QCD vacuum and Instantons

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1 Introduction

In nonabelian gauge theory, and particular in QCD, from topological consideration, there are different vacuum states labeled by different winding number. There are also solutions to classical Yang-Mills theory in Euclidean space, known as instantons. Instantons correspond to tunneling events between different vacuum states. Those tunnelings result the so called "θ vacuum". This parameter θ is the same θ appears in $\int d^4xTr[\frac{ig^2\theta}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}]$, which causes the chiral anomaly we discussed before. This term violates CP symmetry. However, experimental bound of θ is very small. Such small value of θ requires an explanation. This is so called "strong CP problem". We will also see that adding fermions into the pure gauge theory can change the θ dependence of partition function.

2 Vacuum Winding Number

Consider SU(2) gauge theory with gauge fields only $\mathcal{L} = \frac{1}{2}Tr(F^{\mu\nu}F_{\mu\nu})$. The classical field configuration corresponding to the ground state is $F^a_{\mu\nu} = 0$. This implies that the vector potential is $A_\mu = A^a_\mu T^a = \frac{i}{g}U\partial_\mu U^\dagger$, which is a pure gauge transformation of $A_\mu = 0$.

Let us fix temporal gauge $A_0 = 0$, then we can focus on time independent gauge transformations $U(\mathbf{x})$. We also impose the boundary condition that $U(\mathbf{x})$ approaches a particular constant matrix as $|\mathbf{x}| \to \infty$, independent of direction. This is equivalent to adding a spatial "point at infinity" where U has a definite value, and such spatial space has the topology of three-dimensional sphere S^3.

If every $U(\mathbf{x})$ can be smoothly deformed into every other $U(\mathbf{x})$, then all these field configurations are gauge equivalent, and they correspond to a single quantum vacuum state. If every $U(\mathbf{x})$ can not be smoothly deformed into other $U(\mathbf{x})$, then there must be more than one quantum vacuum state. The reason is as follow:

Suppose $U(\mathbf{x})$ cannot be smoothly deformed into $\tilde{U}(\mathbf{x})$. The associated vector potentials $A_\mu = \frac{i}{g}U\partial_\mu U^\dagger$ and $\tilde{A}_\mu = \frac{i}{g}\tilde{U}\partial_\mu \tilde{U}^\dagger$ are both gauge transformations of zero, so both $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ vanish. If we try to smoothly deform A_μ into $\tilde{A}_\mu$, we must pass through vector potentials that are not gauge transformations of zero, and whose field strengths do not vanish. These nonzero field strengths imply nonzero energy, so there is an energy barrier between the field A_μ and $\tilde{A}_\mu$. Therefore they represent two different minima of the hamiltonian in the space of classical field configurations, which correspond to different vacuum states in the quantum theory.

$U(\mathbf{x})$ is a mapping from spatial S^3 to vacuum $SU(2)$ space. Because any elements in $SU(2)$ can be written as $U = u_0 + i\mathbf{u}\cdot\tau$, where τ is the Pauli matrices and u_0 and $\mathbf{u}$ are real satisfying $u_0^2 + \mathbf{u}^2 = 1$ (which follows from $UU^\dagger = 1$). Therefore the manifold of the $SU(2)$ group elements is topologically equivalent to S^3, which we will call vacuum S^3. Thus $U(\mathbf{x})$ provides a map from spatial S^3 to vacuum S^3. From the homotopic classes (see appendix for introduction), we can define a winding number n that counts the number of times the vacuum S^3 covers the spatial $S^3 (\pi_3(S^3) = \mathbb{Z})$.

Given a smooth map $U(\mathbf{x})$, its winding number can be written as

$$n = -\frac{1}{24\pi^2} \int d^3x \epsilon^{ijk} Tr[(U\partial_i U^\dagger)(U\partial_j U^\dagger)(U\partial_k U^\dagger)]$$

From above, we see that SU(2) gauge theory has infinite number of classical field configurations of zero energy, labeled by integer n and separated by energy barriers.

2.1 Simple Example

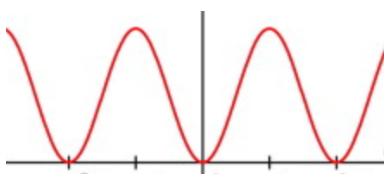

Figure 1. The $V(\phi)$ potential with minima at $\phi = nv$

The infinite number of classical field configurations of zero energy is analogous to a scalar field theory with a potential $V(\phi) = \lambda v^4[1 - cos(2\pi\phi/v)]$. The potential has minima at $\phi = nv$.

Generically, between two quantum states $|n>$ and $|n'>$ that are separated by an energy barrier, there is a tunneling amplitude of the form e^{-S}, where S is the euclidean action for a classical solution of the euclidean field equations that mediates between the field configuration corresponding to n at $t = -\infty$ and the field configuration corresponding to n' at $t = +\infty$. In the scalar field theory, the classical solution of the euclidean field equations is independent of $\mathbf{x}$, therefore S scales like the volume of the space. In infinite volume limit, the tunneling amplitude vanishes. Therefore minima at $\phi = nv$ are degenerated in the quantum theory.

3 Instanton

Here we are trying to find the solution to classical SU(2) Yang-Mills theory in Euclidean space.

At $x_0 = \pm T$, we set $A_\mu(\mathbf{x}) = \frac{i}{g} U_\pm(\mathbf{x}) \partial_\mu U_\pm^\dagger(\mathbf{x})$, where $U_\pm(\mathbf{x})$ has winding number $n_\pm$. At $|\mathbf{x}| = R$, for $-T \leq x_0 \leq T$, we set the boundary condition $A_\mu = 0$, which is equivalent to set $U(\mathbf{x})$ to constant matrix at $|\mathbf{x}| = R$ ($A_\mu = \frac{i}{g} U \partial_\mu U^\dagger$), and we will take T and R to infinity at the end.

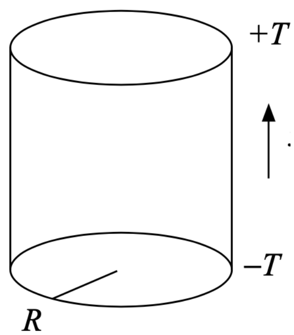

Figure 2. The boundary in euclidean spacetime.

Above cylindrical boundary(caps not included) of four-dimensional spacetime is topologically a S^3. The winding number of the map on this three-sphere is $n_+ - n_-$(think of a line winding around this boundary from bottom to top).

By using the winding number formula given in previous section and noting that the cylindrical wall makes no contribution ($\partial_\mu U^\dagger = 0$), we have the upper cap contributes n_+ and the lower cap contributes $-n_-$(minus sign is due to orientation of the cap).

In large R and T limit, we can consider the boundary of the whole cylinder to be a S^3 at $\rho = \sqrt{x_0^2 + \mathbf{x} \cdot \mathbf{x}} = \infty$. On this boundary, we have map $U(\hat{x} = \frac{x_\mu}{\rho})$ with winding number $n = n_+ - n_-$.

Consider the euclidean action $S = \frac{1}{2} \int d^4x Tr[F^{\mu\nu} F_{\mu\nu}]$ of a field that obeys the boundary condition $lim_{\rho \to \infty} A_\mu(\hat{x}) = \frac{i}{g} U(\hat{x}) \partial_\mu U^\dagger(\hat{x})$ and the field strength is given in terms of the vector potential by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$.

The winding number is

$$n = \frac{1}{24\pi^2}\int dS_\mu \epsilon^{\mu\nu\sigma\tau} Tr[(U\partial_\nu U^\dagger)(U\partial_\sigma U^\dagger)(U\partial_\tau U^\dagger)] = \frac{ig^3}{24\pi^2}\int dS_\mu \epsilon^{\mu\nu\sigma\tau} Tr(A_\nu A_\sigma A_\tau)$$

We want to express the winding number n as volume integral. Let us introduce an unobservable gauge-dependent current (called Chern-Simons current)

$J^\mu = 2\epsilon^{\mu\nu\lambda\rho} Tr[A_\nu F_{\lambda\rho} + \frac{2}{3} ig A_\nu A_\lambda A_\rho]$

It is easy to check that

$\partial_\mu J^\mu = 2Tr[F_{\mu\nu}\tilde{F}^{\mu\nu}]$

where dual of $F_{\mu\nu}$ is $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F^{\lambda\rho}$.

On the surface at infinity, the vector potential is a gauge transformation of 0, so $F_{\mu\nu} = 0$ there, thus

$n = \frac{g^2}{32\pi^2}\int dS_\mu J^\mu = \frac{g^2}{32\pi^2}\int d^4x \partial_\mu J^\mu = \frac{g^2}{16\pi^2}\int d^4x Tr[F_{\mu\nu}\tilde{F}^{\mu\nu}]$

where the integrand in the last expression is gauge-invariant.

Note that $\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} = F^{\mu\nu}F_{\mu\nu}$, therefore

$\frac{1}{2}Tr[\tilde{F}_{\mu\nu} \pm F_{\mu\nu}]^2 = Tr[F^{\mu\nu}F_{\mu\nu}] \pm Tr[\tilde{F}^{\mu\nu}F_{\mu\nu}] \geq 0 \Rightarrow S = \frac{1}{2}\int d^4x Tr[F^{\mu\nu}F_{\mu\nu}] \geq \frac{1}{2}|\int d^4x Tr[\tilde{F}^{\mu\nu}F_{\mu\nu}]| = 8\pi^2|n|/g^2$

We obtain the minimum value of the euclidean action for a solution of the euclidean field equations the mediates between vacuum configurations with winding numbers $n_\pm$ at $x_0 = \pm\infty$, where $n_+ = n_- + n$. And this minimum is achieved when $F_{\mu\nu} = \pm\tilde{F}_{\mu\nu}$, which means that the self-dual of antiself-dual fields are the finite-action solutions to the classical Euclidean Yang-Mills theory.

Notice that the tunneling amplitude is of form $e^{-S} \approx e^{-8\pi^2/g^2}$, therefore this is an effect that cannot be seen in ordinary perturbation theory since $e^{-8\pi^2/g^2}$ is extremely small for small g.

3.1 Example of SU(2) gauge transformation

Consider SU(2) gauge transformation $U(x) = \frac{x_0 + i\mathbf{x}\cdot\sigma}{\rho}$, where σ is the Pauli matrices vector. This gives rise to gauge field $A_\mu(x) = (\frac{\rho^2}{\rho^2+\lambda^2})\frac{i}{g}U\partial_\mu U^\dagger$, where λ is some arbitrary scale parameter called instanton size(the existence of solution of arbitrary sizers is a necessary consequence of the scale invariance of the classical field theory). For $\rho >> \lambda$, we have $A_\mu(x) \rightarrow \frac{i}{g}U\partial_\mu U^\dagger$ as required by the boundary condition. More explicitly, we can write $A_0(x) = \frac{-i\mathbf{x}\cdot\sigma}{\rho^2+\lambda^2}, \mathbf{A}(x) = \frac{-i(\sigma x_0+\sigma\times\mathbf{x})}{\rho^2+\lambda^2}$. One can check the corresponding action integral has value $8\pi^2/g^2$, so it is an instanton centered at the spacetime origin with winding number 1. We say that the instanton is localized in both Euclidean space and time.

We can try to rewrite above gauge field expression in temporal gauge $A_0 = 0$. We can make a gauge transformation first on above gauge field expression $A_\mu(x)$: $A'_\mu(x) = V^\dagger A_\mu(x)V(x) + V^\dagger(x)\partial_\mu V(x)$. The condition of temporal gauge is $A'_0(x) = 0$, which implies $\partial_0 V(x) = -A_0(x)V(x) = \frac{i\mathbf{x}\cdot\sigma}{x_0^2+\mathbf{x}^2+\lambda^2}V(x) \Rightarrow V(x) = exp[\frac{i\mathbf{x}\cdot\sigma}{(\mathbf{x}^2+\lambda^2)^{1/2}}(tan^{-1}(\frac{x_0}{(\mathbf{x}^2+\lambda^2)^{1/2}}) + \theta_0)]$, where θ_0 is the integration constant. We can set $\theta_0 = (n+\frac{1}{2})\pi$. If we take the spatial component $A_i(x)$ to be zero at $x_0 = \pm\infty$, then $A'_i(x_0 = \pm\infty) = V^\dagger(x)\partial_i V(x)$ with $V(x_0 = -\infty) = exp[i\pi\frac{i\mathbf{x}\cdot\sigma}{(\mathbf{x}^2+\lambda^2)^{1/2}}n]$ and $V(x_0 = +\infty) = exp[i\pi\frac{i\mathbf{x}\cdot\sigma}{(\mathbf{x}^2+\lambda^2)^{1/2}}(n+1)]$. Therefore we see $A_0(x) = \frac{-i\mathbf{x}\cdot\sigma}{\rho^2+\lambda^2}, \mathbf{A}(x) = \frac{-i(\sigma x_0+\sigma\times\mathbf{x})}{\rho^2+\lambda^2}$ indeed connects two vacuum states that differ by one unit on winding number.

If we want to have tunneling process whose winding numbers differ by more than 1, we can construct a mediating solution $n = n_+ - n_-$ by patching together n_+ instantons and n_- anti-instantons(with winding number -1) whose centers are widely separated on scale set by their sizes.

Remark: Adding scalar fields has no effect on our analysis. Changing from SU(2) to another simple nonabelian group also has no effect; instanton solutions always reside in an SU(2) subgroup(this is due to a theorem by Raoul Bott, stating that any continuous mapping of S^3 into G can be continuously deformed into a mapping into an SU(2) subgroup of G). If the gauge group is U(1), there are no instantons, and hence no vacuum angle($\pi_3(S^1) = 0$ as stated in appendix).

4 θ vacuum

A gauge transformation with winding number n changes the ground state from $|n_- >$ to $|n_- + n >$. Therefore if we want a gauge invariant ground state, we need to construct a superposition of those ground states with different winding numbers.

Define the θ vacua $|\theta >= \sum_{n=-\infty}^{\infty} e^{-in\theta}|n >$.

If we act on θ vacua with a gauge transformation U_1 with winding number 1, we have

$U_1|\theta >= U_1 \sum_{n=-\infty}^{\infty} e^{-in\theta}|n >= \sum_{n=-\infty}^{\infty} e^{-in\theta}U_1|n >$

$= \sum_{n=-\infty}^{\infty} e^{-in\theta}|n+1> = \sum_{n'=-\infty}^{\infty} e^{i(-n'+1)\theta}|n'> = e^{i\theta}\sum_{n'=-\infty}^{\infty} e^{-in'\theta}|n'>$
$= e^{i\theta}|\theta>$

We see that $|\theta>$ is an eigenstate of U_1 with eigenvalue $e^{i\theta}$. Since the Hamiltonian is invariant under gauge transformations ($[U_1, H] = 0$), $|\theta>$ is also energy eigenstate.

Consider a gauge invariant operator B. Gauge invariance means $[U_1, B] = 0$ or $U_1 B U_1^\dagger = B$, therefore $0 =< \theta|B - U_1 B U_1^\dagger|\theta' >=< \theta|B|\theta' > -e^{-i(\theta'-\theta)} < \theta|B|\theta' >= (1 - e^{-i(\theta'-\theta)}) < \theta|B|\theta' >$, which means that $< \theta|B|\theta' >= 0$ unless $\theta' = \theta + 2\pi l$, with $l \in \mathbb{Z}$. If we restrict $-\pi < \theta \leq \pi$, then $< \theta|B|\theta' >= 0$ unless $\theta' = \theta$, which means that the value of θ can not be changed by a gauge invariant operator, and hamiltonian is gauge invariant operator as well, so the value of θ does not change as time moves on. Therefore $|\theta>$ is really an energy eigenstate. Therefore different θ corresponds to different theory, as different value of coupling constant describes a different theory.

4.1 Understanding θ through Condensed Matter Physics

Consider electron moving in 1d periodic potential, with potential minimum separated by a distance a. From Bloch's theorem, we know that the energy eigenstates for an electron in a crystal can be written as Bloch waves $\psi(x) = e^{ikx}u(x)$, where x is the position, $u(x)$ is a periodic function with the same periodicity as the potential and k is the crystal momentum.

Suppose x_0 is the position of a potential minima, then from Bloch wave, we have the ground state wavefunction $\psi(x_0 + a) = e^{ikx_0}e^{ika}u(x_0 + a) = e^{ika}e^{ikx_0}u(x_0) = e^{ika}\psi(x_0)$, where $x_0 + a$ is another position of potential minima. Then the true ground state wavefunction should be superposition of the wavefunction of all minima. There are different superposition, characterized by crystal momentum k. We see that θ is analogous to crystal momentum k.

4.2 Physics of θ and strong CP problem

Consider the euclidean path integral with boundary condition with states of winding number $n_\pm$ at $x_0 = \pm\infty$. The only field configurations that contribute are those with winding number $n_+ - n_-$. We can write

$$Z_{n_+ \leftarrow n_-}(J) = \int \mathcal{D}A_{n_+ - n_-} e^{-S + \int d^4x Tr[J^\mu A_\mu]}$$

where we only integrate over fields with winding number $n_+ - n_-$, and $Z_{n_+ \leftarrow n_-}(J)$ only depends on $n = n_+ - n_-$ and not separately on n_+ and n_-.

If now we are interested in starting with a particular theta vacua $|\theta>$ and ending with $|\theta'>$, then using the definition of θ vacua, the corresponding path integral is

$$Z_{\theta'\leftarrow\theta}(J) = \sum_{n_-,n_+} e^{i(n_+\theta'-n_-\theta)} Z_{n_+\leftarrow n_-}(J) = \sum_{n_-,n} e^{i((n_-+n)\theta'-n_-\theta)} Z_{n_+\leftarrow n_-}(J)$$
$$= \sum_{n_-,n} e^{in\theta'+in_-(\theta'-\theta)} Z_{n_+\leftarrow n_-}(J)$$

Since $e^{in\theta'} Z_{n_+\leftarrow n_-}(J)$ only depends on n, summing over n_-, we get $\delta(\theta'-\theta)$. Thus we have

$$Z_{\theta'\leftarrow\theta}(J) = \delta(\theta'-\theta) \sum_n e^{in\theta} Z_{n_+\leftarrow n_-}(J)$$

Drop the delta function, and define

$$Z_\theta(J) = \sum_n e^{in\theta} Z_{n_+\leftarrow n_-}(J) = \sum_n e^{in\theta} \int \mathcal{D}A_n e^{-S+\int d^4x Tr[J^\mu A_\mu]}$$

By combing the sum over n and the integral over A_n into an integral over all A, and using $n = \frac{g^2}{16\pi^2}\int d^4x Tr[F_{\mu\nu}\tilde{F}^{\mu\nu}]$, we get

$$Z_\theta(J) = \int \mathcal{D}A e^{\int d^4x Tr[-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{ig^2\theta}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + J^\mu A_\mu]}$$

Putting the path integral back into Minkowski space, we get

$$Z_\theta(J) = \int \mathcal{D}A e^{i\int d^4x Tr[-\frac{1}{2}F^{\mu\nu}F_{\mu\nu} - \frac{g^2\theta}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} + J^\mu A_\mu]}$$

The θ term above causes chiral anomaly we discussed before.

We can write $Tr[F_{\mu\nu}\tilde{F}^{\mu\nu}] = Tr[4E_iB_i]$, where E_i and B_i are the "electric field" and "magnetic field" of $F_{\mu\nu}$ respectively.

Under parity transformation, $E_i \to -E_i, B_i \to B_i$ and under time-reversal transformation, $E_i \to E_i, B_i \to -B_i$, therefore θ term is not invariant under parity(P) or time-reversal(T or CP since CPT is always invariant). Therefore θ term implies CP violation in QCD. However, experimental measurement of θ suggest that $|\theta| < 2\times 10^{-10}$. Such a small value of θ requires for a good explanation, and this is called the strong CP problem.

4.3 QCD with massless fermions

Consider QCD with one flavor of massless quark, represented by a Dirac field Ψ in the fundamental representation of gauge group SU(3). The path integral is

$$Z = \int \mathcal{D}A\mathcal{D}\Psi\mathcal{D}\bar{\Psi} exp[i\int d^4x(i\bar{\Psi}\not{D}\Psi - \frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} - \frac{g^2\theta}{32\pi^2}\tilde{F}^{a\mu\nu}F^a_{\mu\nu})]$$

where we have not written the source terms explicitly.

We know that $U(1)_A$ symmetry($\Psi \to e^{-i\alpha\gamma_5}\Psi, \bar{\Psi} \to \bar{\Psi}e^{-i\alpha\gamma_5}$) is anomalous, meaning that

$$\mathcal{D}\Psi\mathcal{D}\bar{\Psi} \to exp[-i\int d^4x \frac{g^2\alpha}{16\pi^2}\tilde{F}^{a\mu\nu}F^a_{\mu\nu}]\mathcal{D}\Psi\mathcal{D}\bar{\Psi}$$

Therefore the effect of a $U(1)_A$ transformation is to change the value θ to $\theta + 2\alpha$. Since the value of θ can be changed by $U(1)_A$ transformation with arbitrary α(which is equivalent to change of dummy integration variable in the path integral), we conclude that Z does not depend on θ. Therefore adding a massless quark has turned θ into a physically irrelevant, unobservable parameter.

We see that in original pure QCD without matter, Z_θ depends on θ. Why adding a fermion changes the θ dependence of Z? We can perform the integral over the quark field, and we have

$$Z = \int \mathcal{D}A\mathcal{D}\Psi\mathcal{D}\bar{\Psi}exp[i\int d^4x(i\bar{\Psi}\not{D}\Psi - \frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} - \frac{g^2\theta}{32\pi^2}\tilde{F}^{a\mu\nu}F^a_{\mu\nu})] = \int \mathcal{D}A det(i\not{D})e^{-i\int d^4x \frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu}}e^{in\theta}$$

where n is the winding number. We see that Z would be independent of θ if gauge fields with gauge fields with nonzero winding number did not contribute. This will be the case if $det(i\not{D})$ vanishes for gauge fields with $n \neq 0$. Therefore $i\not{D}$ must have a zero eigenvalue or zero mode whenever the gauge field has nonzero winding number.

4.4 QCD with massive fermions

Now consider adding a mass term for the quark. If we write the Dirac field Ψ in terms of two left-handed Weyl fields χ and ξ, $\Psi = \begin{pmatrix}\chi \\ \xi^\dagger\end{pmatrix}$, then the mass term is $\mathcal{L}_{mass} = -m\chi\xi - m^*\xi^\dagger\chi^\dagger = -|m|\bar{\Psi}e^{-i\phi\gamma_5}\Psi$, where $m = |m|e^{i\phi}$. A $U(1)_A$ transformation changes ϕ to $\phi + 2\alpha$ and θ to $\theta + 2\alpha$, we see that $\phi - \theta$ or equivalently $me^{-i\theta}$ is unchanged. Because we want to rotate away the mass phase, we want $-2\alpha = \phi$, then θ parameter is fixed to $\theta - \phi$. Therefore the path integral depends on $me^{-i\theta}$, but not on m and θ separately.

If there are more quark fields, the mass term is $\mathcal{L} = -M_{ij}\chi_i\xi_j + h.c.$. $U(1)_A$ transformation changes the phase of every χ_i and ξ_i by $e^{i\alpha}$, so every matrix element of M picks up a factor $e^{2i\alpha}$. Simultaneously, θ changes to $\theta + 2N\alpha$, where N is the number of quark fields. Thus $(detM)e^{-i\theta}$ is invariant under $U(1)_A$ transformation. Then the path integral depends on $(detM)e^{-i\theta}$, not on $(detM)$ and θ separately.

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A Homotopic classes

Let X and Y be two topological spaces and $f_0(x), f_1(x)$ be two continuous functions from X to Y. Let I denote the unit interval on the real line $0 \leq t \leq 1$. f_0 and f_1 are said to be homotopic if and only if there is a continuous function $F(x,t)$ which maps the direct product of X and I to Y such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. The continuous function $F(x,t)$ which deforms the function $f_0(x)$ continuously into $f_1(x)$ is called the homotopy. We can then divide all functions from X to Y into homotopic classes such that two functions are in the same class if they are homotopic.

A.1 Homotopy groups of spheres

Homotopy groups of spheres describe how spheres of various dimensions can wrap around each other, and they are topological invariants.

The n-dimensional unit sphere - called the n-sphere(denote as S^n) - generalizes the familiar circle(S^1) and the ordinary sphere(S^2). The n-sphere may be defined geometrically as the set of points in a Euclidean space of dimension n+1 located at a unit distance from the origin. The i-th homotopy group $\pi_i(S^n)$ summarizes the different ways in which the i-dimensional sphere S^i can be mapped continuously into the n-dimensional sphere S^n. If two mappings can be continuously deformed into each other, they belong to the equivalence classes of mappings. An "addition" operation defined on these equivalence classes makes the set of equivalence classes into an abelian group.

We notice that $\pi_3(S^1) = 0$, which means that there is no winding number for abelian $U(1)$ group(the manifold of the $U(1)$ group elements is topologically equivalent to S^1).

A.2 Example: $S^1 \to S^1$

Let X be the points on a unit circle labelled by the angle $\{\theta\}$, with θ and $\theta + 2\pi$ identified. Let Y be a set of unimodular complex numbers $u_1 = \{e^{i\sigma}\}$, which is topologically equivalent to a unit circle S^1. Consider the mapping $\{\theta\} \to \{e^{i\sigma}\}$. The continuous functions given by

$$f(\theta) = exp[i(n\theta + a)]$$

form a homotopic class for different values of a and a fixed integer n. This is because we can construct a homotopy

$$F(\theta, t) = exp[i(n\theta + (1-t)\theta_0 + t\theta_1)]$$

such that

$$f_0(\theta) = exp[i(n\theta + \theta_0)]$$

and

$$f_1(\theta) = exp[i(n\theta + \theta_1)]$$

are homotopic. One can visualize $f(\theta)$ as a mapping of a circle onto another circle. In this mapping n points of the first circle ($\frac{2k\pi}{n}$, where $k = 0, 1, ..., n-1$) are mapped into one point of the second circle. We can think of this as "winding around it n times". Thus, each homotopic class is characterized by the winding number n(also called the Pontryargin index). From expression of $f(\theta)$, the winding number n for $f(\theta)$ can be written as

$$n = \int_0^{2\pi} \frac{d\theta}{2\pi} [\frac{-i}{f(\theta)} \frac{df(\theta)}{d\theta}]$$

Of particular interest is the mapping with lowest nontrivial winding number $n = 1$:

$$f^{(1)}(\theta) = e^{i\theta}$$

By taking powers of this mapping, we can get mappings of higher winding numbers. For instance, the mapping $[f^{(1)}(\theta)]^m$ has winding number m.

We can also write $f^{(1)}(\theta) = x + iy$ with $x^2 + y^2 = 1$

We can generalize the domain X of this mapping from the unit circle to the whole real line $-\infty \leq x \leq \infty$, by identifying $x = \infty$ and $x = -\infty$ to be the same point ($f(x = \infty) = f(x = -\infty)$).

Example of this type of mapping with winding number $n = 1$ are

$f_1(x) = exp[i\pi x/(x^2 + \lambda^2)^{1/2}], f_1'(x) = exp[i2sin^{-1}(x/(x^2 + \lambda^2)^{1/2})] = \frac{(\lambda+ix)^2}{\lambda^2+x^2}$

where λ is an arbitrary number. And the winding number for a general mapping is

$n = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx[\frac{-i}{f(x)} \frac{df(x)}{dx}]$.