Symmetry and Conservation Laws

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Symmetries and Conservation Laws

Fundamental Interactions

1. Strong Interaction–Quantum Chromodynamics (QCD)
   Local symmetry(Gauge Theory) based on $SU(3)$ color symmetry
2. Electromagnetic Interaction–Quantum Electrodynamics (QED)
   Local symmetry based on $U(1)$ symmetry
3. Weak interaction–
   Combine with QED to form Electroweak Theory
   Local symmetry based on $SU(2) \times U(1)$ symmetry
4. Gravity–Einstein's General Relativity
   Local symmetry–geneal coordinate transformtion

Symmetries play important roles in high energy physics.
Symmetry $\Longrightarrow$ conservation law

Symmetry transformations $\begin{cases} \text{transformations in space-time} \\ \text{Transformations in internal space} \end{cases}$

Conservation Laws–

- Provide relations between physically measurable quantities
- all come from experiments directly or indirectly
- Can be broken with more accurate measurements

1. **Exact Symmetry**
    1. Energy Conservation–time translation
    2. Momentum Conservation–spatial translation
    3. Electric Charge
    4. Baryon Number
2. **Approximate**–Valid only in some approximations
    1. Parity
    2. Charge Conjugation
    3. Lepton Number
    4. Isospin

Theoretical framework for symmetry—**group theory**

Relation between symmetry and conservation law

Example 1: Energy Conservation

For simplicity, take Newton's equation,

$$m\frac{d^2\vec{x}}{dt^2} = \vec{f}\left(\vec{x}, t\right)$$

Symmetry: if $\vec{f}\left(\vec{x}, t\right)$ is independent of t ,i.e. invariant under time translation and $\vec{f}\left(\vec{x}, t\right) = -\vec{\nabla}V\left(\vec{x}\right)$, then

$$m\frac{d^2\vec{x}}{dt^2}\cdot\frac{d\vec{x}}{dt} = -\vec{\nabla}V\left(\vec{x}, t\right)\cdot\frac{d\vec{x}}{dt} \quad \Longrightarrow \quad \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{d\vec{x}}{dt}\right)^2 + V\right] = 0$$

Thus sum of kinetic energy $\frac{1}{2}m\left(\frac{d\vec{x}}{dt}\right)^2$ and potential energy V is independent of time. This is the content of **energy conservation.**

Example 2 : Angular momentum conservation
Suppose $V\left(\vec{x}\right)$ is rotational invariant, $V\left(\vec{x}\right) = V\left(r\right)$. Newton's equation is then

$$m\frac{d^2\vec{x}}{dt^2} = -\vec{\nabla}V\left(r\right)$$

Or

$$\frac{d\vec{p}}{dt} = -\vec{\nabla}V\left(r\right) = -\frac{\partial V}{\partial r}\frac{\vec{r}}{r}$$

Take cross product with $\vec{r}$,

$$\vec{r}\times\frac{d\vec{p}}{dt} = -\frac{\partial V}{\partial r}\frac{\vec{r}\times\vec{r}}{r} = 0$$

On the other hand,

$$\vec{r}\times\frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\vec{r}\times\vec{p}\right) = 0$$

Thus the angular momentum $\vec{r}\times\vec{p}$ is conserved as a result of rotational invariance.
Example 3 : Momentum conservation

Supppose we have 2 particles interacting with each other

$$\frac{d\vec{p}_1}{dt} = -\vec{\nabla}_1 V\left(\vec{r}_1 - \vec{r}_2\right)$$

$$\frac{d\vec{p}_2}{dt} = -\vec{\nabla}_2 V\left(\vec{r}_1 - \vec{r}_2\right)$$

Then

$$\frac{d}{dt}\left(\vec{p}_1 + \vec{p}_2\right) = -\left(\vec{\nabla}_1 + \vec{\nabla}_2\right) V\left(\vec{r}_1 - \vec{r}_2\right) = 0$$

This implies that total momenta $\vec{p}_1 + \vec{p}_2$is conserved. This is a result of the translational invariance, $\vec{r}_1 \to \vec{r}_1 + \vec{a}$, $\vec{r}_2 \to \vec{r}_2 + \vec{a}$, where $\vec{a}$ is arbitrary.

Historical note
The e^- from nuclei $\beta-$decay,

$$(A, Z) \longrightarrow (A, Z+1) + e^-$$

was observed to have continuous energy spectrum. If basic mechanism for e^- emission were

$$n \longrightarrow p + e^-$$

the energy momentum conservation will require e^- to have a single energy.
Violation of energy momentum conservation ?
Pauli (1930) postulated the presence of **neutrino** which carries away energy and momentum in nuclear β -decay,

$$n \longrightarrow p + e^- + \bar{\nu}_e$$

so that the energy momentum conservations are saved.

Internal Symmetry-symmetry transformation in abstract space

Example: isospin symmetry

1932 Heisenberg: strong interaction seems to be the same for neutron and proton. In analogy with rotational invariance, symmetry transfomations (isospin) were introduced:

$$\begin{pmatrix} n(x) \\ p(x) \end{pmatrix} \rightarrow U \begin{pmatrix} n(x) \\ p(x) \end{pmatrix}, \qquad 2\times 2 \quad \text{unitay matrix indep of } x^{\mu}$$

and assume that strong inteaction is invariant under such transformation.

1935 Yukawa postulated $\pi^+\pi^0\pi^-$ are the mediator of strong interaction

1938 Kemmer introduced isospin triplet and extended to other particles

But these transformations are carried out in some abstract "isospin space" (internal space).

This symmetry can be described by $SU(2)$ *group* which is the same as the symmetry group for angular momentum in Quantum Mechanics.

Isospin symmetry: $m_p = m_n$.

Later this symmetry is extended to other hadrons,

$$(\pi^+, \pi^0, \pi^-) \quad I = 1, \qquad (K^+, K^0),\ (\bar{K^0}, K^-) \quad I = \frac{1}{2}, \qquad \eta, \quad I = 0$$

$$(\Sigma^+, \Sigma^0, \Sigma^-) \quad I = 1, \qquad (\Xi^0, \Xi^-), \quad I = \frac{1}{2}, \qquad \Lambda, \qquad I = 0$$

$$(\rho^+, \rho^0, \rho^-) \quad I = 1,\ (K^{+*}, K^{0*}),\ (\bar{K^{0}}*, K^{*-}) \quad I = \frac{1}{2} \quad \cdots$$

...

This symmetry is clearly not exact,

$$\frac{m_n - m_p}{m_n + m_p} \sim 0.7\times 10^{-3}, \qquad \frac{m_{\pi^+} - m_{\pi^0}}{m_{\pi^+} + m_{\pi^0}} \sim 1.7\times 10^{-2} \quad \cdots$$

Thus isospin symmetry as approximate one and maybe it is good to few %.

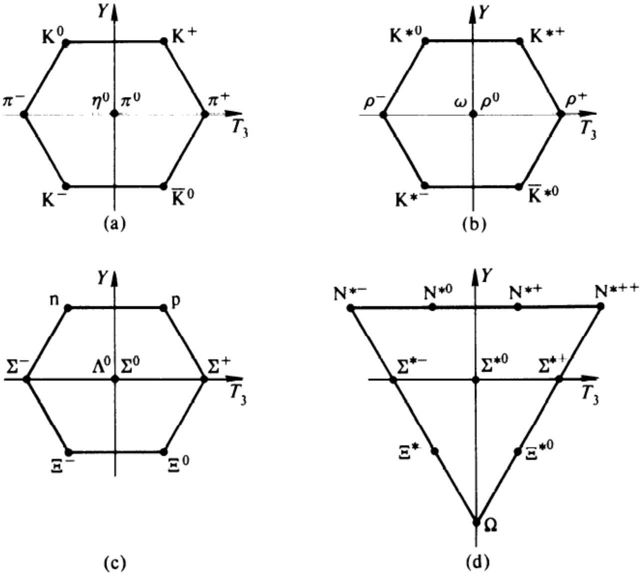
Any symmetry larger than the $SU(2)$ of isospin?
When Λ and k particles were discovered, they were produced in pair (associated production) with longer life time.
It was postulated that these new particles possessed a new additive quantum number, **strangeness** S , conserved by strong interaction but violated in decays,

$$S(\Lambda^0) = -1, \quad S(K^0) = 1 \quad \cdots$$

Extension to other hadrons, we can get a general relation,

$$Q = T_3 + \frac{Y}{2}$$

where $Y = B + S$ is called hyperchargee, and B is the baryon number. This is known as Gell-Mann-Nishijima relation.
Eight-fold way : Gell-Mann, Ne'eman 1961
Group togather mesons or baryons with same spin and parity,



- these particles can be related by $SU(3)$ transformations.

- If symmetry were exact $\Longrightarrow$ all these particles will have the same masses. In reality, their masses are close but not the same. This $SU(3)$ symmetry is not as good as isospin of $SU(2)$. This is known as the **eight-fold way**.

Quark Model

- One peculiar feature of eight fold way : octet and decuplet are not the smallest representation of $SU(3)$ group.
- In 1964, Gell-mann and Zweig independently propose the quark model: all hadrons are built out of spin $\frac{1}{2}$ quarks which transform as the fundamental representation of $SU(3)$,

$$q_i = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} u \\ d \\ s \end{pmatrix}$$

with quantum numbers

	Q	T	T_3	Y	S	B
u	$2/3$	$1/2$	$+1/2$	$1/3$	0	$1/3$
d	$-1/3$	$1/2$	$-1/2$	$1/3$	0	$1/3$
s	$-1/3$	0	0	$-2/3$	-1	$1/3$

In this scheme, mesons are $q\bar{q}$ bound states. For examples,

$$\pi^+ \sim \bar{d}u \quad \pi^0 \sim \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d). \quad \pi^- \sim \bar{u}d$$

$$K^+ \sim \bar{s}u \quad K^0 \sim \bar{s}d \;,\; K^- \sim \bar{u}s. \quad \eta^0 \sim \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s)$$

and baryons are qqq bound states,

$$\begin{aligned} p &\sim uud \ , \ n \sim ddu \\ \Sigma^+ &\sim suu \ , \ \Sigma^0 \sim s\left(\frac{ud+du}{\sqrt{2}}\right) \ , \ \Sigma^- \sim sdd \\ \Xi^0 &\sim ssu \ , \ \Xi^- \sim ssd \ , \ \Lambda^0 \sim \frac{s(ud-du)}{\sqrt{2}}. \end{aligned}$$

- Quantum numbers of hadrons are all carried by the quarks.
- We do not know the dynamics which bound the quarks into hadrons.
- Quarks have not been found.

Paradoxes of simple quark model

1. Quarks have fractional charges. At least one of the quarks is stable. None has been found.
2. Hadrons are exclusively made out $q\bar{q}, qqq$ bound states but $qq, qqqq$ states are absent.
3. The quark content of the baryon N^{*++} is uuu. For spin state in $\left|\frac{3}{2}, \frac{3}{2}\right\rangle$ then all quarks are in spin-up state˜ $\alpha_1\alpha_2\alpha_3$ is totally symmetric. If we assume that the ground state has $l = 0$, then spatical wave function is also symmetric. This will leads to violation of Pauli exclusion principle.

Color degree of freedom

To get around these problems, introduce color degrees for each quark and postulates that only color singlets are physical observables.
3 colors are needed to get antisymmetric wave function for N^{*++} . So each quark comes in 3 colors,

$$u_\alpha = (u_1, u_2, u_3) \quad , \quad d_\alpha = (d_1, d_2, d_3) \cdots$$

All hadrons form singlets under $SU(3)_{color}$ symmetry, e.g.

$$N^{*++} \sim u_\alpha(x_1)\alpha_\beta(x_2)u_\gamma(x_3)\varepsilon^{\alpha\beta\gamma}$$

Futhermore, color singlets can not be formed from the combination qq, $qqqq$ and they are absent from the observed specrum. Also a single quark is not observable.

Baryon number

Why proton is stable? $p \rightarrow e^+ + \gamma$ does not violate any physical law
Baryon number conservation was invented: $B(p) = 1$, $B(e^+) = 0$, $B(\gamma) = 0$,
In the universe at large, only baryons and no anti-baryons are observed
At beginning, maybe $B = 0$ for the universe as whole, because

$$\gamma + \gamma \rightleftarrows p + \overline{p}$$

To get $B \neq 0$ at present time, we need baryon number non-sonservation (Sakharov)
In Grand Unified Theory, it is possible to have the baryon decay,

$$p \rightarrow \pi^0 + e^+$$

Many experiments (IMB, Sudane, Kamiokonde...) search for this decay with null result,

$$\tau\left(p \rightarrow \pi^0 + e^+\right) > 10^{31} \quad \text{years}$$

Lepton number: ν from β-decay

$$n \longrightarrow p + e + \nu$$

ν from π decay are accompanied by μ

$$\pi^+ \longrightarrow \mu^+ + \nu$$

Are these 2 neutrinos the same ?
If they were the same then,

$$n \longrightarrow p + e + \nu$$

$$\nu + p \longrightarrow \mu^+ + n$$

However, only e^+ is observed in the final product and no μ^+. A simple explanation ν_e from β-decayis different from ν_μ in π-decay accompanied by μ and there is also muon number and electron number conservation

$$\begin{array}{ll} e^-, \nu_e & L_e = 1 \\ e^+, \overline{\nu}_e & L_e = -1 \end{array}$$

Similarly, for muon number L_μ

$$\begin{array}{ll} \mu^-, \nu_\mu & L_\mu = 1 \\ \mu^+, \overline{\nu}_\mu & L_\mu = -1 \end{array}$$

As a consequenc, the reaction $\mu^\pm \longrightarrow e^\pm + \gamma$ are forbidden and experimentally this is indeed the case. Lepton number conservations seem to hold up very well until neutrino oscillations have been observed recently,

$$\nu_e \leftrightarrow \nu_\mu$$

Parity violation

$\theta - \tau$ puzzle

In 1950's, it was observed that there are two decays

$$\theta \rightarrow \pi^+ + \pi^-, \qquad \text{(even parity)}$$

$$\tau \rightarrow \pi^+ + \pi^- + \pi^0, \qquad \text{(odd parity)}$$

while θ and τ have same mass, charge and spin. It is difficult to understand these if the parity is a good symmetry.

1956 : Lee and Yang proposed that parity is not conserved.

1957 : C. S. Wu showed that e^- in ^{60}Co decay has the property,

$$\left\langle \vec{\sigma} \cdot \vec{p} \right\rangle \neq 0, \qquad \vec{\sigma},\ \vec{p} \text{ spin and momentum of } e^-$$

This implies that the parity is violated in this decay.

Global symmetry

Global symmetry in Field Theory

Example 1: Self interacting scalar fields
Consider Lagrangian,

$$\mathcal{L} = \frac{1}{2}\left[\left(\partial_\mu \phi_1\right)^2 + \left(\partial_\mu \phi_2\right)^2\right] - \frac{\mu^2}{2}\left(\phi_1^2 + \phi_2^2\right) - \frac{\lambda}{4}\left(\phi_1^2 + \phi_2^2\right)^2$$

this is invariant under rotation in (ϕ_1, ϕ_2) plane, $O(2)$ symmetry,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

θ is independent of x^μ and is called **global** transformation.
Physical consequences:

1. Mass degenercy
2. Relation between coupling constants

Noether's currrent: for $\theta \ll 1$,

$$\delta\phi_1 = -\theta\phi_2, \qquad \delta\phi_2 = \theta\phi_1$$

and

$$J_\mu \sim \frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i = -\left[\left(\partial_\mu\phi_1\right)\phi_2 - \left(\partial_\mu\phi_2\right)\phi_1\right]$$

This current is conserved,

$$\partial_\mu J^\mu = 0$$

and conserved charge is

$$Q = \int d^3x J^0$$

and

$$\frac{dQ}{dt} = \int d^3x \frac{\partial J^0}{\partial t} = -\int d^3x \vec{\nabla}\cdot\vec{J} = -\int d\vec{S}\cdot\vec{J} = 0$$

Another way is to write

$$\phi = \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right)$$

and

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial_\mu\phi - \mu^2\phi^\dagger\phi - \lambda\left(\phi^\dagger\phi\right)^2$$

This is a phase transformation,

$$\phi \longrightarrow \phi' = e^{-i\theta}\phi$$

This is called the $U(1)$ symmetry. Charge conservation. is one such example. Approximate symmetries, e.g. lepton number, isospin, Baryon number,$\cdots$ are probably realized in the form of global symmtries.

Example 2 : Yukawa interaction–Scalar field interacting with fermion field
Lagrangian is of the form

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi + \frac{1}{2}\left(\partial_{\mu}\phi\right)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 + g\bar{\psi}\gamma_5\psi\phi$$

This Lagrangian is invariant under the $U(1)$ transformation,

$$\psi \rightarrow \psi' = e^{i\alpha}\psi, \qquad \phi \rightarrow \phi' = \phi$$

Here fermion number is conserved. Note that if there are two such fermions, ψ_1, ψ_2 with same transformation, then Yukawa interaction will be

$$\mathcal{L}_Y = g_1\bar{\psi}_1\gamma_5\psi_1\phi + g_2\bar{\psi}_2\gamma_5\psi_2\phi$$

Thus we have two independent couplings g_1, g_2.
Example 3 : Global non-abelian symmetry
Consider the case where ψ is a doublet and ϕ a singlet under $SU(2)$,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

and under $SU(2)$

$$\psi \rightarrow \psi' = \exp i\left(\frac{\vec{\tau}\cdot\vec{\alpha}}{2}\right)\psi, \qquad \phi \rightarrow \phi' = \phi$$

$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ are real parameters. The Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi + \frac{1}{2}\left(\partial_{\mu}\phi\right)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 + g\bar{\psi}\psi\phi$$

is $SU(2)$ invariant.
The Noether's currents are of the form,

$$\vec{J}^{\mu} = \bar{\psi}(\gamma^{\mu}\frac{\vec{\tau}}{2})\psi$$

and conserved charges are

$$Q^{i} = \int d^{3}x\ \psi^{\dagger}(\frac{\tau_{i}}{2})\psi$$

One can verify that

$$\left[Q^{i}, Q^{j}\right] = i\varepsilon^{ijk}Q^{k}$$

which is the $SU(2)$ algebra.

Local Symmetry

Local symmetry: transformation parameters, e.g. angle θ, depend on x^μ. This originates from electromagnetic theory.

Maxwell Equations:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}, \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0, \qquad \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \vec{J}$$

Introduce ϕ, $\vec{A}$ to solve those equations without source,

$$\vec{B} = \vec{\nabla} \times \vec{A}, \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

These are not unique because of **gauge** tranformation

$$\phi \longrightarrow \phi - \frac{\partial \alpha}{\partial t}, \qquad \vec{A} \longrightarrow \vec{A} + \vec{\nabla}\alpha$$

or

$$A_\mu \longrightarrow A_\mu - \partial_\mu \alpha$$

will give the same em fields

In quantum mechanics, Schrodinger equation for charged particle,

$$\left[\frac{1}{2m} \left(\frac{\hbar}{i} \vec{\nabla} - e\vec{A} \right)^2 - e\phi \right] \psi = i\hbar \frac{\partial \psi}{\partial t}$$

This requires transformation of wave function,

$$\psi \longrightarrow \exp\left(i\frac{e}{\hbar}\alpha\left(x\right)\right)\psi$$

to get same physics.
Thus gauge transformation is connected to **symmetry** (local) transformation.

1) **Abelian symmetry**

Consider Lagrangian with global $U(1)$ symmetry,

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

Suppose phase transformation depends on x^μ,

$$\phi \longrightarrow \phi' = e^{-ig\alpha(x)} \phi$$

The derivative transforms as

$$\partial^\mu \phi \longrightarrow \partial^\mu \phi' = e^{-i\alpha(x)} [\partial^\mu \phi - ig (\partial^\mu \alpha) \phi],$$

not a phase transformation.

Introduce gauge field A^μ, with transformation

$$A^\mu \longrightarrow A'^\mu = A^\mu - \partial^\mu \alpha$$

The combination

$$D^\mu \phi \equiv (\partial^\mu - igA^\mu) \phi, \qquad \text{covariant derivative}$$

will be transformed by a phase,

$$D^\mu \phi' = e^{-ig\alpha(x)} (D^\mu \phi)$$

and the combination

$$D_\mu \phi^\dagger D^\mu \phi$$

is invarianat under local phase transformation.

Define anti-symmetric tensor for the gauge field

$$(D_\mu D_\nu - D_\nu D_\mu) \phi = gF_{\mu\nu} \phi, \qquad \text{with} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

We can use the property of the covariant derivative to show that

$$F'_{\mu\nu} = F_{\mu\nu}$$

Complete Lagragian is

$$\mathcal{L} = D_\mu \phi^\dagger D^\mu \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\phi)$$

where $V(\phi)$ does not depend on derivative of ϕ.

- mass term $A^\mu A_\mu$ is not gauge invariant $\Rightarrow$ massless particle$\Rightarrow$long range force
- coupling of gauge field to other field is universal

2) Non-Abelian symmetry-Yang Mills fields

1954: Yang-Mills generalized U(1) local symmetry to SU(2) local symmetry.

Consider an isospin doublet $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

Under SU(2) transformation

$$\psi(x) \rightarrow \psi'(x) = exp\{-\frac{i\vec{\tau}\cdot\vec{\theta}}{2}\}\psi(x)$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;,\; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with

$$[\frac{\tau_i}{2}, \frac{\tau_j}{2}] = i\epsilon_{ijk}(\frac{\tau_k}{2})$$

Start from free Lagrangian

$$\mathcal{L}_0 = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi$$

which is invariant under global $SU(2)$ transformation where $\vec{\theta} = (\theta_1, \theta_2, \theta_3)$ are indep of x_μ.

For local symmetry transformation, write

$$\psi(x) \rightarrow \psi'(x) = U(\theta)\psi(x) \qquad U(\theta) = exp\{-\frac{i\vec{\tau}\cdot\theta(\vec{x})}{2}\}$$

Derivative term

$$\partial_\mu\psi(x) \rightarrow \partial_\mu\psi'(x) = U\partial_\mu\psi + (\partial_\mu U)\psi$$

is not invariant. Introduce gauge fields $\vec{A}_\mu$ to form the covariant derivative,

$$D_\mu\psi(x) \equiv (\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2})\psi$$

Require that

$$[D_\mu\psi]' = U[D_\mu\psi]$$

Or

$$(\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu{}'}{2})(U\psi) = U(\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2})\psi$$

This gives the transformation of gauge field,

$$\boxed{\frac{\vec{\tau}\cdot\vec{A}_\mu{}'}{2} = U(\frac{\vec{\tau}\cdot\vec{A}_\mu}{2})U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}}$$

We use covariant derivatives to construct field tensor

$$D_\mu D_\nu\psi = (\partial_\mu - ig\frac{\vec{\tau}\cdot\vec{A}_\mu}{2})(\partial_\nu - ig\frac{\vec{\tau}\cdot\vec{A}_\nu}{2})\psi = \partial_\mu\partial_\nu\psi - ig(\frac{\vec{\tau}\cdot\vec{A}_\mu}{2}\partial_\nu\psi + \frac{\vec{\tau}\cdot\vec{A}_\nu}{2}\partial_\mu\psi)$$

$$-ig\partial_\mu(\frac{\vec{\tau}\cdot\vec{A}_\nu}{2})\psi + (-ig)^2(\frac{\vec{\tau}\cdot\vec{A}_\mu}{2})(\frac{\vec{\tau}\cdot\vec{A}_\nu}{2})\psi$$

Antisymmetrize this to get the field tensor,

$$(D_\mu D_\nu - D_\nu D_\mu)\psi \equiv ig(\frac{\vec{\tau}\cdot\vec{F_{\mu\nu}}}{2})\psi$$

then

$$\frac{\vec{\tau}\cdot\vec{F_{\mu\nu}}}{2} = \frac{\vec{\tau}}{2}\cdot(\partial_\mu \vec{A_\nu} - \partial_\nu \vec{A_\mu}) - ig[\frac{\vec{\tau}\cdot\vec{A_\mu}}{2}, \frac{\vec{\tau}\cdot\vec{A_\nu}}{2}]$$

Or in terms of components,

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g\epsilon^{ijk} A^i_\mu A^k_\nu$$

The the term quadratic in A is new in Non-Abelian symmetry. Under the gauge transformation we have

$$\vec{\tau}\cdot\vec{F_\mu}\nu' = U(\vec{\tau}\cdot\vec{F_\mu}\nu)U^{-1}$$

Infinitesmal transformation $\theta(x) \ll 1$

$$A^{\prime\mu} = A^{\mu} + \epsilon^{ijk}\theta^{j}A^{k}_{\mu} - \frac{1}{g}\partial_{\mu}\theta^{i}$$

$$F^{\prime i}_{\mu\nu} = F^{i}_{\mu\nu} + \epsilon^{ijk}\theta^{j}F^{k}_{\mu\nu}$$

Remarks

1. Again $A^{a}_{\mu}A^{a\mu}$ is not gauge invariant⇒gauge boson massless⇒long range force
2. A^{a}_{μ} carries the symmetry charge (e.g. color —)
3. The quadratic term in $F^{a\mu\nu} \sim \partial A - \partial A + gAA$ is for asymptotic freedom.

Recipe for the construction of theory with local symmetry

1. Write down a Lagrangian with local symmetry
2. Replace the usual derivative $\partial_{\mu}\phi$ by the covariant derivative $D_{\mu}\phi \sim \left(\partial_{\mu} - igA^{a}_{\mu}t^{a}\right)\phi$ where guage fields A^{a}_{μ} have been introduced.
3. Use the antisymmetric combination $\left(D_{\mu}D_{\nu} - D_{\nu}D_{\mu}\right)\phi \sim F^{a}_{\mu\nu}\phi$ to construct the field tensor $F^{a}_{\mu\nu}$ and add $-\frac{1}{4}F^{a}_{\mu\nu}F^{a\mu\nu}$ to the Lagrangian density

Spontaneous Symmetry Breaking

Usually symmetry of Lagrangin or Hamiltonian $\Longrightarrow$ physicial states degenercy.
Spontaneous symmetry breaking (SSB): the symm of interaction > symm of spectrum.

$\Longrightarrow$ massless excitation, called the Nambu-Goldstone boson,

1964 Higgs and others : in local symmetry, SSB convert the long range force in gauge theory into a short range force.
1967 Weinberg construct a model of electromagnetic and weak interactions.
t' Hooft : 1971 it is renomalizable and all the higher order effects are calculable

Example,

Example: ferromagnetism near Curie tempeture T_C.
Landau-Ginzberg's mean field theory
Write free energy density ,

$$u\left(\vec{M}\right) = \left(\partial_t \vec{M}\right)^2 + V\left(\vec{M}\right)$$

where

$$V\left(\vec{M}\right) = \alpha_1\left(T\right)\left(\vec{M}\cdot\vec{M}\right) + \alpha_2\left(\vec{M}\cdot\vec{M}\right)^2$$

u and V rotationally invariant. assume

$$\alpha_1\left(T\right) = \alpha\left(T - T_C\right) \qquad \text{with } \alpha > 0$$

minimize $V\left(\vec{M}\right)$,

$$\frac{\partial V}{\partial M_i} = 0 \qquad \Longrightarrow \qquad M_i\left(\alpha_1 + 2\alpha_2\vec{M}\cdot\vec{M}\right) = 0$$

For $T > T_C$ (i.e. $\alpha_1 > 0$), the solution is at $M_i = 0$. For $T < T_C$ (i.e. $\alpha_1 < 0$), the minimum is at

$$\left|\vec{M}\right| = \sqrt{-\frac{\alpha_1}{2\alpha_2}}$$

direction can be arbitrary. rotational symmetry spontaneously broken.

Nambu-Goldstone theorem

Recall that Noether's theorem says that a continuous symmetry will give conserved charge Q. Suppose there are 2 local operators A, B with property

$$[Q, B] = A \qquad Q = \int d^3x \; j_0(x) \quad \textit{indep of time}$$

Suppose $\langle 0|A|0\rangle = v \neq 0$ (symmetry breaking condition)

$$\Rightarrow 0 \neq \langle 0|[Q, B]|0\rangle = \int d^3 \times \langle 0|[j_0(x), B]|0\rangle$$

$$= \sum_n (2\pi)^3 \delta^3(\vec{P}_n)\{\langle 0|j_0(0)|n\rangle\langle n|B|0\rangle e^{-iE_n t} - \langle 0|B|n\rangle\langle n|j_0(0)|0\rangle e^{-iE_n t}\} = v$$

Since $v \neq 0$ and time-independent, we need a state such that

$$E_n \to 0 \qquad \textit{for} \quad \vec{P}_n = 0$$

massless excitation. For the case of relativistic particle with energy momentum rotation $E = \sqrt{\vec{P}^2 + m^2}$ this implies massless particle- Goldstone boson.

Spontaneous Symmetry Breaking

Global symmetry

Suppose

$$L = \frac{1}{2}\left[\left(\partial_\mu \sigma\right)^2 + \left(\partial_\mu \phi\right)^2\right] - V\left(\sigma^2 + \pi^2\right)$$

with

$$V\left(\sigma^2 + \pi^2\right) = -\frac{\mu^2}{2}\left(\sigma^2 + \pi^2\right) + \frac{\lambda}{4}\left(\sigma^2 + \pi^2\right)^2$$

This is invariant under $O(2)$ rotation

$$\begin{pmatrix} \sigma \\ \pi \end{pmatrix} \longrightarrow \begin{pmatrix} \sigma' \\ \pi' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \sigma \\ \pi \end{pmatrix}$$

rotation angle α independent of spacetime, global transformation. Minimize the potential energy V,

$$\frac{\partial V}{\partial \sigma} = \sigma\left[-\mu^2 + \lambda\left(\sigma^2 + \pi^2\right)\right] = 0$$

$$\frac{\partial V}{\partial \pi} = \pi\left[-\mu^2 + \lambda\left(\sigma^2 + \pi^2\right)\right] = 0$$

For $\mu^2 > 0$, the minimum at

$$\sigma^2 + \pi^2 = v^2, \qquad \text{with} \quad v^2 = \frac{\mu^2}{\lambda}$$

minima is at circle with radius v in the (σ, π) plane.Pick for example,

$$\langle 0 \left|\sigma\right| 0\rangle = \nu, \qquad \langle 0 \left|\pi\right| 0\rangle = 0$$

$O(2)$ symmetry is broken by the vacuum state.
Consider small oscillations around true minimum and define a shifted field

$$\sigma' = \sigma - v$$

Lagrangian density

$$L = \frac{1}{2}\left[\left(\partial_\mu \sigma'\right)^2 + \left(\partial_\mu \phi\right)^2\right] - \mu^2 \sigma'^2 - \lambda v \sigma' \left(\sigma'^2 + \pi^2\right) - \frac{\lambda}{4}\left(\sigma'^2 + \pi^2\right)^2$$

no quadratic term in $\pi-$field and π is the massless **Goldstone boson**.
massless particle $\Longrightarrow$ long range force .

Local Symmetry–Higgs phenomena

Consider local $U(1)$ symmetry

$$L = (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where

$$D_\mu \phi = (\partial_\mu - igA_\mu)\phi, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Local transformation

$$\phi(x) \longrightarrow \phi'(x) = e^{-i\alpha}\phi(x)$$
$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) - \partial_\mu \alpha(x)$$

When $\mu^2 > 0$, minimum of potential

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

at

$$\phi^\dagger \phi = \frac{v^2}{2}, \qquad \text{with} \qquad v^2 = \frac{\mu^2}{\lambda}$$

Thus ϕ has a vacuum expectation value

$$|\langle 0 |\phi| 0\rangle| = \frac{v}{\sqrt{2}}$$

write ϕ as,

$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2)$$

choose

$$\langle 0 |\phi_1| 0\rangle = v, \qquad \langle 0 |\phi_2| 0\rangle = 0$$

define the shifted fields as

$$\phi'_1 = \phi_1 - v, \qquad \phi'_2 = \phi_2$$

ϕ'_2 Goldstone boson.

New feature: covariant derivative term produce mass term for gauge boson,

$$\left|D_\mu \phi\right|^2 = \left|\left(\partial_\mu - igA_\mu\right)\phi\right|^2 \longrightarrow \frac{g^2 v^2}{2} A^\mu A_\mu + \cdots \tag{1}$$

guage boson mass

$$M = gv$$

write scalar field as

$$\phi\left(x\right) = \frac{1}{\sqrt{2}}\left[v + \eta\left(x\right)\right] e^{i\xi(x)/v}$$

use gauge transformation to transform away ξ.

$$\phi^{''} = \exp\left(-i\xi/v\right)\phi = \frac{1}{\sqrt{2}}\left[v + \eta\left(x\right)\right] \tag{2}$$

and

$$B_\mu = A_\mu - \frac{1}{gv}\partial_\mu \xi \tag{3}$$

massless gauge boson+Goldstone boson= massive gaue boson

Milestones of Weak Interaction

1. **Neutrino and Nuclear β decay**,
   The e^- from nuclei decay,
   $$(A, Z) \longrightarrow (A, Z+1) + e^-$$
   was observed to have continuous energy spectrum. If basic mechanism for e^- emission were
   $$n \longrightarrow p + e^-$$
   the energy momentum conservation will require e^- to hav a single energy. Pauli (1930) postulated the presence of **neutrino** which carries away energy and momentum in nuclear β -decay,
   $$n \longrightarrow p + e^- + \bar{\nu}_e$$

2. Parity violation and V - A theory
   $\theta - \tau$ puzzle
   In 1950's, it was observed that there are two decays
   $$\theta \longrightarrow \pi^+ + \pi^-, \qquad \text{(even parity)}$$
   $$\tau \longrightarrow \pi^+ + \pi^- + \pi^0, \qquad \text{(odd parity)}$$
   while θ and τ have same mass, charge and spin. It is difficult to understand these if the parity is a good symmetry.
   1956 : Lee and Yang proposed that parity is not conserved.
   1957 : C. S. Wu showed that e^- in ^{60}Co decay has the property,
   $$\left\langle \vec{\sigma} \cdot \vec{p} \right\rangle \neq 0, \qquad \vec{\sigma},\ \vec{p} \text{ spin and momentum of } e^-$$
   This implies that the parity is violated in this decay.

3. **Two neutrino experiments**: ν from β-decay and ν from π decay are different
If they were the same then,

$$n \longrightarrow p + e + \nu$$

$$\nu + p \longrightarrow \mu^+ + n$$

However, only e^+ is observed in the final product and no μ^+. A simple explanation ν_e from β-decayis different from ν_μ in π-decay accompanied by μ and there is also muon number and electron number conservation

$$\begin{array}{ll} e^-, \nu_e & L_e = 1 \\ e^+, \overline{\nu}_e & L_e = -1 \end{array}$$

Similarly, for the muon number L_μ

$$\begin{array}{ll} \mu^-, \nu_\mu & L_\mu = 1 \\ \mu^+, \overline{\nu}_\mu & L_\mu = -1 \end{array}$$

Then reaction $\mu^\pm \longrightarrow e^\pm + \gamma$ are forbidden and experimentally this is indeed the case. Lepton number conservations seem to hold up very well until neutrino oscillations have been observed recently,

$$\nu_e \leftrightarrow \nu_\mu$$

4. V-A theory (1958 Feynman and Gell-Mann, Sudarshan and Marshak, Sakurai)
As a result of parity violation, weak interaction was casted in term of $V - A$ currents $(left - handed)$,

$$L_{eff} = \frac{G_F}{\sqrt{2}} J^\dagger_\mu J^\mu + h.c.$$

Intermidate Boson Theory(IVB)

In analogy with QED, introduce vector boson W to couple to the V-A current

$$\mathcal{L}_W = g(J_\mu W^\mu + h.c.)$$

Since weak interaction is short range, we need $M_W \neq 0$. Use W-boson propagator in the form

$$\frac{-g^{\mu\nu} + \frac{k^\mu k^\nu}{M_W^2}}{k^2 - M_W^2} \rightarrow \frac{g^{\mu\nu}}{M_W^2} \quad \text{when} \quad |k_\mu| \ll M_W$$

This reproduces 4-fermion interaction with $\frac{g^2}{M_W^2} = \frac{G_F}{\sqrt{2}}$

Standard Model of Electroweak Interaction

- weak interaction is mediated by **massive** vector mesons.
- universality of weak couplings $\Longrightarrow$ local symmetries.
- weak current is left-handed
  spontaneous symmetry breaking in gauge theory has both universality and massive vector mesons.

The gauge group is $SU(2) \times U(1)$ with gauge bosons $\vec{A}_\mu$ and B_μ .
scalar fields is $SU(2)$ doublet with hypercharge $Y = 1$,

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \qquad Y = 1$$

$$V(\phi) = -\mu^2 \left(\phi^\dagger \phi\right) + \lambda \left(\phi^\dagger \phi\right)^2$$

Spontaneous Symmetry Breaking

$$\langle \phi \rangle_0 \equiv \langle 0 | \phi | 0 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \qquad v = \sqrt{\frac{\mu^2}{\lambda}}$$

Write the scalar fields

$$\phi(x) = U^{-1}\left(\vec{\xi}\right) \begin{pmatrix} 0 \\ \frac{v + H(x)}{\sqrt{2}} \end{pmatrix}, \qquad \text{with} \quad U\left(\vec{\xi}\right) = \exp\left[\frac{i\vec{\xi}(x) \cdot \vec{\tau}}{v}\right]$$

where $\vec{\xi}(x)$ Goldstone bosons. Use the gauge transformation to remove $\vec{\xi}(x)$

$$\phi' = U\left(\vec{\xi}\right)\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + H(x) \end{pmatrix}$$

Then $\vec{\xi}(x)$ disappear, left-over field $H(x)$, usually called **Higgs field**.

Fermions

ψ_L are all in $SU(2)$ doublets and ψ_R are all $SU(2)$ singlets $\bar{\psi}_L \psi_R + h.c.$ is not $SU(2)$ invariant$\Longrightarrow$ no bare mass terms

However fermions couple to scalar fields ϕ through Yukawa couplings . ,

$$L_Y = f\left(\bar{\psi}_L \phi \psi_R + h.c.\right)$$

After spontaneous symmetry brreaking

$$L_Y = \left(m\bar{\psi}\psi + \frac{m}{v} H(x)\bar{\psi}\psi\right) \tag{4}$$

Higgs Physics

top priority at LHC is to look for Higgs particle.

1. Higgs coupling to fermion is proportional to fermion mass
2. Higgs coupling to gauge boson is also proportional to gauge boson mass,

$$L_{HVV} = gH(x)\left[M_W W_\mu^+ W + \frac{1}{2\cos\theta_W} M_Z Z^\mu Z_\mu\right]$$

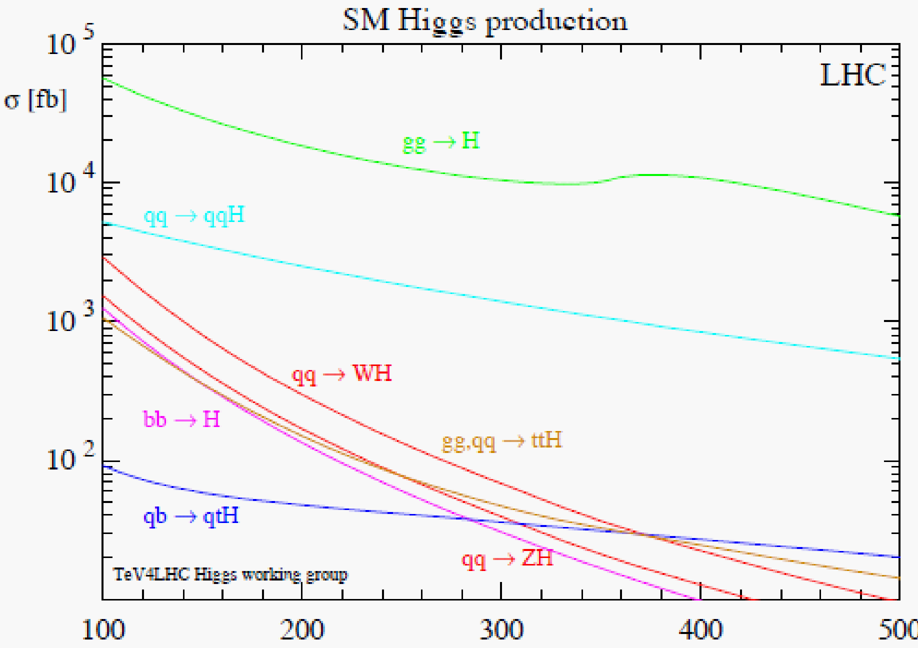
Mass of Higgs particle can be written as

$$m_H = \sqrt{2\mu^2} = \sqrt{2\lambda}v,$$

where $v = 246$ Gev is related to Fermi coupling constant G_F by

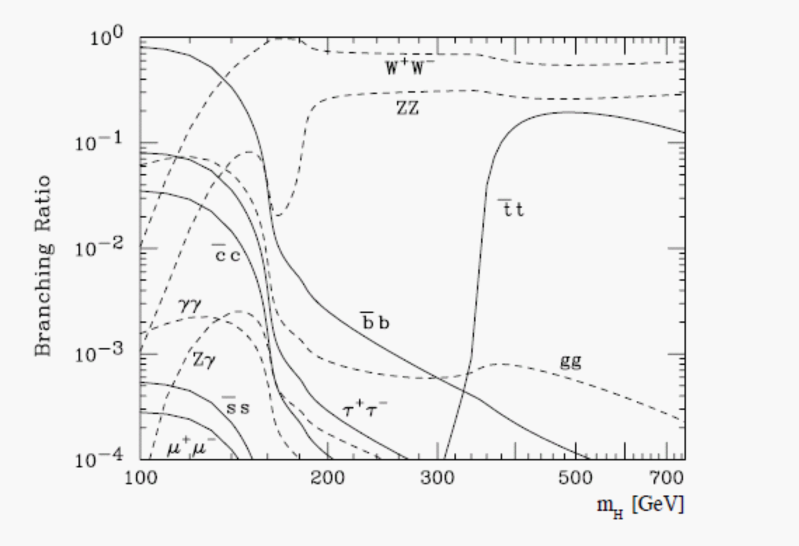
$$v = \sqrt{\frac{\sqrt{2}}{G_F}}$$

Production of Higgs Paticle


SM Higgs production
σ [fb]
10 5
10 4
10 3
10 2
LHC
gg → H
qq → qqH
qq → WH
bb → H
gg,qq → ttH
qb → qtH
qq → ZH
TeV4LHC Higgs working group
100
200
300
400
500

Higgs Decay

relative importance of each decay mode,



- decays of Higgs into WW or ZZ dominate .
- below the WW threshold $H \longrightarrow b\bar{b}$ dominates.
- decay $H \longrightarrow \gamma\gamma$ is of special interest due to their relatively clean experimenal signaure.