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# Advanced Quantum Field Theory

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Taller de Altas Energías 2013, Benasque

CENTRO DE CIENCIAS
DE BENASQUE
PEDRO PASCUAL

# Introduction to anomalies and their phenomenological applications

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# Plan of the course

- First hour: anomalies

  * Classical and quantum symmetries. Anomalies
  * The axial anomaly: a case study.
  * Gauge anomalies and their cancellation.

- Second hour: phenomenological applications

  * The phenomenology of the axial anomaly.
  * Nonperturbative physics from anomalies.
  * Anomaly cancellation and model building.

- Afternoon session: tutorials (see exercise sheet)

# Bibliography (a sample)

## Monographs:


* R.A. Bertlmann, "Anomalies in Quantum Field Theory", Oxford 1996
* K. Fujikawa & H. Suzuki, "Path integrals and Quantum Anomalies", Oxford 2004
* L. Álvarez-Gaumé & M.A. Vázquez-Mozo, "Introduction to Anomalies", Springer (to appear)


## QFT books:


* M.E. Peskin & D.V. Schroeder, "An Introduction to Quantum Field Theory", Perseus Books 1995 (Chapter 19)
* L. Álvarez-Gaumé & M.A. Vázquez-Mozo, "An Invitation to Quantum Field Theory", Springer 2012 (Chapter 9)


## Online Reviews:


* J.A. Harvey, "TASI Lectures on Anomalies", hep-th/0509097

# Part I

# Anomalies: what they are and why we should care about them

# Classical and Quantum Symmetries

Emmy Noether
(1882-1935)

Classically, continuous symmetries are associated with conserved quantities via Noether's theorem. Given a theory with action $S[\phi_i]$ invariant under global transformations

$$\delta_\xi \phi_i(x) = \sum_j \xi_j F_{ij}(\phi_k)$$

We use "Noether's trick": let us make the parameters $\xi_i$ depend on the position $x$. If these functions decrease fast enough at infinity, the variation of the action now reads

$$\begin{aligned} S[\phi_i + \delta_\xi \phi_i] &= S[\phi_i] + \sum_i \int d^4x\, \partial_\mu \xi_i(x)\, j_i^\mu(x) \\ &= S[\phi_i] - \sum_i \int d^4x\, \xi_i(x) \partial_\mu j_i^\mu(x) \end{aligned}$$

If all fields are ***on-shell***, the action is invariant under ***any*** variation of the fields, hence

$$\int d^4x\, \xi_i(x) \partial_\mu j_i^\mu(x) = 0 \quad \Longrightarrow \quad \partial_\mu j_i^\mu(x) = 0$$

Let's move to the ***quantum theory***. Consider the correlation function of a number of operators

$$\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle=\frac{1}{Z}\int\left(\prod_i\mathscr{D}\phi_i\right)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\,e^{iS[\phi_i]}$$

and carry out a ***change of variables*** in the path integral in the right-hand side

$$\phi_i(x)\longrightarrow\phi_i'(x)=\phi_i(x)+\sum_j\xi_j(x)F_{ij}(\phi_k)$$

The variation of the action is given by

$$S[\phi_i+\delta_\xi\phi_i]=S[\phi_i]-\sum_i\int d^4x\,\xi_i(x)\partial_\mu j_i^\mu(x)$$

while the first-order variation of the operators $\mathscr{O}_a(x)$ are

$$\mathscr{O}_a(x)\longrightarrow\mathscr{O}_a'(x)=\mathscr{O}_a(x)+\delta_\xi\mathscr{O}_a(x)$$

Let us assume, besides, that ***the integration measure is invariant*** under these variations

$$\prod_i \mathscr{D}\phi_i' = \prod_i \mathscr{D}\phi_i$$

Collecting the first-order variation of the path integral and setting it to zero (it is just a change of variables!),

$$\frac{1}{Z}\int\left(\prod_i \mathscr{D}\phi_i'\right)\mathscr{O}_1'(x_1)\ldots\mathscr{O}_n'(x_n)\,e^{iS[\phi_i']}\bigg|_{\xi} = 0$$

we arrive at the ***Ward identity*** (for the time being, we have restored $\hbar$)

$$\frac{i}{\hbar}\int d^4x\,\xi_j(x)\partial_\mu^{(x)}\langle\Omega|T\big[j_j^\mu(x)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\big]|\Omega\rangle = \sum_{a=1}^{n}\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle$$

Particularizing this identity to the case $\mathscr{O}_a(x) = 1$ we find

$$\int d^4x\,\xi_i(x)\partial_\mu\langle j_i^\mu(x)\rangle = 0 \qquad \Longrightarrow \qquad \partial_\mu\langle j_i^\mu(x)\rangle = 0$$

The Noether current is ***conserved quantum mechanically***!

We can relax the condition of invariance of the measure and assume that there is a nontrivial Jacobian

$$\prod_i \mathscr{D}\phi_i' = \left[1+\sum_k \int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)\right]\prod_i \mathscr{D}\phi_i$$

This introduces an extra term in the Ward identity

$$\begin{aligned}\frac{i}{\hbar}\int d^4x\,\xi_j(x)\partial_\mu^{(x)}\langle\Omega|T[j_j^\mu(x)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle &= \sum_{a=1}^{n}\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle \\ &\quad -\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)\end{aligned}$$

that ***spoils*** the quantum mechanical conservation of the Noether currents

$$\int d^4x\,\xi_k(x)\partial_\mu\langle j_k^\mu(x)\rangle = i\hbar\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x) \quad\Longrightarrow\quad \partial_\mu\langle j_k^\mu(x)\rangle = i\hbar\,\mathscr{J}_k(x)$$

Whenever this happens, we say that the symmetry in question is ***anomalous*** or that the theory has an ***anomaly***.

We can relax the condition of invariance of the measure and assume that there is a nontrivial Jacobian

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This introduces an extra term in the Ward identity

$$\frac{i}{\hbar}\int d^4x\,\xi_j(x)\partial_\mu^{(x)}\langle\Omega|T[j_j^\mu(x)\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle = \sum_{a=1}^{n}\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\delta_\xi\mathscr{O}_a(x_a)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle$$

$$-\langle\Omega|T\Big[\mathscr{O}_1(x_1)\ldots\mathscr{O}_n(x_n)\Big]|\Omega\rangle\sum_k\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x)$$

that ***spoils*** the quantum mechanical conservation of the Noether currents

quantum effect

$$\int d^4x\,\xi_k(x)\partial_\mu\langle j_k^\mu(x)\rangle = i\hbar\int d^4x\,\xi_k(x)\,\mathscr{J}_k(x) \quad\Longrightarrow\quad \partial_\mu\langle j_k^\mu(x)\rangle = i\hbar\,\mathscr{J}_k(x)$$

Whenever this happens, we say that the symmetry in question is ***anomalous*** or that the theory has an ***anomaly***.

An ***anomaly*** is the ***quantum breakdown*** of a ***classical symmetry***.

Anomalies can be of two very different types:

* When they affect a ***nonfundamental symmetries***, e.g.

  - Scale invariance
  - Global symmetries

These anomalies are at the origin of very interesting physical phenomena:

asymptotic freedom, $\pi^0 \longrightarrow 2\gamma$ ,...

* When they affect ***local (gauge) symmetries***

  - Gauge anomalies
  - Gravitational anomalies

These are very dangerous anomalies that have to be ***cancelled*** somehow, otherwise the whole theory becomes ***inconsistent***.

# First example: Scale invariance

Let us consider a massless φ$^4$ theory which is invariant under scale transformations

$$S = \int d^4x \left( \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{g}{4!}\phi^4 \right)$$

$$x^\mu \longrightarrow x'^\mu = \lambda x^\mu$$

$$\phi(x) \longrightarrow \phi'(x) = \lambda^{-\Delta}\phi(\lambda^{-1}x) \quad \text{(with } \Delta = 1\text{)}$$

This invariance is broken by ***quantum corrections***. Regularization and renormalization requires the introduction of an energy scale that breaks scale invariance

This is reflected in the running of the coupling constant. At one loop:

$$\beta(g) = \frac{3g^2}{16\pi^2}$$

$$g(\mu) = \frac{g(\mu_0)}{1 - \frac{3}{16\pi^3} g(\mu_0) \log\left(\frac{\mu}{\mu_0}\right)}$$

so physics at different scales "does not look the same".

In ***QCD*** this quantum breaking of scale invariance is responsible for the most interesting features of the theory, such as ***asymptotic freedom*** and ***confinement***.

# Second example: The axial anomaly

Let us focus on QED:

$$S_{\mathrm{QED}} = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \overline{\psi}(i\partial\!\!\!/ - m)\psi - e\overline{\psi}A\!\!\!/\psi \right]$$

The theory has a $\mathrm{U}(1)$ ***gauge symmetry***

$$\psi(x) \longrightarrow e^{i\alpha(x)}\psi(x)$$

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \alpha(x) \qquad \alpha(x) \in \mathbb{R}$$

with a conserved vector current

$$J^\mu_{\mathrm{V}} = \overline{\psi}\gamma^\mu\psi \qquad \Longrightarrow \qquad \partial_\mu J^\mu_{\mathrm{V}} = 0$$

This invariance is crucial for the internal consistency of the theory (e.g. unitarity).

In addition, the ***massless*** theory has a ***global*** axial-vector symmetry

$$\psi(x) \longrightarrow e^{i\beta\gamma_5}\psi(x) \qquad\qquad \beta \in \mathbb{R}.$$

The associated conserved axial-vector current is

$$J^{\mu}_{\mathrm{A}} = \overline{\psi}\gamma^{\mu}\gamma_5\psi \qquad \Longrightarrow \qquad \partial_\mu J^{\mu}_{\mathrm{A}} = 0$$

In the quantum theory, both the axial and the vector-axial current are ***composite operators*** that need to be ***defined***.

The question is whether these operators can be defined to satisfy the ***quantum conservation equations***

$$\partial_\mu \langle J^{\mu}_{V}(x)\rangle \stackrel{?}{=} 0$$

$$\partial_\mu \langle J^{\mu}_{A}(x)\rangle \stackrel{?}{=} 0$$

To analyze this problem, we look at a massless Dirac fermion coupled to an ***classical external*** $\mathrm{U}(1)$ ***gauge field***

$$S = \int d^4x \left( i\overline{\psi}\partial\!\!\!/\psi - eJ^\mu_{\rm V}\mathscr{A}^\mu \right) \qquad \text{(remember that } J^\mu_{\rm V} = \overline{\psi}\gamma^\mu\psi\text{)}$$

The ***expectation value*** of the axial current in this background is given by

$$\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} = \frac{\int \mathscr{D}\psi\mathscr{D}\overline{\psi}\, J^\mu_{\rm A}(x) e^{i\int d^4x\left(i\overline{\psi}\partial\!\!\!/\psi - eJ^\mu_{\rm V}\mathscr{A}_\mu\right)}}{\int \mathscr{D}\psi\mathscr{D}\overline{\psi}\, e^{i\int d^4x\left(i\overline{\psi}\partial\!\!\!/\psi - eJ^\mu_{\rm V}\mathscr{A}_\mu\right)}}$$

This correlation function can be computed in perturbation theory

$$\begin{aligned}\langle J^\mu_{\rm A}(x)\rangle_{\mathscr{A}} &= -ie\int d^4y\langle 0|T[J^\mu_{\rm A}(x)J^\alpha_{\rm V}(y)]|0\rangle\mathscr{A}_\alpha(y) \\ &- \frac{e^2}{2}\int d^4y_1 d^4y_2\,\langle 0|T[J^\mu_{\rm A}(x)J^\alpha_{\rm V}(y_1)J^\beta_{\rm V}(y_2)]|0\rangle\mathscr{A}_\alpha(y_1)\mathscr{A}_\beta(y_2) + \ldots\end{aligned}$$

To analyze this problem, we look at a massless Dirac fermion coupled to an ***classical external*** $\mathrm{U}(1)$ ***gauge field***

$$S=\int d^4x\left(i\overline{\psi}\partial\!\!\!/\psi - eJ^{\mu}_{\mathrm{V}}\mathscr{A}^{\mu}\right)$$

(remember that $J^{\mu}_{\mathrm{V}}=\overline{\psi}\gamma^{\mu}\psi$)

The ***expectation value*** of the axial current in this background is given by

$$\langle J^{\mu}_{\mathrm{A}}(x)\rangle_{\mathscr{A}}=\frac{\displaystyle\int\mathscr{D}\psi\mathscr{D}\overline{\psi}\,J^{\mu}_{\mathrm{A}}(x)e^{i\int d^4x\left(i\overline{\psi}\partial\!\!\!/\psi-eJ^{\mu}_{\mathrm{V}}\mathscr{A}_{\mu}\right)}}{\displaystyle\int\mathscr{D}\psi\mathscr{D}\overline{\psi}\,e^{i\int d^4x\left(i\overline{\psi}\partial\!\!\!/\psi-eJ^{\mu}_{\mathrm{V}}\mathscr{A}_{\mu}\right)}}$$

This correlation function can be computed in perturbation theory

$$\begin{aligned}\langle J^{\mu}_{\mathrm{A}}(x)\rangle_{\mathscr{A}} &= \cancelto{0}{-ie\int d^4y\langle 0|T[J^{\mu}_{\mathrm{A}}(x)J^{\alpha}_{\mathrm{V}}(y)]|0\rangle\mathscr{A}_{\alpha}(y)} \\ &- \frac{e^2}{2}\int d^4y_1d^4y_2\,\langle 0|T[J^{\mu}_{\mathrm{A}}(x)J^{\alpha}_{\mathrm{V}}(y_1)J^{\beta}_{\mathrm{V}}(y_2)]|0\rangle\mathscr{A}_{\alpha}(y_1)\mathscr{A}_{\beta}(y_2)+\ldots\end{aligned}$$

We are faced with the calculation of the following ***free-field*** correlation function

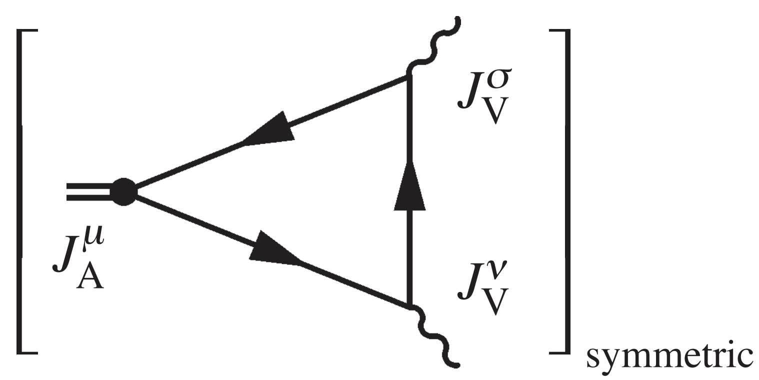
$$C^{\mu\nu\sigma}(x,y) = \langle 0|T[J^{\mu}_{\rm A}(x) J^{\nu}_{\rm V}(y) J^{\sigma}_{\rm V}(0)]|0\rangle$$

Which, applying Wick's theorem gives

$$C^{\mu\nu\sigma}(x,y) = \langle 0|\overline{\psi}\gamma^{\mu}\gamma_5\psi(x)\,\overline{\psi}\gamma^{\nu}\psi(y)\,\overline{\psi}\gamma^{\sigma}\psi(0)|0\rangle + \langle 0|\overline{\psi}\gamma^{\mu}\gamma_5\psi(x)\,\overline{\psi}\gamma^{\nu}\psi(y)\,\overline{\psi}\gamma^{\sigma}\psi(0)|0\rangle$$

These contractions are codified in the celebrated ***triangle diagram***:

$$C^{\mu\nu\sigma}(x,y) =$$




The sought conservation equation is then

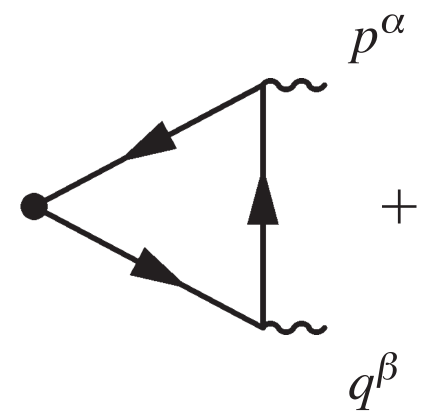
$$\partial_{\mu}\langle J^{\mu}_{A}(x)\rangle_{\mathscr{A}} = -\frac{e^2}{2}\int d^4y_1 d^4y_2\, \partial^{(x)}_{\mu} C^{\mu\nu\sigma}(x,y)\mathscr{A}(x-y_1+y_2)\mathscr{A}_{\sigma}(x-y_2)$$

It is convenient to work in ***momentum space***

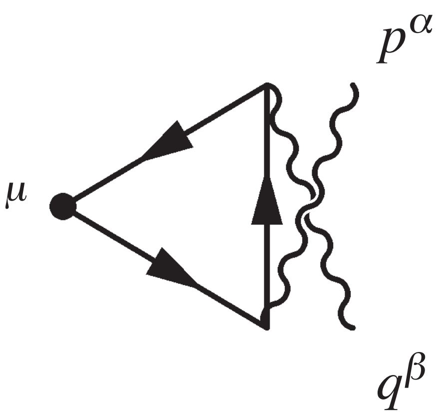
$$e^2\langle 0|T[J_{\rm A}^{\mu}(0)J_{\rm V}^{\alpha}(x_1)J_{\rm V}^{\beta}(x_2)]|0\rangle \;=\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}i\Gamma^{\mu\alpha\beta}(p,q)e^{ip\cdot x_1+iq\cdot x_2}$$

where

$$i\Gamma_{\mu\alpha\beta}(p,q) = \;(p+q)^\mu$$




$$+ \quad (p+q)^\mu$$




and the ***anomaly equation*** to be computed is

$$(p+q)_\mu i\Gamma^{\mu\alpha\beta}(p,q) = ?$$

Applying the Feynman rules of QED, we have

$$i\Gamma_{\mu\alpha\beta}(p,q) = -e^2\int\frac{d^4\ell}{(2\pi)^4}\mathrm{Tr}\left(\frac{i}{\not\ell-\not p+i\varepsilon}\gamma_\mu\gamma_5\frac{i}{\not\ell+\not q+i\varepsilon}\gamma_\alpha\frac{i}{\not\ell+i\varepsilon}\gamma_\beta\right) \;+\begin{pmatrix} p\leftrightarrow q\\ \alpha\leftrightarrow\beta\end{pmatrix}$$

However, the relevant Feynman integrals are ***linearly divergent***.

***Why is this a problem?*** Let us look at the simple one-dimensional integral

$$I(a) = \int_{-\infty}^{\infty} dx\, f(x+a) \qquad \begin{cases} \lim\limits_{|x|\to\infty} f(x) = \text{constant} & \Longrightarrow \quad \text{linearly divergent} \\[2ex] \lim\limits_{|x|\to\infty} f(x) = 0 & \Longrightarrow \quad \text{logarithmically divergent or convergent} \end{cases}$$

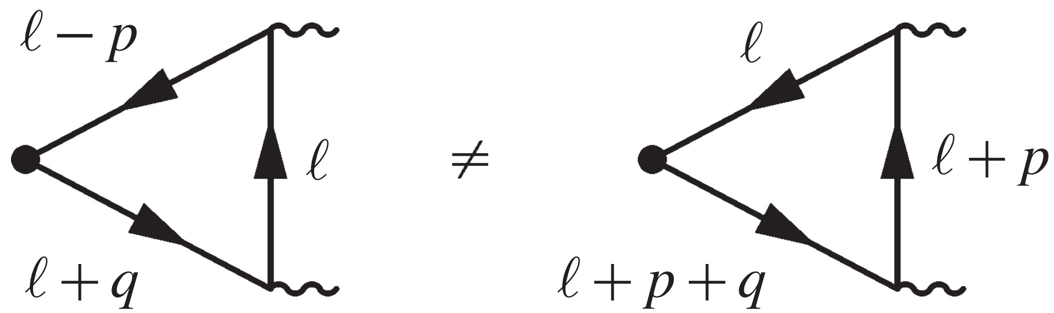
Computing the derivative

$$I'(a) = \int_{-\infty}^{\infty} dx\, f'(x+a) = f(\infty) - f(-\infty) \quad \begin{cases} \neq 0 & \text{if linearly divergent} \\[2ex] = 0 & \text{if logarithmically divergent or convergent} \end{cases}$$

Hence, if the integral is linearly divergent the result of the integration depends on a shift in the integration variable!

The same happens for multidimensional integrals.

Thus, the triangle diagram is ***ambiguous*** because its contribution depends on how we ***label*** the loop momentum!




From ***Lorentz invariance***, the most general form of $i\Gamma_{\mu\alpha\beta}(p,q)$ is (the ***Levi-Civita tensor*** is due to $\gamma_5$ )

$$
\begin{aligned}
i\Gamma_{\mu\alpha\beta}(p,q) = {} & f_1 \varepsilon_{\mu\alpha\beta\sigma} p^\sigma + f_2 \varepsilon_{\mu\alpha\beta\sigma} q^\sigma + f_3 \varepsilon_{\mu\alpha\sigma\lambda} p_\beta p^\sigma q^\lambda \\
& + f_4 \varepsilon_{\mu\alpha\sigma\lambda} q_\beta p^\sigma q^\lambda + f_5 \varepsilon_{\mu\beta\sigma\lambda} p_\alpha p^\sigma q^\lambda \\
& + f_6 \varepsilon_{\mu\beta\sigma\lambda} q_\alpha p^\sigma q^\lambda + f_7 \varepsilon_{\alpha\beta\sigma\lambda} p_\mu p^\sigma q^\lambda + f_8 \varepsilon_{\alpha\beta\sigma\lambda} q_\mu p^\sigma q^\lambda
\end{aligned}
$$

Besides, from ***Bose symmetry***

$$
i\Gamma_{\mu\alpha\beta}(p,q) = i\Gamma_{\mu\beta\alpha}(q,p) \quad \Longrightarrow \quad
\begin{array}{ll}
f_1(p,q) = -f_2(q,p), & f_3(p,q) = -f_6(q,p), \\
f_4(p,q) = -f_5(q,p), & f_7(p,q) = -f_8(q,p).
\end{array}
$$

A bit of dimensional analysis:

$$
\begin{aligned}
i\Gamma_{\mu\alpha\beta}(p,q) = {} & f_1 \varepsilon_{\mu\alpha\beta\sigma} p^\sigma + f_2 \varepsilon_{\mu\alpha\beta\sigma} q^\sigma + f_3 \varepsilon_{\mu\alpha\sigma\lambda} p_\beta p^\sigma q^\lambda \\
& + f_4 \varepsilon_{\mu\alpha\sigma\lambda} q_\beta p^\sigma q^\lambda + f_5 \varepsilon_{\mu\beta\sigma\lambda} p_\alpha p^\sigma q^\lambda \\
& + f_6 \varepsilon_{\mu\beta\sigma\lambda} q_\alpha p^\sigma q^\lambda + f_7 \varepsilon_{\alpha\beta\sigma\lambda} p_\mu p^\sigma q^\lambda + f_8 \varepsilon_{\alpha\beta\sigma\lambda} q_\mu p^\sigma q^\lambda
\end{aligned}
$$

Dimensions = (energy)$^0$

Dimensions = (energy)$^{-2}$

Thus, only $f_1(p,q)$ and $f_2(p,q)$ are (logarithmically) ***divergent*** and their values depend on the regularization scheme used.

The remaining integrals $f_3(p,q)$ to $f_8(p,q)$ are ***convergent*** and free of ambiguities.

Is there a ***wise way of fixing*** these regularization ambiguities?

So far we have ignored the issue of ***gauge invariance***. The relevant gauge ***Ward identities*** reads

$$i\Gamma_{\mu\alpha\beta}(p,q)$$

$$J_A^{\mu} \quad J_V^{\alpha} \quad J_V^{\beta}$$

$$\Longrightarrow \quad \begin{cases} p^{\alpha}\, i\Gamma_{\mu\alpha\beta}(p,q) = 0 \\ q^{\beta}\, i\Gamma_{\mu\alpha\beta}(p,q) = 0 \end{cases}$$

These conditions impose further constraints on the functions $f_i(p,q)$

$$p^{\alpha}\, i\Gamma_{\mu\alpha\beta}(p,q) = \Big(f_2 - p^2 f_5 - p\cdot q f_6\Big)\varepsilon_{\mu\beta\alpha\sigma} q^{\alpha} p^{\sigma} \quad \Longrightarrow \quad f_2(p,q) = p^2 f_5(p,q) + p\cdot q f_6(p,q)$$

$$q^{\beta}\, i\Gamma_{\mu\alpha\beta}(p,q) = \Big(f_1 - q^2 f_4 - p\cdot q f_3\Big)\varepsilon_{\mu\alpha\beta\sigma} q^{\beta} p^{\sigma} \quad \Longrightarrow \quad f_1(p,q) = q^2 f_4(p,q) - p\cdot q f_3(p,q)$$

Hence, ***gauge invariance completely fixes the ambiguities*** and the anomaly is completely determined by ***finite integrals***

$$(p+q)^{\mu}\, i\Gamma_{\mu\alpha\beta}(p,q) = \Big[p^2(f_5+f_7) + q^2(-f_4+f_8) + p\cdot q(-f_3+f_6+f_7+f_8)\Big]\varepsilon_{\alpha\beta\sigma\lambda} q^{\sigma} p^{\lambda}$$

Now we only have to evaluate the integrals

$$f_3(p,q) = -f_6(q,p) = \frac{ie^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{xy}{x(1-x)p^2 + y(1-y)q^2 + 2xyp\cdot q}$$

$$f_4(p,q) = -f_5(q,p) = \frac{ie^2}{\pi^2}\int_0^1 dx \int_0^{1-x} dy \frac{y(1-y)}{x(1-x)p^2 + y(1-y)q^2 + 2xyp\cdot q}$$

$$f_7(p,q) = -f_8(q,p) = 0$$

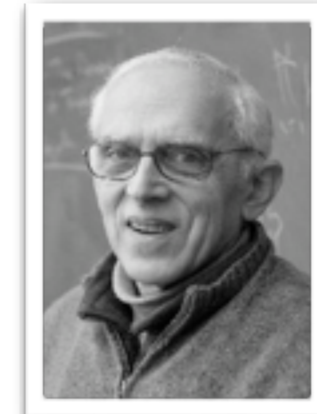
to find the result

$$(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) = -\frac{ie^2}{2\pi^2}\varepsilon_{\alpha\beta\sigma\lambda}q^\sigma p^\lambda$$

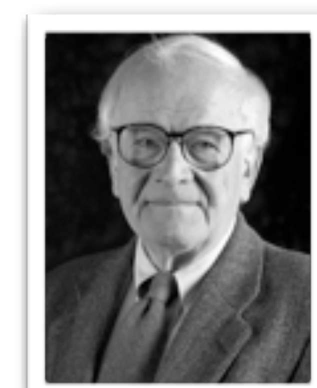
Back in position space, we arrive at the famous ***Adler-Bell-Jackiw anomaly***


Jack Steinberger
(b. 1921)

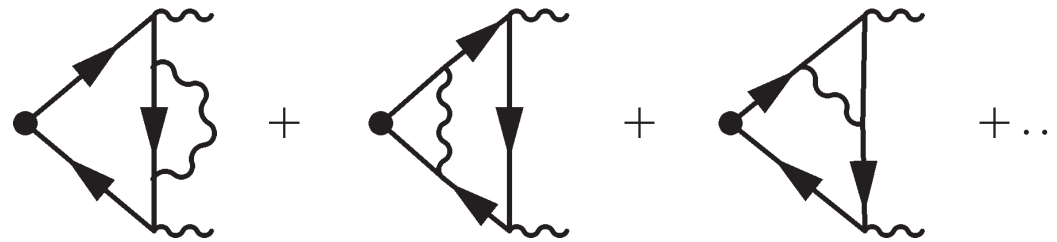
$$\boxed{\partial_\mu \langle J_A^\mu(x)\rangle_{\mathscr{A}} = \frac{e^2}{16\pi^2}\varepsilon^{\mu\nu\alpha\beta}\mathscr{F}_{\mu\nu}\mathscr{F}_{\alpha\beta}}$$


Steven Adler
(b. 1939)


John S. Bell
(1928-1990)

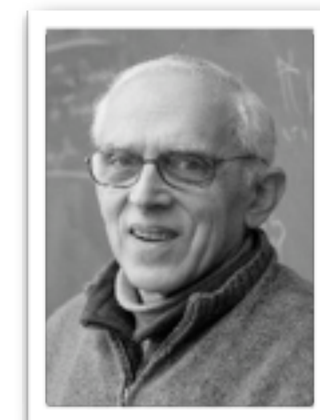

Roman Jackiw
(b. 1939)

***Is the one-loop result enough?*** We can ask about the contribution of higher loop diagrams to the anomaly, e.g.



These diagrams contain ***five*** fermion propagator. The integration over the "triangle momentum" has the structure

$$\cdots \int \frac{d^4\ell}{(2\pi)^4} \prod_{i=1}^{5} \frac{i}{\not\ell + \not A_i + i\varepsilon} \cdots$$



Steven Adler
(b. 1939)

and it is ***unambiguous***. The integration over the ***photon momentum*** can be regularized in a gauge-invariant way, for example adding the term

$$\Delta S = \frac{1}{\Lambda^2} \int d^4x F_{\mu\nu} \Box F^{\mu\nu} \quad \Longrightarrow \quad G_{\mu\nu}(p) \sim \frac{\Lambda^2}{p^4}$$

William A. Bardeen
(b. 1941)

Hence, higher-loop triangles ***do not contribute*** to the anomaly.

This is known as the ***Adler-Bardeen theorem*** (the rigorous proof is more involved that this back-of-the-envelope argument)

To summarize...

* The result

$$\partial_\mu \langle J_A^\mu(x) \rangle_{\mathscr{A}} = A(e^2) \varepsilon^{\mu\nu\alpha\beta} \mathscr{F}_{\mu\nu} \mathscr{F}_{\alpha\beta} \qquad \text{with} \qquad A(e^2) = \frac{e^2}{16\pi^2}$$

is ***exact*** to all orders in perturbation theory.

* Our derivation stresses the fact that, once ***Lorentz invariance***, ***Bose symmetry*** and ***gauge invariance*** are imposed, the anomaly is determined only by ***finite integrals***

* There is a ***tension*** between global chiral transformations and gauge invariance. In order to fix the ambiguities we cound have chosen to impose the ***axial Ward identity***

$$(p+q)^\mu i\Gamma_{\mu\alpha\beta}(p,q) = \Big[ -f_1 + f_2 + p^2 f_7 + p\cdot q(f_7+f_8) + q^2 f_8 \Big] \varepsilon_{\alpha\beta\sigma\lambda} q^\sigma p^\lambda = 0$$

This leads however to a violation of the gauge Ward identities

$$p^\alpha i\Gamma_{\mu\alpha\beta}(p,q) \neq 0, \qquad q^\beta i\Gamma_{\mu\alpha\beta}(p,q) \neq 0.$$

The true meaning of the anomaly is that there exists ***no regularization scheme*** leading to the ***simultaneous conservation*** of ***both*** the vector and the axial current.

* The anomaly cannot be "renormalized away", i.e. removed by adding a ***local counterterm***.

* The anomaly is ***not*** the result of a ***poor choice of regulator***. It reflects a fundamental incompatibility between the conservation of the vector and the axial currents.

As a matter of fact, the anomaly admits a ***topological interpretation***:

$$\int d^4x\,\partial_\mu \langle J_{\rm A}^\mu(x)\rangle_{\mathscr{A}} = -2i\Big(\dim\ker D_+ - \dim\ker D_+^\dagger\Big)$$

where $D_+ = i\not{D}(\mathscr{A})\left(\frac{1+\gamma_5}{2}\right)$. Here

$\dim\ker D_+ =$ # of positive-chirality zero modes of the Dirac operator $i\not{D}(\mathscr{A})$

$\dim\ker D_+^\dagger =$ # of negative-chirality zero modes of the Dirac operator $i\not{D}(\mathscr{A})$

The difference between these two integers defines the ***index*** of the operator $D_+$. Its value is given in terms of the Chern character $\mathrm{ch}(F)$ by the celebrated ***Atiyah-Singer index theorem***

$$\mathrm{Index}\,D_+ = \int [\mathrm{ch}(F)]_{\rm vol} = \frac{1}{32\pi^2}\int d^4x\,\varepsilon^{\mu\nu\sigma\lambda}\mathscr{F}_{\mu\nu}\mathscr{F}_{\sigma\lambda} \quad \text{where} \quad \begin{cases} \mathrm{ch}(F) = \mathrm{tr}\exp\left(\frac{i}{2\pi}F\right) \\ F = \frac{1}{2}\mathscr{F}_{\mu\nu}dx^\mu \wedge dx^\nu \end{cases}$$

***Beware!*** this gives the axial anomaly in Euclidean space (thus the extra $i$)

# UV or IR?

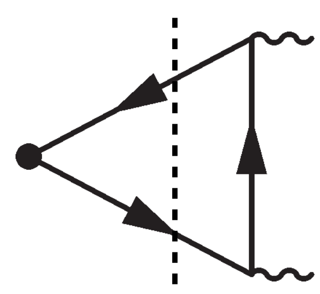
Usually, the anomaly is presented as resulting from the ***UV behavior*** of the theory. The need to regularize the theory clashes with the invariance under chiral transformations and the conservation of the axial current remains broken after the UV cutoff is removed.

The axial anomaly can also be seen as the consequence of the existence of a ***zero-momentum pole*** in the three-current correlation function

$$i\Gamma_{\mu\alpha\beta}(p,q) = -\frac{ie^2}{2\pi^2}\frac{(p+q)_\mu}{(p+q)^2+i\varepsilon}\varepsilon_{\alpha\beta\sigma\lambda}p^\sigma q^\lambda + \text{higher order terms}$$

whose residue gives the anomaly.

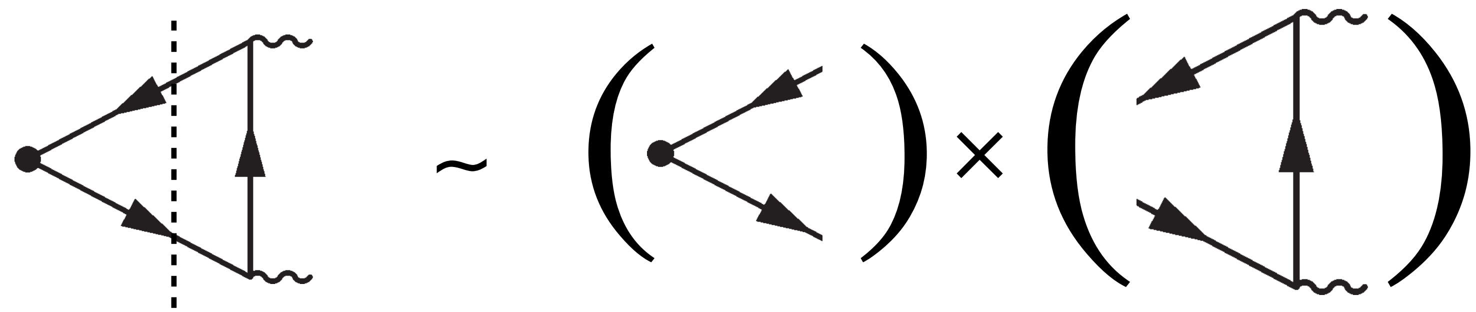
It is instructive to look at the ***imaginary part*** of the triangle diagram, where the anomaly appears as a delta function at zero momentum



$$\sim \frac{e^2}{2\pi}(p+q)_\mu \delta\Big((p+q)^2\Big)$$

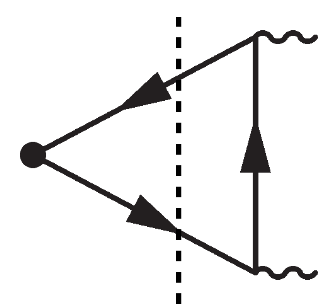
Whereas the real part of the amplitude depends on possible subtractions, the imaginary part is ***unambiguous***. Remember that in our previous calculation, the anomaly was given by unambiguous integrals (once we imposed Lorentz invariance, Bose symmetry, and vector current conservation).

This delta function (or the pole in the full amplitude) can be regularized by giving a mass to the fermion. Applying ***unitarity***



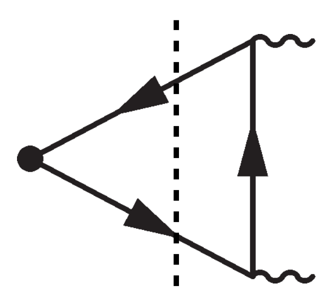
The two on-shell processes on the right-hand side are ***forbidden*** for ***massless*** fermions due to ***axial charge conservation***. Hence, ***naively*** we would expect

$$\lim_{m\to 0}$$



$$= 0$$

However, the actual calculation gives



$$\sim \frac{e^2}{2\pi}\delta\left((p+q)^2\right)$$

Thus, the anomaly is signaled by a ***discontinuity*** in the imaginary part of the amplitude at zero fermion mass. This is the ***infrared face*** of the axial anomaly.

# Third example: The gauge anomaly

The axial anomaly was not dangerous because the global axial current does not couple to a gauge field. In a ***chiral gauge theory*** (e.g. a V-A theory), the axial current ***does couple*** to a gauge fields and its not conservation leads to a ***breaking*** of gauge invariance.

This is potentially ***disastrous*** because the anomaly spoils unitarity (and renormalizability)

Nonchiral theories, however, are save. If the left- and right-handed fermions transform in the ***same representation***, a Dirac mass term can be constructed and the theory can be regulated using ***Pauli-Villars fields***, which preserves gauge invariance.

We look at a theory of a chiral fermion coupled to an external nonabelian gauge field

$$\begin{aligned} S &= \int d^4x\,\overline{\psi}i\not{D}(\mathscr{A})P_+\psi \\ &= \int d^4x\left[i\overline{\psi}\not{\partial}\psi + g_{\rm YM}\overline{\psi}T_+^a\gamma^\mu\left(\frac{1+\gamma_5}{2}\right)\psi\mathscr{A}_\mu^a\right] \end{aligned}$$

Gauge invariance requires, at the quantum level,

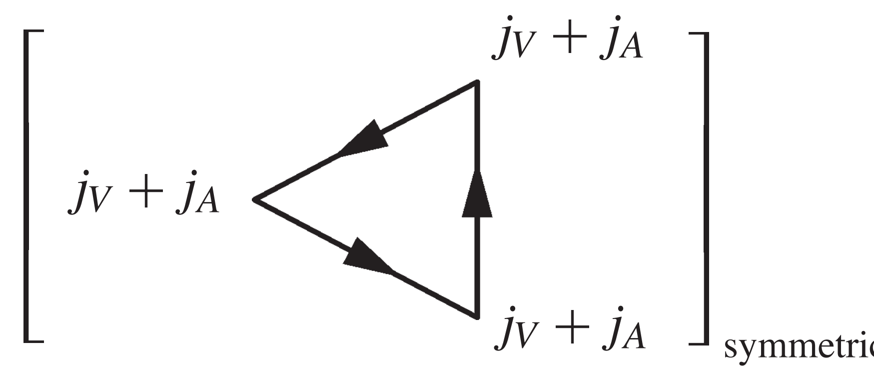
$$D_\mu\langle(j_{\rm V}^\mu + j_{\rm A}^\mu)\rangle_{\mathscr{A}} = 0 \qquad \text{with} \qquad \begin{cases} j_{\rm V}^{\mu\,a}(x) = \overline{\psi}T_+^a\gamma^\mu\psi \\ j_{\rm A}^{\mu\,a}(x) = \overline{\psi}T_+^a\gamma^\mu\gamma_5\psi \end{cases}$$

The relevant quantity to compute is

$$\langle (j_{\mathrm{V}}^{\mu\,a}+j_{\mathrm{A}}^{\mu\,a})\rangle_{\mathscr{A}} = \frac{1}{Z}\int \mathscr{D}\overline{\psi}\mathscr{D}\psi\,(j_{\mathrm{V}}^{\mu\,a}+j_{\mathrm{A}}^{\mu\,a})\,e^{i\int d^4x\left[i\overline{\psi}\not{\partial}\psi+\frac{1}{2}g_{\mathrm{YM}}(j_{\mathrm{V}}^{\mu\,a}+j_{\mathrm{A}}^{\mu\,a})\mathscr{A}_{\mu}^{a}\right]}$$

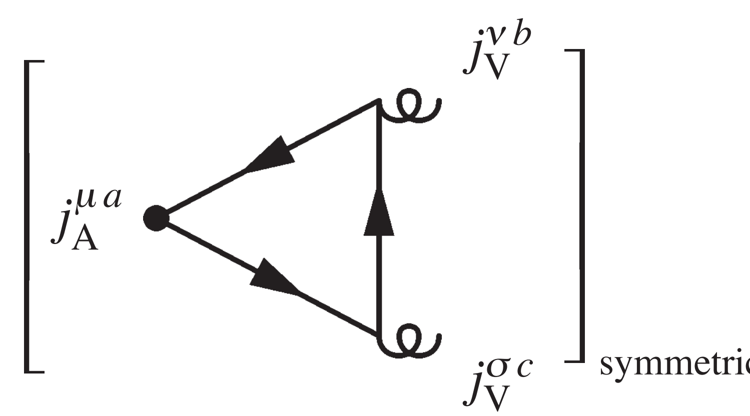
and in perturbation theory the relevant term is the three-current correlation function

$$\langle 0|T\Big[(j_{\mathrm{V}}^{\mu\,a}+j_{\mathrm{A}}^{\mu\,a})(0)(j_{\mathrm{V}}^{\nu\,b}+j_{\mathrm{A}}^{\nu\,b})(x)(j_{\mathrm{V}}^{\sigma\,c}+j_{\mathrm{A}}^{\sigma\,c})(y)\Big]|0\rangle =$$

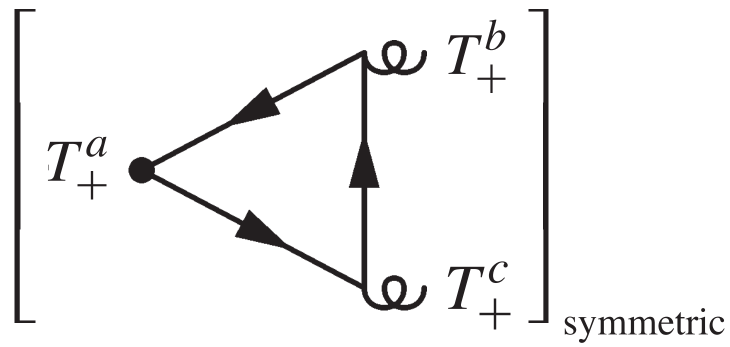



Here we are not interested in computing the anomaly, just in finding the conditions for its ***cancellation***. For this purpose it is enough to compute

$$\langle 0|T\Big[j_{\mathrm{A}}^{\mu\,a}(0)j_{\mathrm{V}}^{\nu\,a}(x)j_{\mathrm{V}}^{\sigma\,c}(y)\Big]|0\rangle =$$



This three-current correlations function is proportional to the group-theoretic factor




$$\sim \mathrm{Tr}\left[T^a_+\left\{T^b_+, T^c_+\right\}\right]$$

Thus, the condition for the cancellation of the gauge anomaly reads

$$\mathrm{Tr}\left[T^a_+\left\{T^b_+, T^c_+\right\}\right] = 0$$

In a theory with $N_+$ positive chirality fermions and $N_-$ negative chirality fermions, the ***anomaly cancellation condition*** takes the form

$$\boxed{\sum_{i=1}^{N_+}\mathrm{Tr}\left[T^a_{i,+}\left\{T^b_{i,+}, T^c_{i,+}\right\}\right] - \sum_{j=1}^{N_-}\mathrm{Tr}\left[T^a_{j,-}\left\{T^b_{j,-}, T^c_{j,-}\right\}\right] = 0}$$

For an arbitrary representation $\mathbf{R}$, we can define the invariant $A(\mathbf{R})$ by

$$\mathrm{Tr}\left[T^a_{\mathbf{R}}\left\{T^b_{\mathbf{R}},T^c_{\mathbf{R}}\right\}\right]=A(\mathbf{R})d^{abc}$$

where $d^{abc}$ is independent of $\mathbf{R}$ and $A(\mathbf{R})=1$ for the fundamental representation.

Representations for which $A(\mathbf{R})=0$ are safe because chiral fermions transforming under them do not give rise to gauge anomalies. This is the case, for example, when ($S$ unitary)

$$(T^a_{\mathbf{R}})^*=-S\,T^a_{\mathbf{R}}\,S^{-1}$$

$S=S^T$ (***real*** representation)

$S=-S^T$ (***pseudoreal*** representation)

If the generators are taken Hermitian, we have

$$\mathrm{Tr}\left[T^a_{\mathbf{R}}\left\{T^b_{\mathbf{R}},T^c_{\mathbf{R}}\right\}\right]=\mathrm{Tr}\left[T^a_{\mathbf{R}}\left\{T^b_{\mathbf{R}},T^c_{\mathbf{R}}\right\}\right]^T=\mathrm{Tr}\left[(T^a_{\mathbf{R}})^*\left\{(T^b_{\mathbf{R}})^*,(T^c_{\mathbf{R}})^*\right\}\right]$$

and for real or pseudoreal representations

$$\mathrm{Tr}\left[(T^a_{\mathbf{R}})^*\left\{(T^b_{\mathbf{R}})^*,(T^c_{\mathbf{R}})^*\right\}\right]=-\mathrm{Tr}\left[ST^a_{\mathbf{R}}S^{-1}\left\{ST^b_{\mathbf{R}}S^{-1},ST^c_{\mathbf{R}}S^{-1}\right\}\right]=-\mathrm{Tr}\left[T^a_{\mathbf{R}}\left\{T^b_{\mathbf{R}},T^c_{\mathbf{R}}\right\}\right]$$

so we conclude that $A(\mathbf{R})=0$.

Thus, ***real*** and ***pseudoreal*** representations are ***safe***. This includes ***all*** representations of the groups

$$\mathrm{SU}(2),\ \mathrm{SO}(2N+1),\ \mathrm{SO}(4N),\ \mathrm{Sp}(2N),\ G_2,\ F_4,\ E_7,\ E_8$$

In addition, the ***adjoint*** representation of any group is real and therefore ***safe***.

In conclusion, anomalies can only appear when chiral fermions transform in ***complex*** representations, for which there is no unitary equivalence between the representation and its complex conjugate. This is the case, for example, of $\mathrm{SU}(N)$ with $N \geq 3$.

Thus, if the representation is complex we have to ***explicitly check*** whether $A(\mathbf{R}) = 0$ or not.

If the gauge group is a direct product, $G_1 \otimes \ldots \otimes G_n$, there might be mixed gauge anomalies associated with triangles with "different group factors" at each vertex

$\Longrightarrow$ more on this later.

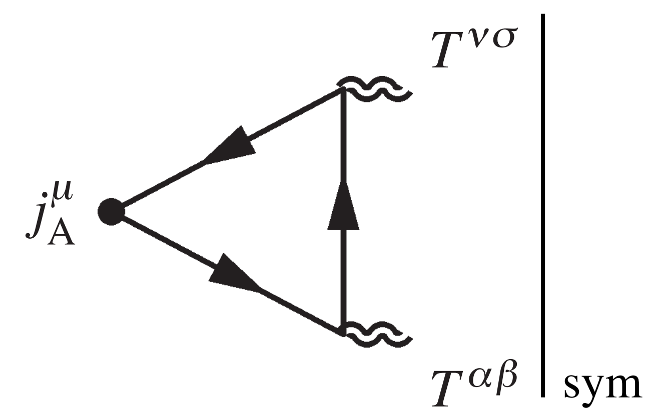
***Gravity*** can also contribute to the gauge anomaly...

If the gauge theory is coupled to gravity, there is a new interaction term in the action coupling the ***graviton field*** $h_{\mu\nu}$ to the ***energy-momentum tensor*** of the gauge theory

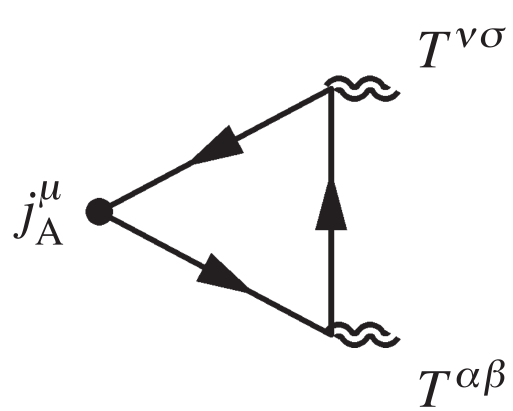
$$\Delta S = \kappa \int d^4x\, h_{\mu\nu} T^{\mu\nu}$$

Expanding in powers of the gravitational coupling, this extra term gives an additional contribution to the anomaly given by the triangle anomaly with two energy-momentum tensor couplings

$$\langle 0|T\Big[ j_{\rm A}^{\mu}(0) T^{\nu\sigma}(x) T^{\alpha\beta}(y)\Big]|0\rangle =$$




Since the gravitational coupling does not affect the gauge quantum numbers, the ***mixed anomaly*** is proportional to the ***trace*** of the generators of the gauge group




$$\sim \sum_{i=1}^{N_+} \mathrm{Tr}\,(T^a_{i,+}) - \sum_{i=1}^{N_-} \mathrm{Tr}\,(T^a_{i,-})$$

Source of trouble for $\mathrm{U}(1)$'s!

# Part II

# Anomalies: how to take advantage of them to learn about Nature

# Anomalies and pion decay

The axial anomaly has very important ***consequences*** for the physics of strong interactions, and in particular for the ***pion decay***.

Let us begin by studying the ***global symmetries*** of QCD in the chiral limit

$$\mathscr{L}_{\rm QCD} = -\frac{1}{4}F^A_{\mu\nu}F^{A\,\mu\nu} + \sum_{f=1}^{N_f}\left(i\overline{Q}{}^f_L \not{D} Q^f_L + i\overline{Q}{}^f_R \not{D} Q^f_R\right)$$

The action is invariant under the $\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R$ symmetry

$$\mathrm{U}(N_f)_L : \begin{cases} Q^f_L \to \displaystyle\sum_{f'=1}^{N_f}(U_L)_{ff'}Q^{f'}_L \\ Q^f_R \to Q^f_R \end{cases} \qquad \mathrm{U}(N_f)_R : \begin{cases} Q^f_L \to Q^f_L \\ Q^r_R \to \displaystyle\sum_{f'=1}^{N_f}(U_R)_{ff'}Q^{f'}_R \end{cases} \qquad U_L, U_R \in \mathrm{U}(N_f)$$

Since $\mathrm{U}(N)$=$\mathrm{U}(1)\times\mathrm{SU}(N)$ the symmetry group can be written as

$$\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_L \times \mathrm{U}(1)_R$$

The left-right global transformations can be now decomposed into ***vector*** and ***vector-axial*** parts

$$\mathrm{U}(N_f)_L \times \mathrm{U}(N_f)_R = \mathrm{SU}(N_f)_\mathrm{V} \times \mathrm{SU}(N_f)_\mathrm{A} \times \mathrm{U}(1)_B \times \mathrm{U}(1)_\mathrm{A}$$

with

$$\mathrm{U}(1)_B : \begin{cases} Q_L^f \to e^{i\alpha} Q_L^f \\ \\ Q_R^f \to e^{i\alpha} Q_R^f \end{cases} \qquad \mathrm{U}(1)_\mathrm{A} : \begin{cases} Q_L^f \to e^{i\alpha} Q_L^f \\ \\ Q_R^f \to e^{-i\alpha} Q_R^f \end{cases}$$

$$\mathrm{SU}(N_f)_\mathrm{V} : \begin{cases} Q_L^f \to \sum_{f'=1}^{N_f} U_{ff'} Q_L^{f'} \\ \\ Q_R^f \to \sum_{f'=1}^{N_f} U_{ff'} Q_R^{f'} \end{cases} \qquad \mathrm{SU}(N_f)_\mathrm{A} : \begin{cases} Q_L^f \to \sum_{f'=1}^{N_f} U_{ff'} Q_L^{f'} \\ \\ Q_R^f \to \sum_{f'=1}^{N_f} U_{ff'}^{-1} Q_R^{f'} \end{cases}$$

The associated classically conserved currents are

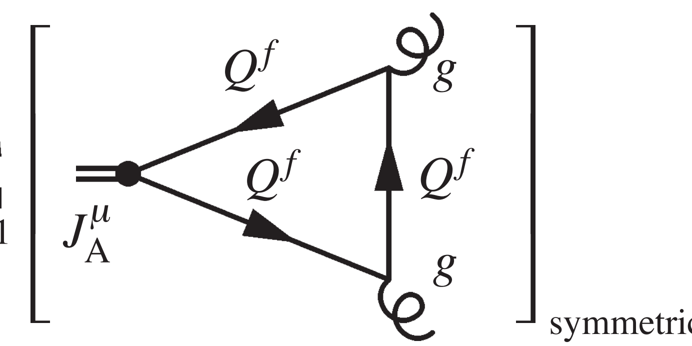
$$J_\mathrm{V}^\mu = \sum_{f=1}^{N_f} \overline{Q}^f \gamma^\mu Q^f \qquad J_\mathrm{A}^\mu = \sum_{f=1}^{N_f} \overline{Q}^f \gamma^\mu \gamma_5 Q^f$$

$$J_\mathrm{V}^{I\,\mu} \equiv \sum_{f,f'=1}^{N_f} \overline{Q}^f \gamma^\mu (T^I)_{ff'} Q^{f'} \qquad J_\mathrm{A}^{I\,\mu} \equiv \sum_{f,f'=1}^{N_f} \overline{Q}^f \gamma^\mu \gamma_5 (T^I)_{ff'} Q^{f'}$$

Axial currents are ***potentially anomalous***. To settle the question we compute the correlation function of one axial current and two gauge currents.

For the ***abelian*** current we have:

$$C^{\mu\nu\sigma}(x,x') \equiv \langle 0|T\left[J^{\mu}_{\mathrm{A}}(x)\, j^{A\,\nu}_{\mathrm{gauge}}(x')\, j^{B\,\sigma}_{\mathrm{gauge}}(0)\right]|0\rangle \;=\; \sum_{f=1}^{N_f}$$




where $j^{A\,\mu}_{\mathrm{gauge}} \equiv \sum_{f=1}^{N_f} \overline{Q}^f \gamma^\mu \tau^A Q^f$.

The diagram contains two $\mathrm{SU}(N_c)$ generators at the gauge current vertices, so the group theoretical factor multiplying this diagram is

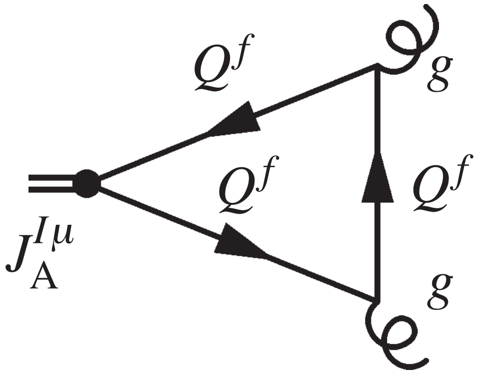
$$\mathrm{Tr}\{\tau^A, \tau^B\} \neq 0$$

so the anomaly does not cancel. An explicit calculation gives

$$\partial_\mu J^\mu_A = -\frac{g^2 N_f}{32\pi^2}\varepsilon^{\mu\nu\sigma\lambda} F^a_{\mu\nu} F^a_{\sigma\lambda}$$

We study next the $\mathrm{SU}(N_f)_\mathrm{A}$ current. Looking at the group theory factor, we find

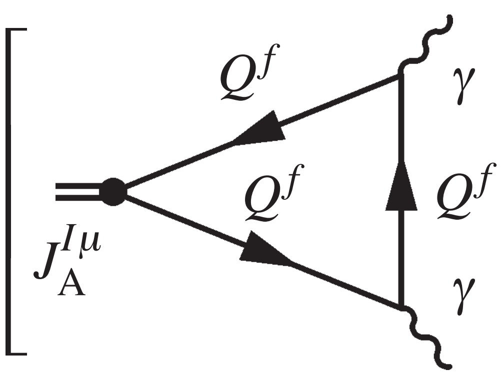
$$\Big[\quad\Big]_{\text{symmetric}} \sim \operatorname{Tr} T^I \operatorname{Tr}\{\tau^A, \tau^B\} = 0 \qquad \text{since} \quad \operatorname{Tr} T^I = 0$$




where $T^I$ are the generators of $\mathrm{SU}(N_f)_\mathrm{A}$.

This however is ***not enough*** to conclude that $\mathrm{SU}(N_f)_\mathrm{A}$ is anomaly free. Quarks also couple to the electromagnetic field, so there is second contribution to the anomaly

$$\langle 0|T\left[J_\mathrm{A}^{I\mu}(x) j_\mathrm{em}^\nu(x') j_\mathrm{em}^\sigma(0)\right]|0\rangle = \sum_{f=1}^{N_f} \Big[\quad\Big]_{\text{symmetric}} \qquad \text{with} \quad j_\mathrm{em}^\mu = \sum_{f=1}^{N_f} q_f \overline{Q}^f \gamma^\mu Q^f$$




The computation of the diagram gives

$$\partial_\mu J_\mathrm{A}^{I\mu} = -\frac{N_c}{16\pi^2}\left[\sum_{f=1}^{N_f} (T^I)_{ff}\, q_f^2\right] \varepsilon^{\mu\nu\sigma\lambda} F_{\mu\nu} F_{\sigma\lambda}$$

We particularize the result to the case of QCD ($N_c = 3$) with the two light flavors $u$ and $d$ ($N_f = 2$). Taking into account that

$$q_u = \frac{2}{3}e \qquad\qquad q_d = -\frac{1}{3}e$$

we have

$$\sum_{f=u,d} (T^1)_{ff}\, q_f^2 = \sum_{f=u,d} (T^2)_{ff}\, q_f^2 = 0 \qquad\qquad \sum_{f=u,d} (T^3)_{ff}\, q_f^2 = \frac{e^2}{6}$$

so only the third component is anomalous.

To summarize:

* $\mathrm{U}(1)_{\mathrm{A}}$ is always anomalous. This is not too problematic, since this axial symmetry is explicitly broken by ***instantons***. This is the idea behind the famous resolution of the U(1) problem by 't Hooft.

* Of $\mathrm{SU}(2)_{\mathrm{A}}$ only the third isospin component $J_{\mathrm{A}}^{3\,\mu}$ is anomalous. This is interesting, because the field $\partial_\mu J_{\mathrm{A}}^{3\,\mu}$ has precisely the quantum numbers of the neutral pion $\pi^0$.

  In fact, we are going to see how the existence of this anomaly is the way to understand the ***electromagnetic decay*** of the pion, which is the dominant channel despite the Sutherland-Veltman suppression.

Due to the dynamics of QCD, the axial symmetry $\mathrm{SU}(2)_{\mathrm{A}}$ is spontaneously broken by quark condensates

$$\langle\overline{Q}^f Q^f\rangle \neq 0 \qquad \text{(no summation on } f\text{)}$$

The three pions $\pi^{\pm}$, $\pi^0$ are the ***(pseudo)Goldstone bosons*** associated with this $\chi$SB. By the Goldstone theorem his means that

$$\langle 0|J_{\mathrm{A}}^{a\,\mu}(x)|\pi^b(p)\rangle \neq 0 \quad \Longrightarrow \quad \langle 0|J_{\mathrm{A}}^{a\,\mu}(x)|\pi^b(p)\rangle = f_\pi p^\mu \delta^{ab} e^{-ip\cdot x}$$

Moreover, taking the divergence here we have

$$\langle 0|\partial_\mu J_{\mathrm{A}}^{a\,\mu}(x)|\pi^b(p)\rangle = -if_\pi m_\pi^2 \delta^{ab} e^{-ip\cdot x}$$

This means that the (canonically normalized) interpolating field for the neutral pion is

$$\varphi_\pi(x) = \frac{1}{f_\pi m_\pi^2}\partial_\mu J^{3\mu}(x)$$

Using the LSZ formula, we can compute the physical amplitude for $\pi^0 \longrightarrow 2\gamma$

$$\langle k_1,\varepsilon_1;k_2,\varepsilon_2|\pi^0(q)\rangle = (2\pi)^4\delta^{(4)}(q-k_1-k_2)\varepsilon_1^{\mu}(k_1)\varepsilon_2^{\nu}(k_2)\Gamma_{\mu\nu}(q,k_1,k_2)$$

where

$$\Gamma^{\mu\nu} \sim e^2 m_\pi^2 \int d^4x_1 \int d^4x_2\, e^{ik_2\cdot y - iq\cdot x}\langle 0|\underbrace{T\Big[\varphi_\pi(x)J_{\rm V}^{\mu}(y)J_{\rm V}^{\nu}(0)\Big]|0\rangle}_{\text{the } J_{\rm A}^{3\,\mu} \text{ anomaly}}$$

Evaluating the amplitude in the soft pion limit $q \longrightarrow 0$

$$\Gamma_{\mu\nu}(q^2=0) = \frac{e^2 N_c}{12\pi^2 f_\pi}\varepsilon_{\mu\nu\sigma\lambda}k_1^{\sigma}k_2^{\lambda}$$

Averaging over the photon polarizations, we find the width of the decay to be

$$\Gamma(\pi^0 \to 2\gamma) = \frac{\alpha^2 m_\pi^3 N_c^2}{576\pi^3 f_\pi^2} \qquad \Longrightarrow \qquad \text{color counting observable}$$

and substituting the numerical values we find

$$\Gamma(\pi^0 \to 2\gamma) = 7.64\,\text{eV} \qquad\qquad \Gamma(\pi^0 \to 2\gamma)_{\rm exp} = 7.37\,\text{eV}$$

# Nonperturbative physics from anomalies

In theories whose spectrum changes with the scale (such a QCD), the ***matching of anomalies*** between the high- and low-energy theories provide nontrivial nonperturbative information.

Let us take a gauge theory with ***massless*** fermions ***confining*** below certain energy scale $\Lambda$ and with a ***nonanomalous global*** symmetry group $G$ with generators

$$T^a \qquad\qquad (a=1,\ldots,\dim G)$$

Below $\Lambda$, the theory will contain a number of ***composite*** massless fermions transforming in some representation of the global group $G$

$$\widetilde{T}^a \qquad\qquad (a=1,\ldots,\dim G)$$

We ***gauge*** now the global group $G$ by adding a ***new set*** of nonabelian gauge field $B^a_\mu(x)$ with the coupling

$$\Delta S = g' \int d^4x B^a_\mu(x) J^{a\,\mu}_G(x)$$

If, however,

$$\sum_L \mathrm{tr}\left[T_L^a\{T_L^b, T_L^c\}\right] - \sum_R \mathrm{tr}\left[T_R^a\{T_R^b, T_R^c\}\right] \neq 0$$

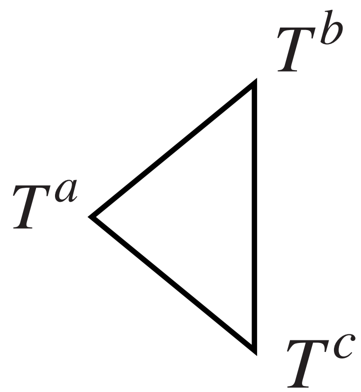
the “new” gauge symmetry will be anomalous.

To prevent this, we ***add new fermions*** coupling only to $B_\mu$ that cancel this anomaly. Since we can take the coupling $g'$ as ***small*** as we like, these ***spectator fermions*** do not modify the dynamics of the original gauge theory!

an example will come later

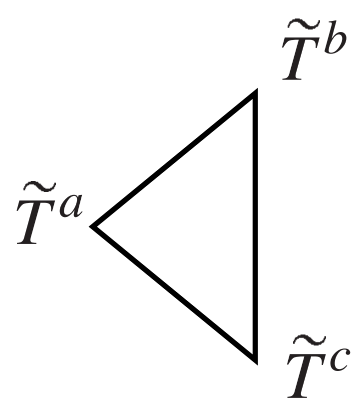
This cancellation of the anomaly should work both in the unconfined and confined theories.

***UV:***




$$\sim \sum_L \mathrm{tr}\left[T_L^a\{T_L^b, T_L^c\}\right] - \sum_R \mathrm{tr}\left[T_R^a\{T_R^b, T_R^c\}\right] + \begin{pmatrix}\text{spectator}\\ \text{fermions}\end{pmatrix} = 0$$

***IR:***




$$\sim \sum_L \mathrm{tr}\left[\widetilde{T}_L^a\{\widetilde{T}_L^b, \widetilde{T}_L^c\}\right] - \sum_R \mathrm{tr}\left[\widetilde{T}_R^a\{\widetilde{T}_R^b, \widetilde{T}_R^c\}\right] + \begin{pmatrix}\text{spectator}\\ \text{fermions}\end{pmatrix} = 0$$

Since the spectator fermions are weakly coupled to the original gauge theory, they have the same contribution in the UV and the IR. Thus, we arrive at ***'t Hooft's anomaly matching condition***

Gerard 't Hooft (b. 1946)

$$\sum_L \mathrm{tr}\left[T_L^a\{T_L^b,T_L^c\}\right]-\sum_R \mathrm{tr}\left[T_R^a\{T_R^b,T_R^c\}\right]=\sum_L \mathrm{tr}\left[\tilde{T}_L^a\{\tilde{T}_L^b,\tilde{T}_L^c\}\right]-\sum_R \mathrm{tr}\left[\tilde{T}_R^a\{\tilde{T}_R^b,\tilde{T}_R^c\}\right]$$

This identity is still true in the limit $g' \longrightarrow 0$ in which all spectators fermions decouple.

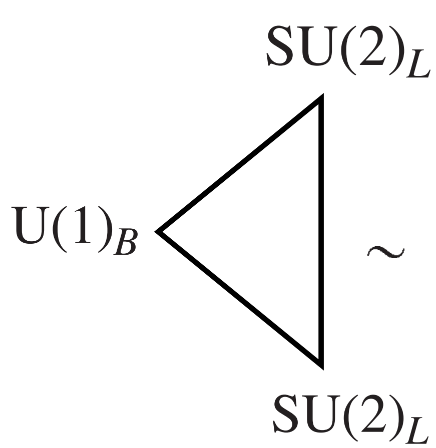
To see the power of this matching condition, we look to a couple of examples:

***QCD with $N_f$ = 2:*** The global symmetry group is $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_B$

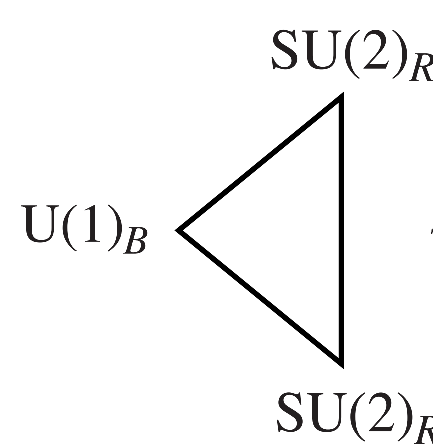
At ***high energies***, the fermionic spectrum is composed by the massless quarks $u$ and $d$ [we use the notation $(r_L,r_R)_B$]. The anomalous triangles are

$q_L$: $(2,1)_{\frac{1}{3}}$

$q_R$: $(1,2)_{\frac{1}{3}}$




$$\sim \left(3\times\frac{1}{3}\right)\delta^{ab}=\delta^{ab}$$



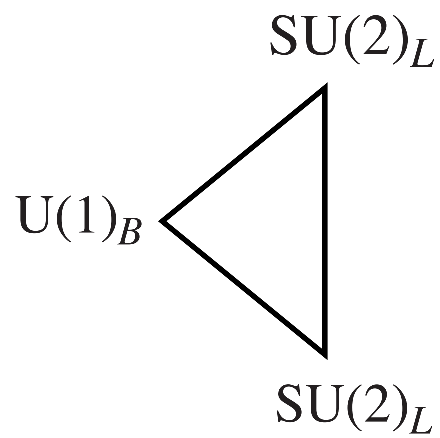

$$\sim -\left(3\times\frac{1}{3}\right)\delta^{ab}=-\delta^{ab}$$

At ***low energies***, there are two possibilities:

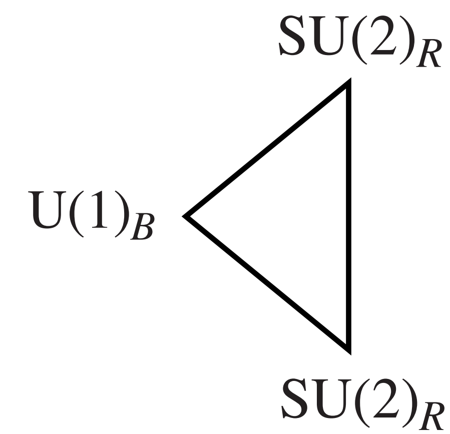
- ✱ The global group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_B$ remains unbroken and we have massless protons and neutrons

$N_L$: $(2,1)_1$

$N_R$: $(1,2)_1$




$\sim \delta^{ab}$

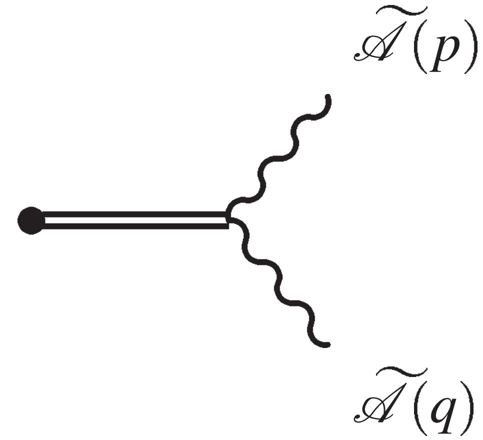



$\sim -\delta^{ab}$

Using the IR interpretation of the anomaly, the pole at zero momentum is associated with the massless composite chiral fermions (its residue matches the UV anomaly).

- ✱ The global symmetry is spontaneously broken to the diagonal group $\mathrm{SU}(2)_V \times \mathrm{U}(1)_B$. protons and neutrons are massive and we have three Goldstone bosons contributing to the pole.

The axial current interpolates between the vacuum and the Goldstone boson, and the IR pole is associated with the propagation of this massless state (equivalently, the delta function in the imaginary part)

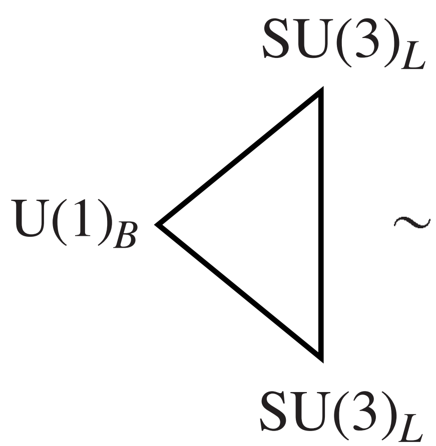
$\langle 0|J_A^\mu(0)|p,q\rangle_{\mathscr{A}} = \; k^\mu$ 

$= \langle 0|J_{\mathrm{A}}^\mu(0)|k;\phi\rangle \dfrac{i}{k^2}\langle k;\phi|p,q\rangle_{\mathscr{A}}$

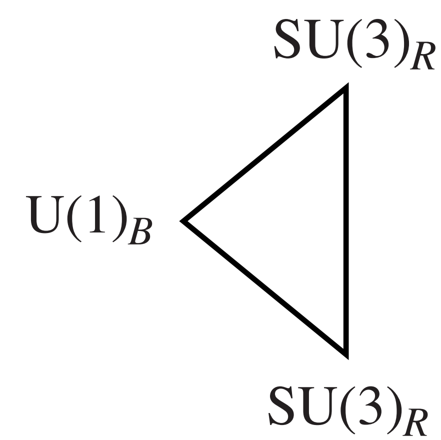
In the case of ***QCD with two flavors***, the matching of anomalies ***cannot distinguish*** between massless composites and spontaneous chiral symmetry breaking.

***QCD with $N_f$ = 3:*** Now, the global symmetry group is $SU(3)_L \times SU(3)_R \times U(1)_B$.

At high energies we have three massless quarks ($u$, $d$, and $s$). The potentially anomalous diagrams are now



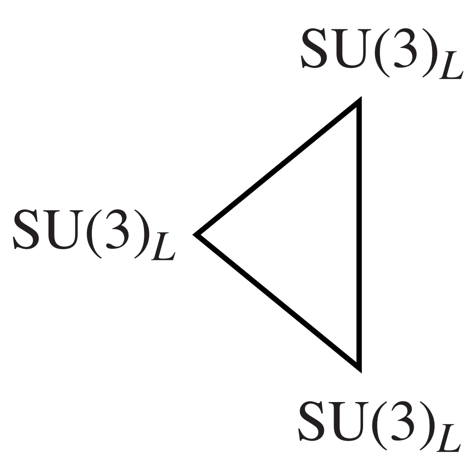

$$\sim \left(3\times\frac{1}{3}\right)\delta^{ab} = \delta^{ab}$$



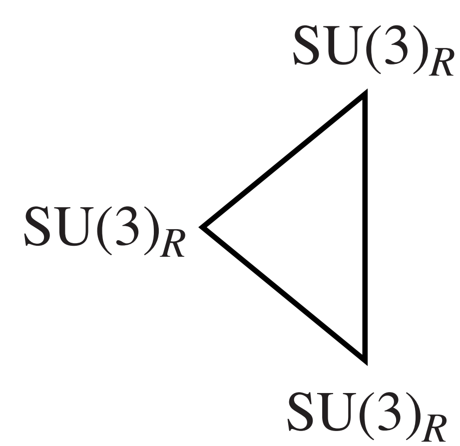

$$\sim -\left(3\times\frac{1}{3}\right)\delta^{ab} = -\delta^{ab}$$

$q_L$: $\quad (3,1)_{\frac{1}{3}}$

$q_R$: $\quad (1,3)_{\frac{1}{3}}$

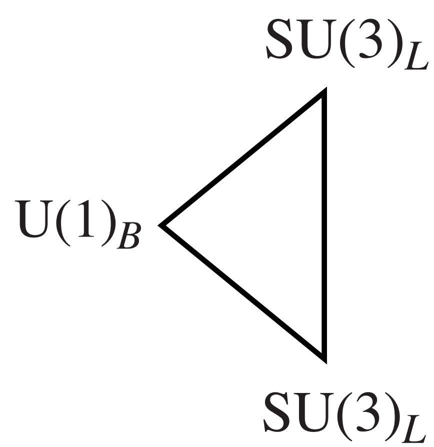



$$\sim d^{abc}$$




$$\sim -d^{abc}$$

At ***low energies*** there is the octet of composite fermions ($p$, $n$, $\Sigma^+$, $\Sigma^-$, $\Sigma^0$, $\Lambda$, $\Xi^0$, $\Xi^-$). If they are massless, their contribution to the anomaly ***does not match*** the result for the high energy theory. For example




$$\sim \mathrm{Tr}\left(T^a_{\mathbf{8}} T^b_{\mathbf{8}}\right) = 3\delta^{ab}$$

Since the ***anomalies do not match***, the bound state fermions cannot remain massless and $\mathrm{SU}(3)_A$ has to be spontaneously broken. We conclude that for ***$N_f$ = 3 chiral symmetry breaking takes place***.

The resulting massless Goldstone bosons are responsible for the singularities at zero momentum.

't Hooft anomaly matching allows to extract a very ***nonperturbative*** piece of information from purely algebraic arguments. This tool is very useful in analyzing a wide class of theories.

# Constraints from anomaly cancellation

We have seen how in ***chiral gauge theories*** the absence of gauge anomalies are determined by the condition

$$\underbrace{\sum_{i=1}^{N_+}\mathrm{Tr}\left[\tau_{i,+}^{A}\{\tau_{i,+}^{B},\,\tau_{i,+}^{C}\}\right]}_{\text{positive chirality}}-\underbrace{\sum_{j=1}^{N_-}\mathrm{Tr}\left[\tau_{j,-}^{A}\{\tau_{j,-}^{B},\,\tau_{j,-}^{C}\}\right]}_{\text{negative chirality}}=0$$

This provides a nontrivial constraint on the matter content and group representations of a theory and it is a useful tool in ***model building*** since it selects both matter content and group representations.

Here we analyze two cases:

- ✱ The standard model
- ✱ Minimal supersymmetric standard model

***The standard model:*** We only have to care about the chiral fermions in the standard model (i.e. leptons and quarks). Denoting their representations of $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)_Y$ by

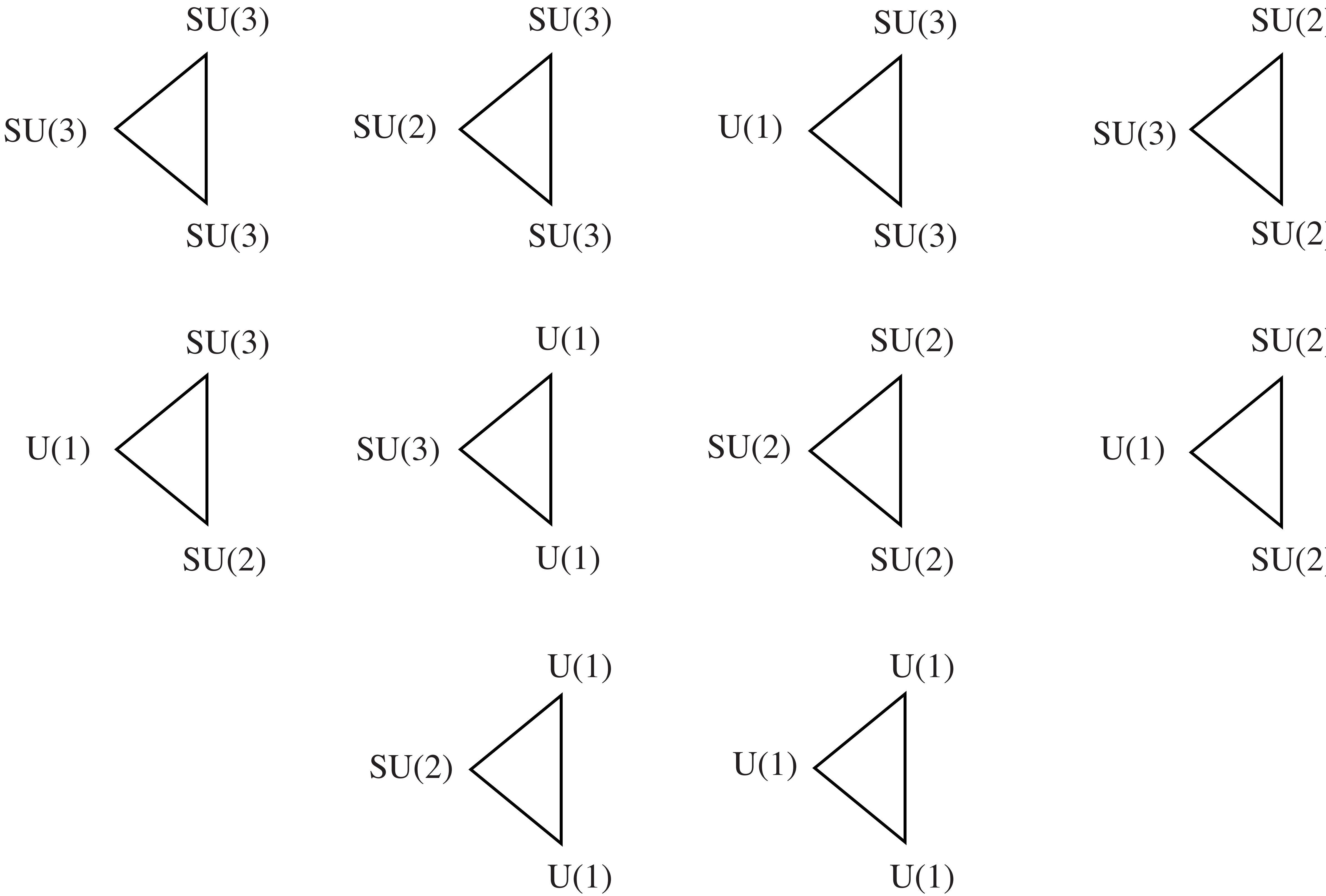
$$(n_c, n_w)_Y$$

the ***matter content*** of the theory is

$$\left.\begin{array}{llll} \text{Left-handed fermions:} & (3,2)^L_{\frac{1}{6}} & & (1,2)^L_{-\frac{1}{2}} \\[2em] \text{Right-handed fermions:} & (3,1)^R_{\frac{2}{3}} & (3,1)^R_{-\frac{1}{3}} & (1,1)^R_{-1} \end{array}\right\} \times 3 \text{ families}$$

$$\underbrace{(3,2)^L_{\frac{1}{6}} \quad (3,1)^R_{\frac{2}{3}} \quad (3,1)^R_{-\frac{1}{3}}}_{\text{quarks}} \qquad \underbrace{(1,2)^L_{-\frac{1}{2}} \quad (1,1)^R_{-1}}_{\text{leptons}}$$

Symbolically, the ten anomaly coefficients to compute are:



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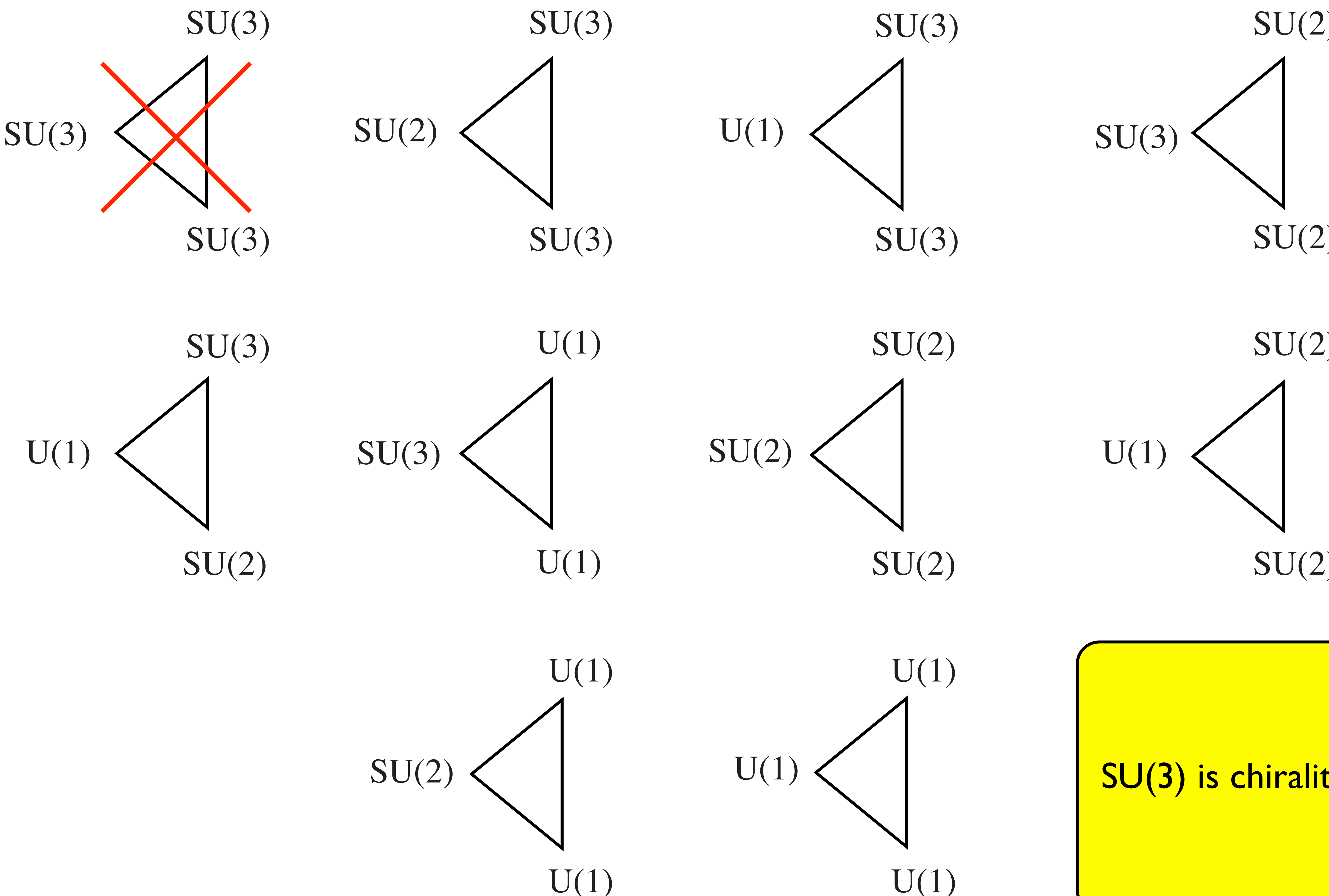
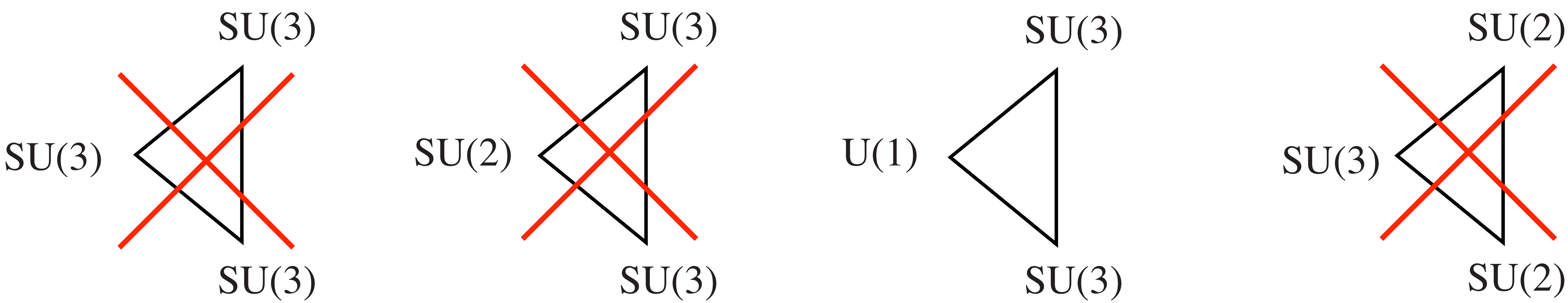
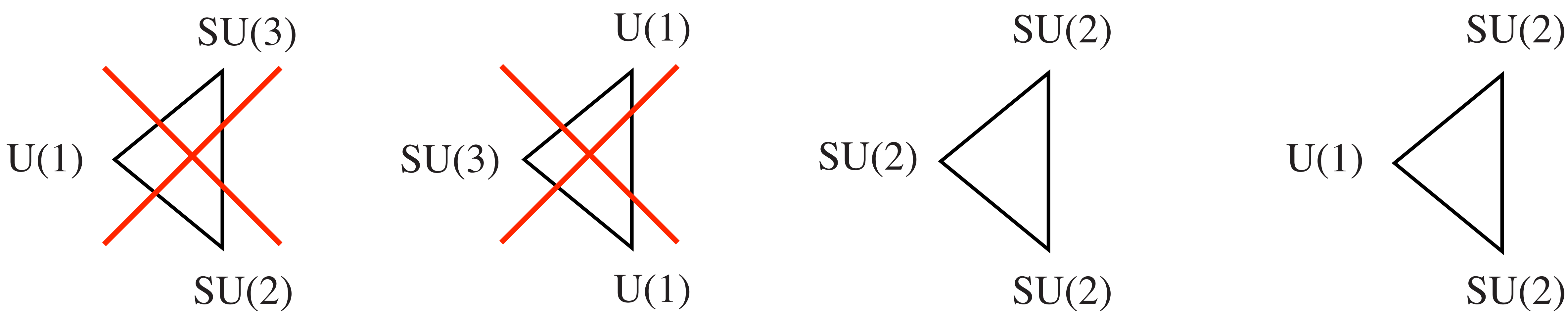
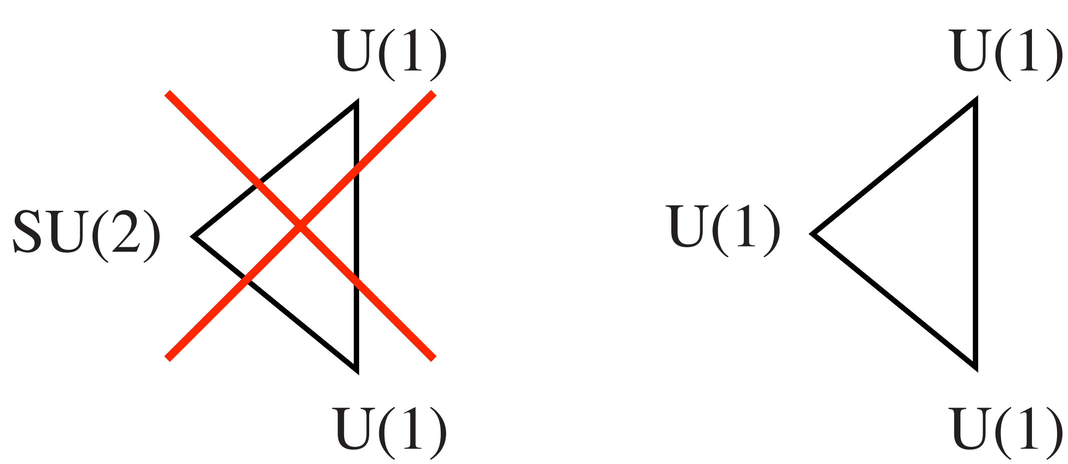



SU(3) is chirality-blind

Symbolically, the ten anomaly coefficients to compute are:




The generator of SU(2) and SU(3) are traceless

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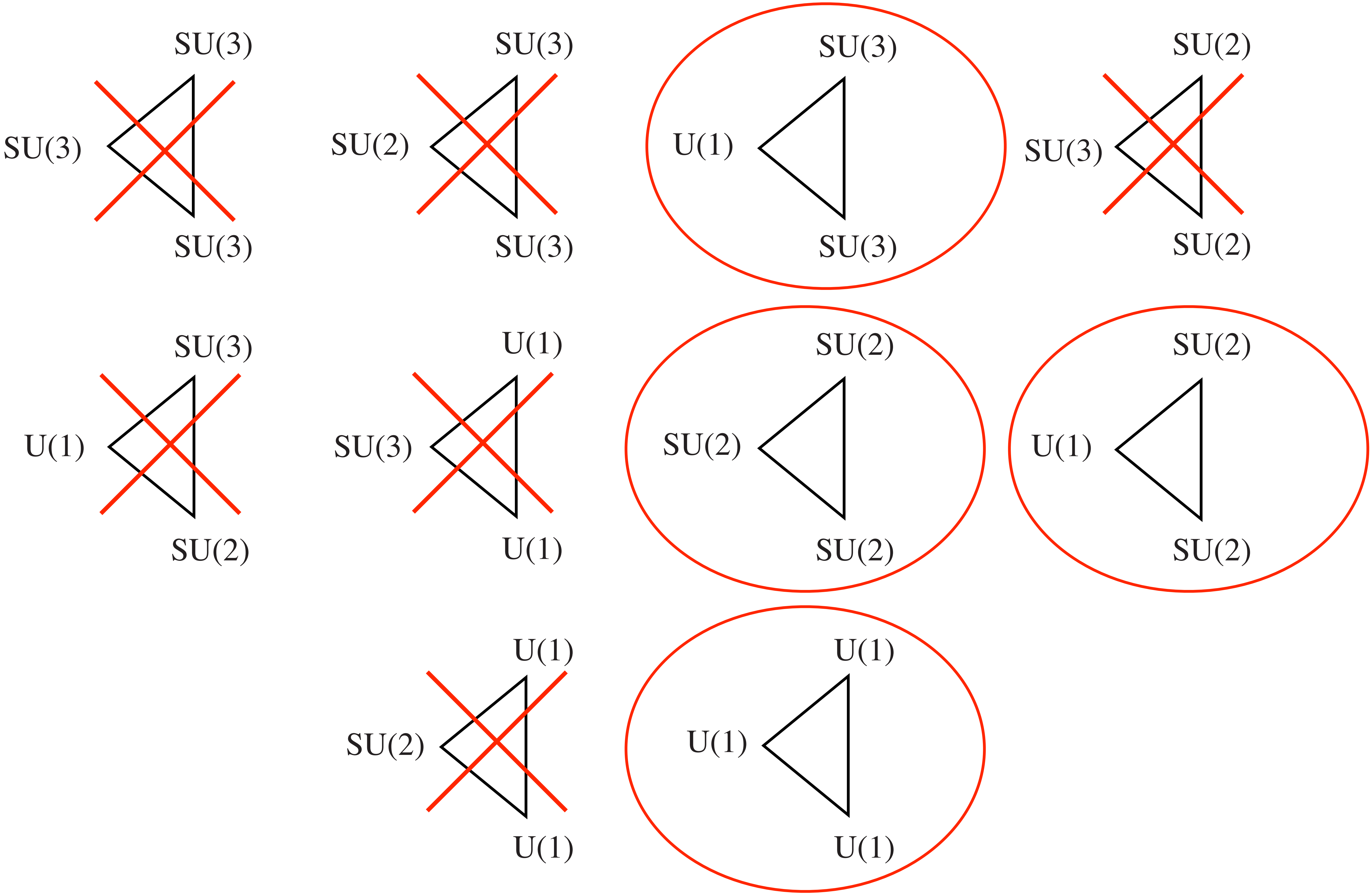
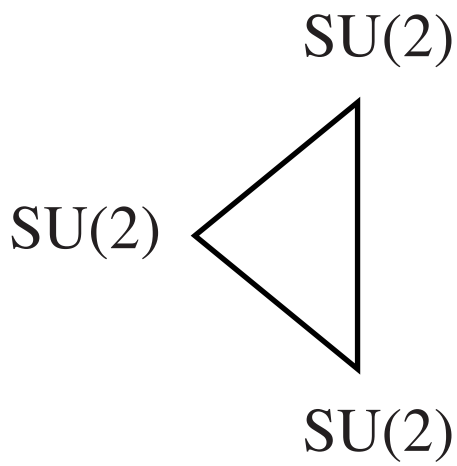
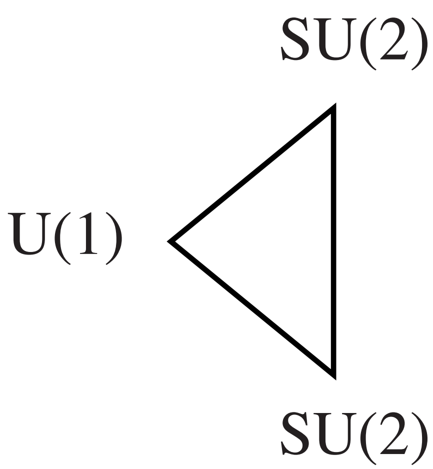


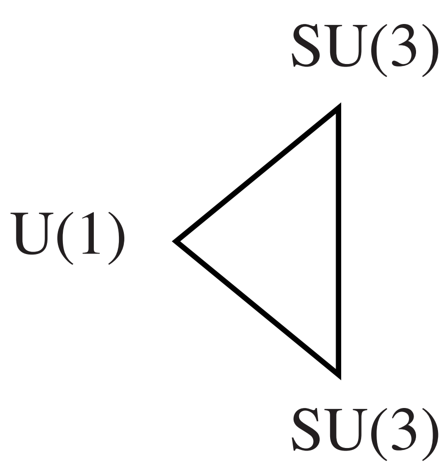



$$= \mathrm{Tr}\left[\sigma_i\{\sigma_j,\sigma_k\}\right] \underset{\substack{\uparrow \\ \{\sigma_j,\sigma_k\} = 2\delta_{jk}}}{=} 2(\mathrm{Tr}\,\sigma_i)\delta_{jk} = 0$$




$$\sim \sum_L Y_L = \underbrace{3\times 2\times\left(\frac{1}{6}\right)}_{\text{quarks}} + \underbrace{2\times\left(-\frac{1}{2}\right)}_{\text{leptons}} = 0$$

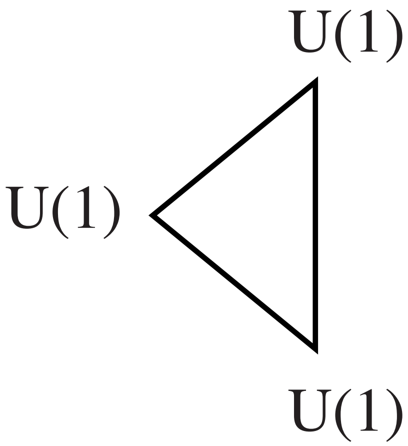
[right-handed fermions do not couple to SU(2)]




$$\sim \sum_{q_L} Y_L - \sum_{q_R} Y_R = 3\times 2\times\left(\frac{1}{6}\right) - 3\times\left(\frac{2}{3}\right) - 3\times\left(-\frac{1}{3}\right) = 0$$

(leptons do not contribute)

The strongest condition comes from:

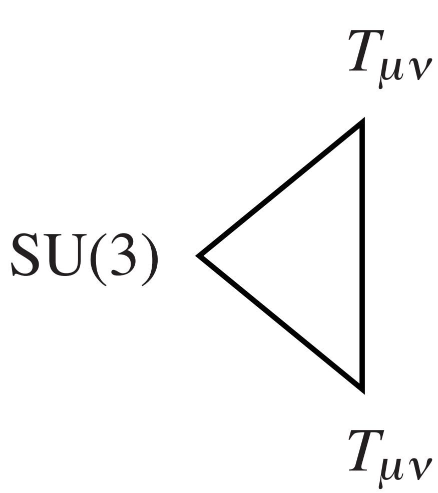


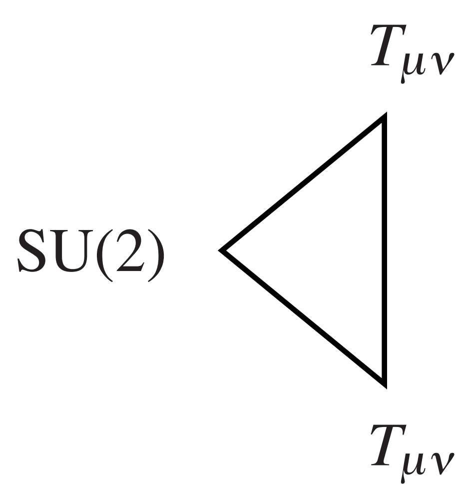

$$= \sum_L Y_L^3 - \sum_R Y_R^3 = 3\times 2\times\left(\frac{1}{6}\right) + 2\times\left(-\frac{1}{2}\right)^3 - 3\times\left(\frac{2}{3}\right)^3$$

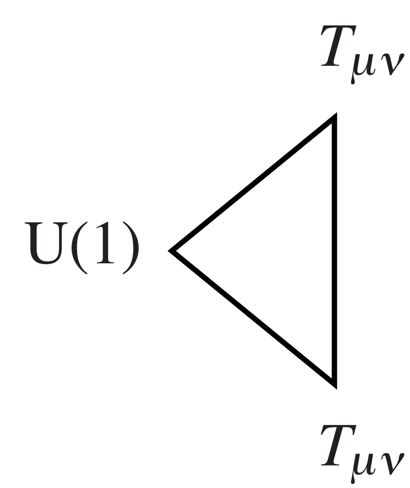
$$- 3\times\left(-\frac{1}{3}\right)^3 - (-1)^3 = \underbrace{\left(-\frac{3}{4}\right)}_{\text{quarks}} + \underbrace{\left(\frac{3}{4}\right)}_{\text{leptons}} = 0$$

Hence, all ***pure gauge anomalies*** cancel in the standard model within each family!

However, we still have to deal with ***mixed gauge-gravitational anomalies***:



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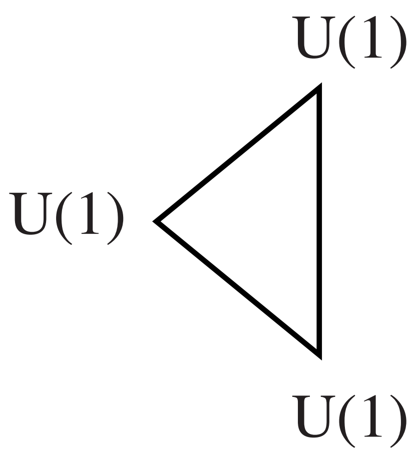


$$= \sum_L Y_L^3 - \sum_R Y_R^3 = 3 \times 2 \times \left(\frac{1}{6}\right) + 2 \times \left(-\frac{1}{2}\right)^3 - 3 \times \left(\frac{2}{3}\right)^3$$

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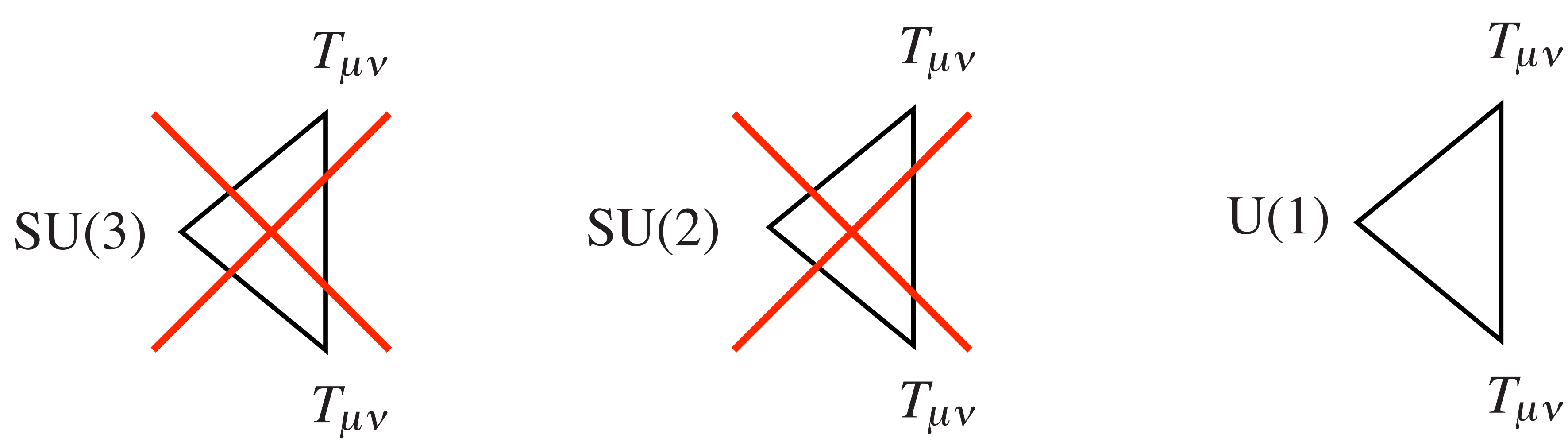


The generator of SU(2) and SU(3) are traceless

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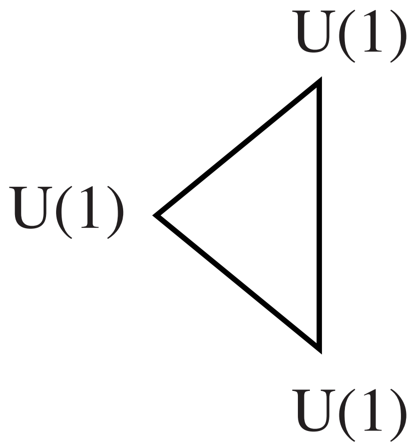


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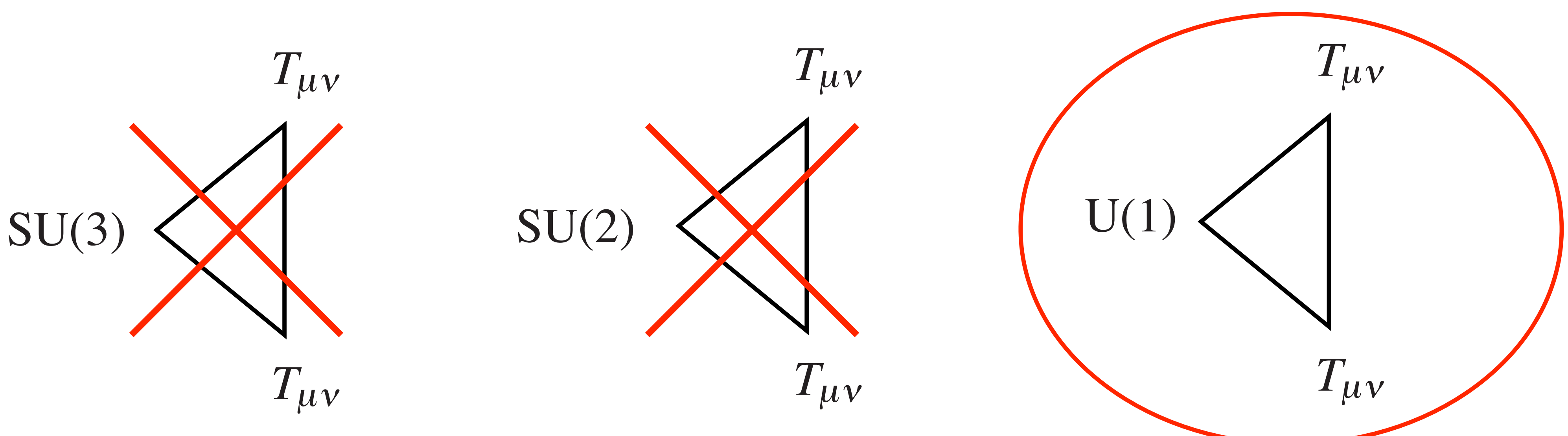


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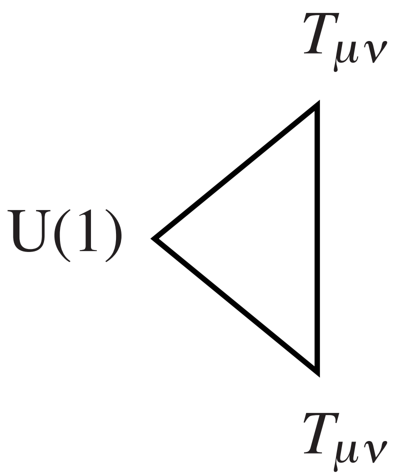



$$= \sum_L Y_L - \sum_R Y_R = 3\times 2\times\left(\frac{1}{6}\right) + 2\times\left(-\frac{1}{2}\right) - 3\times\left(\frac{2}{3}\right) - 3\times\left(-\frac{1}{3}\right) - (-1) = 0$$

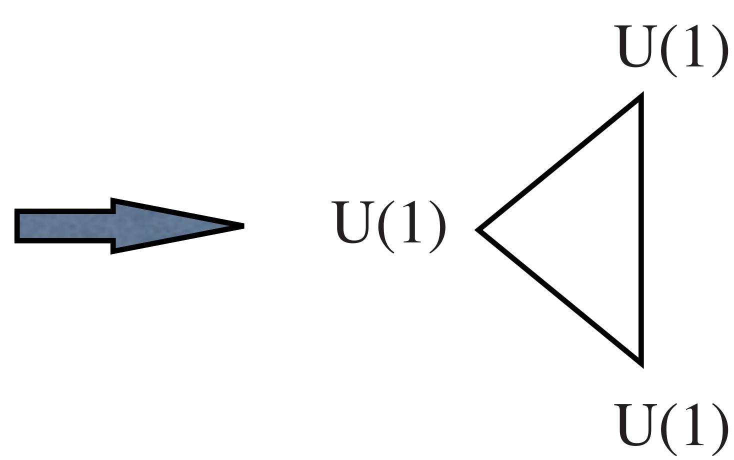
Thus, all ***pure*** and ***mixed*** gauge anomalies ***cancel*** and the standard model is ***anomaly free***!

This cancellation is very delicate and severely constraints any extension of the standard model. For example, the addition of a ***sterile right-handed neutrino*** is innocuous, since it does not contribute to the triangle:

right-handed neutrino: $(1,1)_0^R$

We can also add any number of extra families. An extra lepton (quark), however, makes the theory inconsistent, e.g.

$(1,2)^L_{-\frac{1}{2}}$



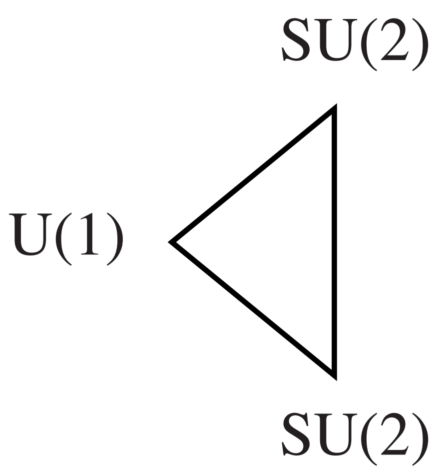

$\neq 0$

## ***Flashback!***

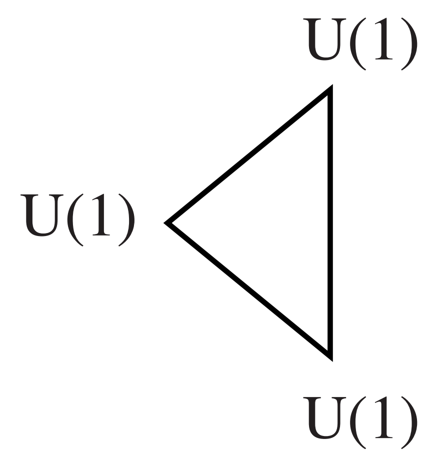
We have seen how quarks and leptons are needed to cancel the anomalies associated with the $\mathrm{SU}(2) \times \mathrm{U}(1)_Y$ sector of the standard model. This illustrate the idea of ***"spectator fermions"*** introduced when talking about ***anomaly matching***.

From the QCD point of view $\mathrm{SU}(2)\times\mathrm{U}(1)_Y$ can be seen as a global symmetry that is gauged by coupling quarks to the electroweak gauge bosons.

This theory, by itself, is anomalous:




$$\sim \sum_{q_L} Y_{q_L} = 3 \times 2 \times \left(\frac{1}{6}\right) = 1$$




$$\sim \sum_{q_L} Y_{q_L}^3 - \sum_{q_R} Y_{q_R}^3 = -\frac{3}{4}$$

To cancel the anomaly, we add the spectator fermions, i.e. the standard model ***leptons***!

These new fermions, however, do not modify the strongly coupled IR dynamics of the quarks.

**MSSM**: let us consider now the minimal supersymmetric extension of the standard model. The spectrum now is doubled

gauge bosons ( $s = 1$) ⟺ gauginos ($s = \frac{1}{2}$)

quarks ( $s = \frac{1}{2}$ ) ⟺ squarks ($s = 0$ )

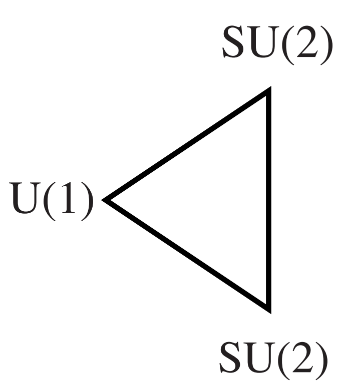
leptons ($s = \frac{1}{2}$ ) ⟺ sleptons ($s = 0$ )

Higgs ($s = 0$ ) ⟺ Higgsino ($s = \frac{1}{2}$)

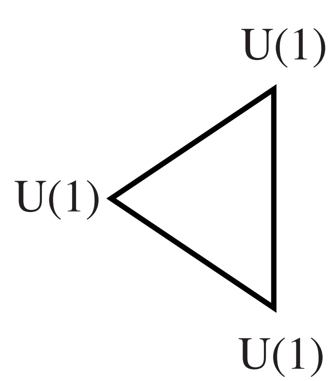
We have learned that all anomalies cancel in the standard model. So we only have to worry about the new chiral fermions:

***gauginos***: $\left\{ \begin{array}{ll} \text{gluino} & (8,1)_0 \\ \text{wino} & (1,3)_0 \\ \text{bino} & (1,1)_0 \end{array} \right.$ ⟹ since the adjoint representation is ***real*** there are no anomalies!

***Higgsino***: $(1,2)_{\frac{1}{2}}$




$\sim \frac{1}{2}$

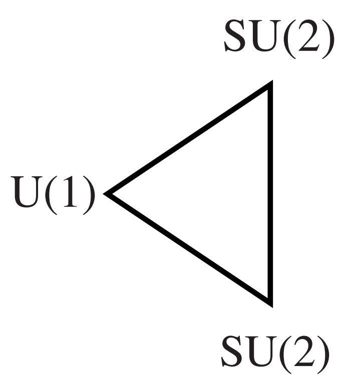



$\sim \left(\frac{1}{2}\right)^3$ ⟹ Thus, the MSSM with a single Higgsino is ***anomalous***!

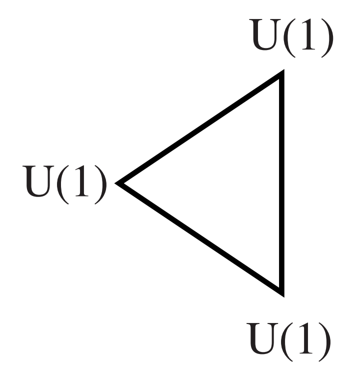
Solving this requres the addition of a ***second Higgs doublet*** with opposite hypercharge. Thus, the MSSM has ***two Higgsinos*** with the same helicity

$$H_1 : \quad (1,2)_{\frac{1}{2}} \qquad\qquad H_2 : \quad (1,2)_{-\frac{1}{2}}$$

and ***all anomalies cancel*** (remember that adding the second Higgs scalar doublet does not contribute to the SM anomaly!)

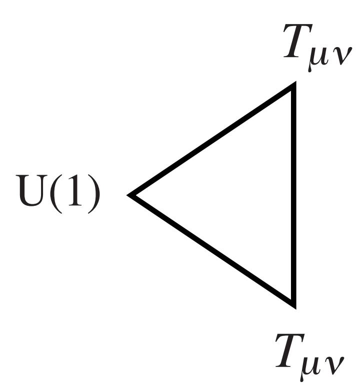



$$\sim \frac{1}{2} + \left(-\frac{1}{2}\right) = 0$$




$$\sim \left(\frac{1}{2}\right)^3 + \left(-\frac{1}{2}\right)^3 = 0$$

as well as the ***mixed gauge-gravitational anomaly***




$$\sim \frac{1}{2} + \left(-\frac{1}{2}\right) = 0$$

Incidentally, the second Higgsino also cancels ***Witten's global anomaly*** (a theory with an odd number of SU(2) doublets is anomalous under "large" gauge transformations)

# Some topics left out of this lectures

## ∗ *Functional methods*

Here we have studied anomalies from a diagrammatic point of view. Functional methods, however, offer a very powerful tool to compute anomalies in arbitrary dimensions.

Given a chiral fermion coupled to a gauge field, we define the (Euclidean) ***fermion effective action***

$$e^{-\Gamma[\mathscr{A}_\mu]_{\text{eff}}} = \int \mathscr{D}\psi \mathscr{D}\overline{\psi} \exp\left[-\int d^{2n}x\, \overline{\psi} \not{D}(\mathscr{A}_\mu) P_+ \psi\right]$$

The gauge anomaly is determined by the ***gauge variation*** of the effective action

$$\begin{aligned}\delta_\varepsilon \Gamma[\mathscr{A}_\mu]_{\text{eff}} &\equiv \Gamma[\mathscr{A}_\mu + D_\mu \varepsilon]_{\text{eff}} - \Gamma[\mathscr{A}_\mu]_{\text{eff}} \\ &= -\int d^{2n}x\, \varepsilon^A(x) \Big(D_\mu \langle j^\mu(x)\rangle_{\mathscr{A}}\Big)^A\end{aligned}$$

The anomaly is associated with the existence of a ***nontrivial transformation*** of the functional integration ***measure***.

The real part of the effective action is always gauge invariant, so the anomaly can only occur in its ***complex part***.

## ∗ *Gravitational anomalies*

When chiral fermions are coupled to gravity they may produce anomalies in the conservation of the energy-momentum tensor

$$\nabla_\mu \langle T^{\mu\nu}(x)\rangle_g \neq 0$$

When can we expect to have pure gravitational anomalies? To answer the question we have to study two different cases:

$D = 4k$: Particles and antiparticles have opposite helicity. Since gravity does not distinguish between them, the gravitational coupling of the fermions "looks" vector-like

$\Longrightarrow$ ***no gravitational anomalies***

$D = 4k+2$: Now, particles and antiparticles have the same helicity. The gravitational coupling is chiral and anomalies might arise.

This includes two important cases: $\begin{cases} D = 2 \\ D = 10 \end{cases}$ $\Longrightarrow$ Green-Schwarz mechanism

Other fields such as gravitinos and self-dual antisymmetric tensors also contribute.

* ***The topological theory of anomalies:***

Axial, gauge, and gravitational anomalies in ***D dimensions*** can be understood in terms of the ***topology*** of the gauge bundle that defines the gauge theory.

- The ***global axial anomaly*** is given by the index of the Dirac operator and can be computed using the Atiyah-Singer index theorem.

- ***Gauge anomalies*** in $2n$ dimensions are related to the index of a Dirac operators defined in $2n+2$ dimensions.

The anomaly can be computed then using the appropriate ***index theorem***.

* ***Global anomalies***

A SU(2) chiral gauge theory can be anomalous with respect to gauge transformations ***not in the connected component of the identity***. If this anomaly is not cancelled, all correlation functions vanish

Witten showed that the theory is ***anomalous*** if the number of SU(2) doublets is ***odd*** (not the case of the standard model!).

# ***Thank you***