^2}^{(n)}(p; p_1, p_2, p_3, ..., p_n) = \int d^4x \, e^{-ipx} \int \prod_{i=1}^n d^4x_i e^{-ip_i x_i} G_\Omega^{(n)}(x; x_1, x_2, x_3, ..., x_n)$$

In perturbation theory, we can use Wick's theorem to work out these Green's functions in terms of Feynman diagram.
Example, to lowest order in λ the 2-point function with one composite operator $\Omega(x) = \frac{1}{2}\phi^2(x)$ is, after using the Wick's theorem,

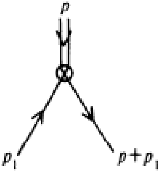
$$G^{(2)}_{\phi^2}(x; x_1, x_2) = \frac{1}{2}\langle 0| T\{\phi^2(x)\phi(x_1)\phi(x_2)\}|0\rangle = i\Delta(x - x_1)i\Delta(x - x_2)$$

or in momentum space

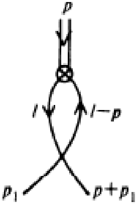
$$G^{(2)}_{\phi^2}(p; p_1, p_2) = i\Delta(p_1)i\Delta(p + p_1)$$

If we truncate the external propagators, we get

$$\Gamma^{(2)}_{\phi^2}(p, p_1, -p_1 - p) = 1$$



(a)



(b)

To first order in λ, we have

$$G^{(2)}_{\phi^2}(x, x_1, x_2) = \int \left\langle 0 \right| T\{\frac{1}{2}\phi^2(x)\phi(x_1)\phi(x_2)\frac{(-i\lambda)}{4!}\phi^4(y)\}\left|0\right\rangle d^4y$$

$$= \int d^4y \frac{-i\lambda}{2}[i\Delta(x-y)]^2 i\Delta(x_1-y)i\Delta(x_2-y)$$

The amputated 1PI momentum space Green's function is

$$\Gamma^{(2)}_{\phi^2}(p; p_1, -p-p_1) = \frac{-i\lambda}{2}\int \frac{d^4l}{(2\pi)^4}\frac{i}{l^2-\mu^2+i\epsilon}\frac{i}{(l-p)^2-\mu^2+i\epsilon}$$

To calculate this type of Green's functions systematically, we can add a term $\chi(x)\Omega(x)$ to $\mathcal{L}$

$$\mathcal{L}[\chi] = \mathcal{L}[0] + \chi(x)\Omega(x)$$

where $\chi(x)$ is a c-number source function. We can construct the generating functional $W[\chi]$ in the presence of this external source. We obtain the connected Green's function by differentiating $\ln W[\chi]$ with respect to χ and then setting χ. to zero.

Renormalization of composite operators

Superficial drgrees of divergence for Green 's function with one composite operator is,

$$D_\Omega = D + \delta_\Omega = D + (d_\Omega - 4)$$

where d_Ω is the canonical dimension of Ω. For the case of $\Omega(x) = \frac{1}{2}\phi^2(x)$, $\quad d_{\phi^2} = 2$ and $D_{\phi^2} = 2 - n \Rightarrow$ only $\Gamma^{(2)}_{\phi^2}$ is divergent. Taylor expansion takes the form,

$$\Gamma^{(2)}_{\phi^2}(p;p_1) = \Gamma^{(2)}_{\phi^2}(0,0) + \Gamma^{(2)}_{\phi^2 R}(p,p_1)$$

We can combine the counter term

$$\frac{-i}{2}\Gamma^{(2)}\phi^2(0,0)\chi(x)\phi^2(x)$$

with the original term to write

$$\frac{-i}{2}\chi\phi - \frac{i}{2}\Gamma^2_{\phi^2}(0,0)\chi\phi^2 = -\frac{i}{2}Z_{\phi^2}\chi\phi^2$$

In general, we need to insert counterterm $\Delta\Omega$ into the original addition

$$L \to L + \chi(\Omega + \Delta\Omega)$$

If $\Delta\Omega = C\Omega$, as in the case of $\Omega = \frac{1}{2}\phi^2$,we have

$$L[\chi] = L[0] + \chi Z_\Omega \Omega = L[0] + \chi\Omega_0$$

with

$$\Omega_0 = Z_\Omega\Omega = (1 + C)\Omega$$

Such composite operators are said to be mutiplicative renormalizable and Green's functions of unrenormalized operator Ω_0 is related to that of renormalized operator Ω by

$$G_{\Omega_0}^{(n)}(x;x_1,x_2,...x_n) = \langle 0|T\{\Omega_0(x)\phi(x_1)\phi(x_2)...\phi(x_n)\}|0\rangle$$

$$= Z_\Omega Z_\phi^{n/2} G_{IR}^{(n)}(x;x_1,...x_n)$$

For more general cases, $\Delta\Omega \neq c\Omega$ and renormalization of a composite operator may require counterterm proportional to other composite operators.
Example: Conside 2 composite operators A and B. Denote the counterterms by ΔA and ΔB. Including the counter terms we can write,

$$L[\chi] = L[0] + \chi_A(A + \Delta A) + \chi_B(B + \Delta B)$$

Very often with counterterms ΔA and ΔB are linear combinations of A and B

$$\Delta A = C_{AA}A + C_{AB}B$$

$$\Delta B = C_{BA}A + C_{BB}B$$

We can write

$$L[\chi] = L[0] + (\chi_A\, \chi_B)\,\{C\}\begin{pmatrix} A \\ B \end{pmatrix} \quad where\ \ \{C\} = \begin{pmatrix} 1 + C_{AA} & C_{AB} \\ C_{BA} & 1 + C_{BB} \end{pmatrix}$$

Diagonalized $\{C\}$ by bi-unitary transformation

$$U\{C\}V^{+} = \begin{pmatrix} Z_{A'} & 0 \\ 0 & Z_{B'} \end{pmatrix}$$

Then

$$L[\chi] = L[0] + Z_{A'}\chi_{A'}A' + Z_{B'}\chi_{B'}B'$$

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = V\begin{pmatrix} A \\ B \end{pmatrix} \quad \left(\chi_{A'}\ \ \chi_{B'}\right) = (\chi_A\ \ \chi_B)\,U$$

A' , B' are multiplicatively renormalizable.

Renormalization group

Discussion will be brief. Renormalization scheme requires specification of substraction points which introduce new mass scales. This introduces the concept of energy dependent "coupling constants",

$$e.g \quad \lambda = \lambda(s)$$

even though the coupling constants in the original Lagranggian are independent of energies.

Renormalization group equation

In general, there is arbitrariness in choosing the renormalization schemes (or the substraction points). Nevertheless, the physical results should be the same, i.e. independent of renormalization schemes. In essence this is the physical content of the renormalization group equation. Suppose we have different renormalizarion scheme R and R'. From the point of view of BPH renormalization, we can write

$$\mathcal{L} = \mathcal{L}_R(R - quantities) = \mathcal{L}_{R'}(R' - quantities)$$

Recall that

$$\phi_R = Z_{\varphi R}^{-\frac{1}{2}}\phi_0, \qquad \lambda_R = Z_{\lambda R}^{-1} Z_{\phi R}^2 \lambda_0 \qquad \mu_R^2 = \mu_0^2 + \delta\mu_R^2$$

Similarly,

$$\phi_{R'} = Z_{\varphi R'}^{-\frac{1}{2}}\phi_0, \qquad \lambda_{R'} = Z_{\lambda R'}^{-1} Z_{\phi R'}^2 \lambda_0 \qquad \mu_{R'}^2 = \mu_0^2 + \delta\mu_{R'}^2$$

Since ϕ_0, λ_0 and μ_0 are the same, we can finite relations between $R-$ and R' quantities

Callan-Symanzik equation

Here to conform with the standard notation, we make a change of notation. We will use m_0 and m for bare and renormalized masses instead of μ_0 and μ. The paramter μ is now used to denote the substraction point. In general the renormalized Green's functions are related to bare Green's by

$$Z_\phi^{\frac{n}{2}}\Gamma_R^{(n)}(P_i,\lambda,m,\mu)=\Gamma^{(n)}(P_i,\lambda_0,m_0)$$

The renormalized $\Gamma_R^{(n)}(P_i,\lambda,m,\mu)$ depend on the substraction point μ, while the unrenormalied one $\Gamma^{(n)}(P_i,\lambda_0,\mu_0^2)$ do not,

$$\mu\frac{\partial}{\partial\mu}\Gamma^{(n)}(P_i,\lambda_0,\mu_0^2)=0,\qquad \text{or}\qquad \mu\frac{\partial}{\partial\mu}[Z_\phi^{-\frac{n}{2}}\Gamma_R^{(n)}(P_i,\lambda,\mu)]=0$$

Using the μ dependence of Z,λ,m we get

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(\lambda)\frac{\partial}{\partial\lambda}+m(\lambda)\frac{\partial}{\partial m}-n\gamma(\lambda)\right]\Gamma_R^{(n)}(P_i,\lambda,\mu)=0$$

where

$$\beta(\lambda)=\mu\frac{\partial\lambda}{\partial\mu},\qquad m(\lambda)=\mu\frac{\partial m}{\partial\mu},\qquad \gamma(\lambda)=\frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_\phi$$

This is usually referred to as **Callan-Symanzik** equation.

For simplicity, set $m = 0$.
Define a dimensionless quantity $\bar{\Gamma}$ by

$$\Gamma_{as}^{(n)}(P_i, \lambda, \mu) = \mu^{4-n}\bar{\Gamma}_R^{(n)}(\frac{P_i}{\mu}, \lambda)$$

Since $\bar{\Gamma}$ is dimensionless, as we scale up the momenta to write

$$(\mu\frac{\partial}{\partial\mu} + \sigma\frac{\partial}{\partial\sigma})\bar{\Gamma}_R^{(n)}(\frac{\sigma p_i}{\mu}, \lambda) = 0$$

and

$$[\mu\frac{\partial}{\partial\mu} + \sigma\frac{\partial}{\partial\sigma} + (n-4)]\Gamma_{as}^{(n)}(\frac{\sigma P_i}{\mu}, \lambda) = 0$$

From Callan-Symanzik equation we get

$$[\sigma\frac{\partial}{\partial\sigma} - \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda) + (n-4)]\Gamma_{as}^{(n)}(\sigma p_i, \lambda, \mu) = 0$$

To solve this equation, we remove the non-derivative terms by the transformation

$$\Gamma_{as}^{(n)}(\sigma p_i, \lambda, \mu) = \sigma^{4-n}\exp[n\int_0^\lambda \frac{\gamma(x)}{\beta(x)}dx]\Gamma^{(n)}(\sigma p_i, \lambda, \mu)$$

Then $F^{(n)}$ satisfies the equation

$$[\sigma\frac{\partial}{\partial\sigma} - \beta(\lambda)\frac{\partial}{\partial\lambda}]\Gamma^{(n)}(\sigma p_i, \lambda, \mu) = 0$$

or

$$[\frac{\partial}{\partial t} - \beta(\lambda)\frac{\partial}{\partial \lambda}]\Gamma^{(n)}(e^t p_i, \lambda, \mu) = 0 \quad \text{where} \quad t = \ln \sigma$$

Introduce the effective, or running constant $\bar{\lambda}$ as solution to the equation

$$\frac{d\bar{\lambda}(t,\lambda)}{dt} = \beta(\bar{\lambda}) \qquad \text{with initial condition} \quad \bar{\lambda}(0,\lambda) = \lambda$$

This equation has the solution

$$t = \int_{\lambda}^{d\bar{\lambda}(t,\lambda)} \frac{dx}{\beta(x)}$$

It is straightforward to show that

$$\frac{1}{\beta(\bar{\lambda})}\frac{d\bar{\lambda}}{d\lambda} = \beta(\lambda) \qquad \text{and} \qquad [\frac{\partial}{\partial t} - \beta(\lambda)\frac{\partial}{\partial \lambda}]\bar{\lambda}(t,\lambda) = 0$$

In other words, $F^{(n)}$ depends on t and λ only through the combination $\bar{\lambda}(t,\lambda)$

$$F^{(n)} = F^{(n)}(p_i, \bar{\lambda}(t,\lambda), \mu)$$

Also

$$\begin{aligned} \exp[n\int_0^{\lambda}\frac{\gamma(\lambda)}{\beta(\lambda)}d\lambda] \quad &\sim \quad \exp[n\int_0^{\bar{\lambda}}\frac{\gamma(x)}{\beta(x)}dx + n\int_{\bar{\lambda}}^{\lambda}\frac{\gamma(x)}{\beta(x)}dx] \\ &= \quad H(\bar{\lambda})\exp[-n\int_{\lambda}^{\bar{\lambda}}\frac{\gamma(x)}{\beta(x)}dx] \end{aligned}$$

where

$$H(\bar{\lambda}) = exp[n \int_0^{\bar{\lambda}} \frac{\gamma(x)}{\beta(x)} dx]$$

The solution is then

$$\Gamma_{as}^{(n)}(\sigma p_i, \lambda, \mu) = \sigma^{4-n} exp[-n \int_0^t \gamma(\bar{\lambda}(x', \lambda)) dx'] H(\bar{\lambda}) F^{(n)}(p_i, \bar{\lambda}(t, \lambda), \mu)$$

If we set $t = 0$ (or $\sigma = 0$), we see that

$$\Gamma_{as}^{(n)}(p_i, \lambda, \mu) = H(\lambda) F^{(n)}(p_i, \lambda, \mu)$$

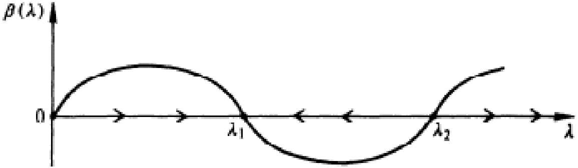
Thus the solution has the simple form

$$\Gamma_{as}^{(n)}(\sigma p_i, \lambda, \mu) = \sigma^{4-n} exp[-n \int_0^t \gamma(\bar{\lambda}(x', \lambda)) dx' \, \Gamma_{as}^{(n)}(p_i, \bar{\lambda}(t, \lambda), \mu)$$

Effective coupling constant $\bar{\lambda}$

$$\frac{d\bar{\lambda}(t,\lambda)}{dt} = \beta(\bar{\lambda}) \qquad \text{initial condition} \quad \bar{\lambda}(0,\lambda) = \lambda$$

Suppose $\beta(\lambda)$ has the following simple behavior



Suppose $0 < \lambda < \lambda_1$,then at $t = 0$, $\frac{d\bar{\lambda}}{dt}\,|_{t=0} > 0 \Rightarrow \bar{\lambda}$ increases as t increases

This increase will continue until $\bar{\lambda}$ reaches λ_1, where $\frac{d\bar{\lambda}}{dt} = 0$

On the other hand, if initially $\lambda_1 < \lambda < \lambda_2$, then $\frac{d\bar{\lambda}}{dt}\,|_{t=0} < 0$,$\bar{\lambda}$will decrease until it reaches λ_1. Thus as $t \to \infty$, we get

$$\lim_{t\to\infty} \bar{\lambda}(t,\lambda) = \lambda_1 \qquad \lambda_1 : \text{ultraviolet stable fixed point}$$

and

$$\Gamma_{as}^{(n)}(p_i, \bar{\lambda}(t,\lambda), \mu) \to_{t\to\infty} \Gamma_{as}^{(n)}(p_i, \lambda_1, \mu)$$

Example: Suppose $\beta(x)$ has a simple zero at $\lambda = \lambda_1$,

$$\beta(\lambda) \simeq a(\lambda_1 - \lambda) \qquad a > 0$$

Then

$$\frac{d\bar{\lambda}}{dt} = a(\lambda_1 - \lambda) \Rightarrow \bar{\lambda} = \lambda_1 + (\lambda - \lambda_1)e^{-at}$$

i.e. the approach to fixed point is exponential in $t = \ln \sigma$, or power in σ . Also the prefactor can be simplified,

$$\int_0^t \gamma(\bar{\lambda}(x,\lambda))dx = \int_\lambda^{\bar{\lambda}} \frac{\gamma(y)dy}{\beta(y)} \approx \frac{-\gamma(\lambda_1)}{a} \int_\lambda^{\bar{\lambda}} \frac{d\lambda'}{\lambda' - \lambda_1} = \frac{-\gamma(\lambda_1)}{a} \ln(\frac{\bar{\lambda} - \lambda_1}{\lambda - \lambda_1})$$

$$= \gamma(\lambda_1)t = \gamma(\lambda_1) \ln \sigma$$

$$\lim_{\sigma \to \infty} \Gamma_{as}^{(n)}(\sigma p_i, \lambda, \mu) = \sigma^{4-n[1+\gamma(\lambda_1)]} \Gamma_{as}^{(n)}(p_i, \lambda_1, \mu)$$

Thus the asymptotic behavior in field theory is controlled by the fixed point λ_1 and $\gamma(\lambda_1)$ anomalous dimension.

Quantum Field Theory

Ling-Fong Li

Asymptotic Freedom

1. $\lambda\phi^4$ theory ——The Lagrangian is

$$\mathcal{L} = \frac{1}{2}[(\partial_\mu\phi)^2 - m^2\phi^2] - \frac{\lambda}{4!}\phi^4$$

Effective coupling constant $\bar{\lambda}$ satisfies the differential equation

$$\frac{d\bar{\lambda}}{dt} = \beta(\bar{\lambda}) \ , \quad \beta(\lambda) \approx \frac{3\lambda^2}{16\pi^2} + O(\lambda^3)$$

It is not asymptotically free. The generalization to more than one scalar fields is the replacement,

$$\lambda\phi^4 \to \lambda_{ijkl}\phi_i\phi_j\phi_k\phi_l \ , \qquad \lambda_{ijkl} \text{ is totally symmetric}$$

Then the differential equations are of the form,

$$\beta_{ijkl} = \frac{d\lambda_{ijkl}}{dt} = \frac{1}{16\pi^2}\left[\lambda_{ijmn}\lambda_{mnkl} + \lambda_{ikmn}\lambda_{mnjl} + \lambda_{ilmn}\lambda_{mnjk}\right]$$

For the special case, $i = j = k = l = 1$, we wee that$\beta_{1111} = \frac{3}{16\pi^2}\lambda_{iimn}\lambda_{mn11} > 0$ and theory is not asymptotically free.

2. Yukawa interaction

Here we need to include the scalar self interaction $\lambda\phi^4$ in order to be renormalizable

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}[(\partial_\mu\phi)^2 - \mu^2\phi^2] - \lambda\phi^4 + f\bar{\psi}\psi\phi$$

Now we have a coupled differential equations,

$$\beta_\lambda = \frac{d\lambda}{dt} = A\lambda^2 + B\lambda f^2 + Cf^4, \qquad A > 0$$

$$\beta_f = \frac{df}{dt} = Df^3 + E\lambda^2 f, \qquad D > 0$$

To get $\beta_\lambda < 0$, with $A > 0$, we need $f^2 \sim \lambda$. This means we can drop E term in β_f. With $D > 0$, Yukawa coupling f is not asymptotically free. Generalization to the cases of more than one fermion fields or more scalar fields will not change the situation.

3. Abelian gauge theory(QED)

The Lagrangian is of the usual form,

$$\mathcal{L} = \bar{\psi} i \gamma^{\mu} (\partial_{\mu} - ieA_{\mu}) \psi - m\bar{\psi}\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

The effective coupling constant $\bar{e}$ satisfies the equation,

$$\frac{d\bar{e}}{dt} = \beta_e = \frac{\bar{e}^3}{12\pi^2} + O(e^5)$$

For the scalar *QED* we have

$$\frac{d\bar{e}}{dt} = \beta'_e = \frac{\bar{e}^3}{48\pi^2} + O(e^5)$$

Both are not asymptotically free.

4. Non-Abelian gauge theories

It turns out that only non-Abelian gauge theories are asymptotically free. Write the Lagrangian as

$$\mathcal{L} = -\frac{1}{2} T_r (F_{\mu\nu} F^{\mu\nu})$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \qquad A_\mu = T_a A^a_\mu$$

and

$$[T_a, T_b] = if_{abc} T_c \ , \quad T_r(T_a, T_b) = \frac{1}{2}\delta_{ab}$$

The evolution of the effective coupling constant is governed by

$$\frac{dg}{dt} = \beta(g) = -\frac{g^3}{16\pi^2}(\frac{11}{3})t_2(V) < 0$$

Since $\beta(g) < 0$ for small g, this theory is asymptotically free. Here

$$t_2(V)\delta^{ab} = t_r[T_a(V)T_b(V)] \qquad t_2(V) = n \ \ for \ \ SU(n)$$

If gauge fields couple to fermions and scalars with representation matrices, $T^a(F)$ and $T^a(s)$ respectively, then

$$\beta_g = \frac{g^3}{16\pi^2}\left[-\frac{11}{3}t_2(V) + \frac{4}{3}t_2(F) + \frac{1}{3}t_2(s)\right]$$

where

$$t_2(F)\delta^{ab} = t_r(T^a(F)T^b(F))$$

$$t_2(S)\delta^{ab} = t_r(T^a(S)T^b(S))$$