Lecture Notes of Quantum Field Theory

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Lecture 1

1.1 Introduction

At first, I will introduce a cube of physics shown below, where the horizontal axis is about the speed of motion in a physical system, leftside for slow and rightside for fast, and where the vertical axis is about the scale of a physcical system, upside for small and downside for big. Considering this property, we can fill the cube with dominant theories in physics, namely left-downside for classical physics, right-downside for special relativity, left-upside for quantum mechanics and finally right-upside for quantum field theory(QFT), a marrigae of special relativity and quantum mechanics. However, with all above, we can just fill up a planar table but not a cube, which must have the other dimension behind. And thus it is gravity theory, such as general relavity. It is challenging to build a quantum theoretical gravity today. Roughly, we can say Einstein has two daughters, special relativity and general relativity, among which one has married with quantum but the other has not.

small	Quantum Mechanics	Quantum Field Theory
big	Classical Physics	Special Relativity
	slow	fast

Then let us focus on quantum field theory. It is a qualitatively new physics for its combination of special relativity and quantum mechanics and for the valid discription of creation and annihilation of particles. In special relativity, we have the well-known mass energy relation $E = mc^2$ which means energy can be transformed into particle in principle. However, since we have energy conservation, this transformation process is not allowed. On the other hand, in quantum mechanics, we have Heisenberg uncertainty relation: $\Delta t = \frac{\hbar}{\Delta E}$, which means energy fluctuation is allowed and can be very large when time counted is short enough. Nevertheless, in (non-relativistic) mechanics, energy is not equivalent with mass and thus transformation between particle and energy is also forbidden. We all knows when we write down Schrodinger equation, $i\hbar\frac{\partial}{\partial t}\psi = H\psi$, the number of particles will keep constant forever even though the state can vary a lot. Only combining these two theories can we build a valid theory to discribe creation and annihilation of particles, namely quantum field theory. Moreover, combining both theories, a large range of phenomena can be exposed, such as Hawking radiation, which originated from the fluctuation of energy near blackhole.

Here, I should draw our attention to the new physics around recent 20 years-condensed matter physics, in which we have gradually use quantum field theory to handle with relative problems. We will not give lectures about condensed matter but instead show how important and effective application of quantum field theory to this field. It can be found in any standard textbook for condensed matter that the irons vibrate around their equilibrium lattice position. And these vibrations can be translated as excitiated state(or elementary excitation) and dipicted as disturbation of ground state, which be can calculated with the method of Feynman diagrams in quantum field theories. They are essentially equivalent.

1.2 Path Integral

Quntum field theory can be formulated in various ways, our focus is the one invented by Dirac and Feynman, path integral formulation. Maybe there will be lots of student knowing path integral clearly, I will continue introducing it considering some of you are not familar with this formulation.

At first, we know Newton's law $F = m\ddot{q}$, where $\dot{q} = \frac{dq}{dt}$ and $\ddot{q} = \frac{d^2q}{dt^2}$, and from which we can solve the dynamical variable $q(t)$ as function of time t. However, we can use lagrangian formalism to rewrite this law. In this formalism, we should first write down action

$$S=\int_0^T dt\mathcal{L}(q,\dot{q}) \tag{1.2.1}$$

and then extremize S(action principle) to obtain the equation of motion. Extremization can be both maximum and minimum. Although it is easy to check that this extremized action S yields the Newton's law for motion, the attitudes for physical fomalism are completely different and so for comprehension. Indeed action extremization method can be seeds of quantum mechanics because the classical limit of path integral is action extremizaiton, even though encountering classical problems, action extremization is much messy than Newton's simple law $F=m\ddot{q}$.

There is a little story about Feynman. It is said that when Feynman was in high school and studied $F=m\ddot{q}$, he was amazed. I always critize today's American students for their disinterest when they learn new knowledge. I think we should not only be amazed, but also be amused by such wonderful equations. I remember that Mr. Yang likes to remark 'miao' in Chinese to praise such significant physical theories. In my opinion, 'miao' can be understood as amazed and amused, to some extent. I will remark the mysterious role of formalism in theoretical physics. According to my research experience, papers about new formalism are not attractive enough for others because they are just equivalent forms of pre-existing theories. However, a good fomalism can deeply influence future physics, such as the variation method in lagrangian formalism which can be used even in quantum mechanics and general relativity though it is invented by Euler in 18th century. As an example, Einstein has spent 10 years to build GR(general relativity). However the action of GR is surprisingly simple, if Einstein had ever used lagrangian, he would bring out GR in a much easier way. Even for today's gauge theory, maybe we also need a better formalism to rewrite it in a more neat way, as there are lots of gauge redundent degrees of freedom should be fixed in temporary gauge theory.

From action principle, we can deduce Euler equation:

$$\frac{\delta\mathcal{L}}{\delta q}=\frac{d}{dt}\frac{\delta\mathcal{L}}{\delta\dot{q}} \tag{1.2.2}$$

which is equivalent with $F=m\ddot{q}$. And given such a lagrangian $\mathcal{L}(q,\dot{q})$, we can write down Hamiltonian

$$H(p,q)=p\dot{q}-\mathcal{L}(q,\dot{q}) \tag{1.2.3}$$

where $p=\frac{\delta\mathcal{L}}{\delta\dot{q}}$ and which is indeed a Legendre transformation. Method of Hamiltonian is another formalism, which is substantially important in quantum mechanics even in classical physics, it is insignificant. In hindsight, formalism can be foreshadow of later development. For instance, the extremization of action in phase space indicates path integral; when operators are permitted in Hamiltonian, it becomes quantum Hamiltonian; Poisson bracket $\{A,B\}$ inspired Dirac to deform it to Heisenberg commutation bracket $[A,B]$ and so on. Sometimes, a good formalism is not panacea. Path integral is a salient example. When I was a postgraduate, nobody uses path integral to calculate quantum problems. Instead, we all use Schwinger formalism. This is because path integral is too messy when solving simple problems, such as finite deep well problem. Nonetheless, no one can deny the central position of path integral in quantum field theory.

Let us deduce path integral formalism. I will firstly clarify that path integral formalism is invented by Dirac and then recovered by Feynman independently. That we customarily name path integral as Feynman path integral is not fair for Dirac. (Please forgive me for neglecting the prevalent old story about Feynman and his high school physics teacher and turn to Dirac's work directly) Dirac rewrite the Heisenberg formalism

$$\langle q_f|e^{-iHt}|q_i\rangle \tag{1.2.4}$$

in the form where he splits $\exp(-iHt)$ into many peices, each one of which occupies $\delta t=T/N$ and $N\to\infty$, namely in the form of

$$\langle q_f|e^{-iHt}|q_i\rangle=\langle q_f|e^{-iH\delta t}...e^{-iH\delta t}|q_i\rangle \tag{1.2.5}$$

Then we insert identity

$$1=\int dq_j|q_j\rangle\langle q_j| \tag{1.2.6}$$

between every two $e^{-iH\delta t}$s and thus we have

$$\langle q_f|e^{-iHt}|q_i\rangle=\langle q_f|e^{-iH\delta t}|q_{N-1}\rangle\langle q_{N-1}|...\langle q_{j+1}|e^{-iH\delta t}|q_j\rangle...|q_1\rangle\langle q_1|e^{-iH\delta t}|q_i\rangle \tag{1.2.7}$$

For each part $\langle q_{j+1}|e^{-iH\delta t}|q_j\rangle$, we have Hamiltonian

$$H=\frac{\hat{p}^2}{2m}+V(\hat{q}) \tag{1.2.8}$$

and inserting identity $\int \frac{dp}{2\pi}|p\rangle\langle p| = 1$, for simplicity setting $V(\hat{q}) = 0$(leaving the calculation about nontrivial potential as an excercise) we obtain

$$\begin{aligned}
\langle q_{j+1}|e^{-iH\delta t}|q_j\rangle &= \int \frac{dp}{2\pi}\langle q_{j+1}|e^{-iH\delta t}|p\rangle\langle p|q_j\rangle \\
&= \int \frac{dp}{2\pi}\langle q_{j+1}|e^{-i\frac{p^2}{2m}\delta t}|p\rangle\langle p|q_j\rangle \\
&= \int \frac{dp}{2\pi}e^{-i\frac{p^2}{2m}\delta t}e^{ip(q_{j+1}-q_j)} \\
&= \int \frac{dp}{2\pi}\exp\left[-i\frac{\delta t}{2m}\left(p-\frac{(q_{j+1}-q_j)}{\delta t}m\right)^2+i\frac{m}{2\delta t}(q_{j+1}-q_j)^2\right] \\
&= \sqrt{\frac{-im}{2\pi\delta t}}\exp\left[i\frac{m}{2\delta t}(q_{j+1}-q_j)^2\right] \qquad (1.2.9)
\end{aligned}$$

where we have used the Gausian integral $\int_{-\infty}^{\infty}dxe^{-ax^2} = \sqrt{\frac{\pi}{a}}$. Thus we have

$$\begin{aligned}
\langle q_f|e^{-iHt}|q_i\rangle &= \left(\frac{-im}{2\pi\delta t}\right)^{\frac{N}{2}}\prod\int\cdots\int dq_1dq_2...dq_N\exp\left[i\frac{m}{2\delta t}(q_{j+1}-q_j)^2\right] \\
&= \left(\frac{-im}{2\pi\delta t}\right)^{\frac{N}{2}}\prod\int\cdots\int dq_1dq_2...dq_N\exp\left[i\frac{m}{2}\dot{q}^2\delta t\right] \\
\text{(including potential terms)} &\to \left(\frac{-im}{2\pi\delta t}\right)^{\frac{N}{2}}\prod\int\cdots\int dq_1dq_2...dq_N\exp\left[i\frac{m}{2}\dot{q}^2\delta t - V(q)\right] \qquad (1.2.10)
\end{aligned}$$

The terms in exponent is indeed the lagrangian $\mathcal{L}(q,\dot{q})$, thus we have

$$\langle q_f|e^{-iHt}|q_i\rangle = \int\limits_{\text{all paths}}\mathcal{D}qe^{i\int_0^T dt\mathcal{L}(q,\dot{q})} \qquad (1.2.11)$$

where $\mathcal{D}q = \left(\frac{-im}{2\pi\delta t}\right)^{\frac{N}{2}}\prod\int\cdots\int dq_1dq_2...dq_N$ is the measure of path integral. A particularly nice feature of path integral formalism is the classical limit can be easier recovered. In (1.2.11) we have omitted $\hbar$, which should appear on the exponent as $\frac{i}{\hbar}\int_0^T dt\mathcal{L}(q,\dot{q})$. In classical limit, $\hbar \to 0$, the integral asymptotically tends to be $e^{(i/\hbar)\int_0^T dt\mathcal{L}(\dot{q}_c,q_c)}$ where $q_c(t)$ is the classical path determined by solving the Euler-Lagrangian equation(refering to Anthony Zee's book chapter I.2). This means in classical limit contributions from all paths cancel out except for the classical one. (A little story: when I was student, I had never heard of path integral in America. It was two Russian bring it to America during 1960s)

Finally let us talk about the concept of field. We have such a mattress shown in Figure 1.2.1.

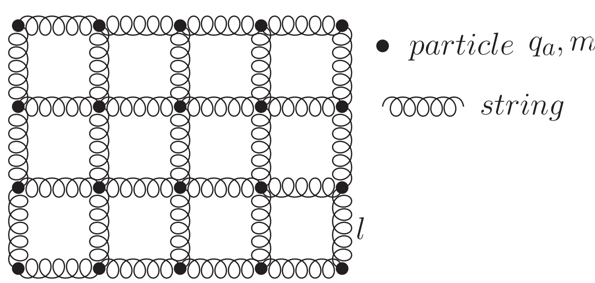

Figure 1.2.1: the mattress model for fields

where black blocks are masses located at q_a, and spiral lines are strings whose lenght is l. The lagrangian can be easily write down for this system:

$$L = \sum_a \frac{1}{2}m\dot{q}_a^2 - \sum_{ab}k_{ab}(q_a-q_b)^2 - \sum_{abc}g_{abc}q_aq_bq_c - ... \qquad (1.2.12)$$

To reach the classical limit, we should just simply take the limit $l \to 0$ and thus the mattress bocomes a robber sheet whose size length $A \gg l$. In this way, we can list a dictionary for analogy of field and strings and masses.

$a \to \vec{x}$, label$\to$location
$q_a(t) \to \psi(t, \vec{x})$
$\sum \to \int d^D x$, D is the spatial dimension

In this table, there is one thing we should pay special attention that in this figuration, $\vec{x}$ namely former $q(t)$ is NOT the dynamical variable, unlike the case in classical mechanics, but just a location label. The actual dynamical variable is $\psi(t, \vec{x})$. We furthermore introduce relativistic notation $x = (t, \vec{x})$ and thus the action becomes

$$\begin{aligned} S &= \int dt L(q, \dot{q}) \\ &= \int dt \int d\vec{x} \mathcal{L}(\psi, \frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial x^i}) \end{aligned} \tag{1.2.13}$$

where $\mathcal{L}$ is the lagrangian density which can be rewrite from (1.2.12) as

$$\mathcal{L} = \frac{1}{2}\left(\frac{\partial \psi}{\partial t}\right)^2 - \frac{1}{2}\sum_{i=1}^{D}\left(\frac{\partial \psi}{\partial x^i}\right)^2 - \frac{1}{2}m^2\psi^2 - g\psi^3 - \lambda\psi^4 - ... \tag{1.2.14}$$

where the first term results from the first term in (1.2.12) except absorbing m into ψ; the second term is written in the same way except absorbing k_{ab} into x^i; other terms cannot be written without any coefficients as first two terms for no extra variables to absorbing coefficients. Here comes one confusion. There are two ms appearing both in (1.2.12) and (1.2.14). We must be careful that they are different-the one in (1.2.12) is the mass of black point in the mattress and the one in (1.2.14) is the mass of particle discribed by field ψ. Indeed, in quantum field theory, particles are activation of vacuum and fields are the disciption of such activation whereas the black blocks are not activation. When we push our black blocks up and down, we will set up wave packet which is kind of activation, and we use such wave packets to denote particles with definite features. Furthermore, we can rewrite (1.2.14) in a more neat way when we introducing Mikovski metric $\eta^{\mu\nu} = diag(1, -1, -1, -1)$. This yields

$$\frac{1}{2}\left(\frac{\partial \psi}{\partial t}\right)^2 - \frac{1}{2}\sum_{i=1}^{D}\left(\frac{\partial \psi}{\partial x^i}\right)^2 \equiv \frac{1}{2}\partial^\mu\psi\partial_\mu\psi = \frac{1}{2}(\partial^\mu\psi)^2 \tag{1.2.15}$$

where $\partial_\mu\psi = \frac{\partial \psi}{\partial x^\mu}$.

Finally, I will put forward my final remark about ground state. In quantum mechanics, ground state of a system is highly nontrivial and need to be solved with Schrodinger equation. However, in quantum field theory, ground state is vacuum which seems nothing, but in fact is very interesting. Our disposal with vacuum is introduced by Dyson-disturbing the vacuum. As stated above, when we stress the strings we can set up wave packets which are a kind of disturbation. In formalism, we change lagrangian L as

$$L \to L + \sum_a q_a(t) J_a(t) \tag{1.2.16}$$

where $q_a(t)$ is the dynamical variable and $J_a(t)$ represents our choice how we disturb the vacuum. In path integral, we write it in lagrangian density form

$$\int d^d x \mathcal{L} \to \int d^d x \left[\mathcal{L}(\psi(x), \partial_\mu\psi(x)) + \psi(x)J(x)\right] \tag{1.2.17}$$

We can define the deformed path integral

$$Z(J) \equiv \int \mathcal{D}\psi e^{i\int d^4x\left(\frac{1}{2}(\partial_\mu\psi)^2 - V(\psi) + J\psi\right)} \tag{1.2.18}$$

which is called Schwinger functional integral.

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Lecture 2

2.1 Path Integral(continued)

Last lecture, we reach the Schwinger functional integral(also called Schwinger 'sorcery' when I was a student)

$$Z(J) \equiv \int \mathcal{D}\psi e^{i\int d^4x\left(\frac{1}{2}(\partial_\mu\psi)^2 - V(\psi) + J\psi\right)} \tag{2.1.1}$$

where J is a source. However, our path integral cannot be calculated exactly except $V = \frac{1}{2}m^2\psi^2$, which is called harmonic oscillator potential. Indeed, most theoretical physics are based on harmonic paradigm, and implement disturbation to it.

Before we calculate the functional integral with harmonic oscillator potential, I will give a review of Gausian integral, namely,

$$\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}ax^2} = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} \tag{2.1.2}$$

where the coefficient a is not important unless we do dimensional analysis. However, it is much more difficult for us to calculate some integral even a little more complicated, such as the one with quartic term: $\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}ax^2 - \lambda x^4}$. We are only able to evaluate the interal with at most additional linear term Jx. Thus we have

$$\begin{aligned}\int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}ax^2 + Jx} &= \int_{-\infty}^{+\infty} dx e^{-\frac{1}{2}a(x^2 - \frac{2Jx}{a})} \\ &= \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} e^{+\frac{J^2}{2a}}\end{aligned} \tag{2.1.3}$$

This is called self reproducing that a Gaussian integral produce anther Gaussian term $e^{+\frac{J^2}{2a}}$, except an inversed sign(this is very important). Ok, then we can generalize this approach to N dimensional integral:

$$\int \cdots \int_{-\infty}^{+\infty} dx_1 \cdots dx_N e^{-\frac{1}{2}x\cdot A\cdot x + J\cdot x} \tag{2.1.4}$$

where A is a $N \times N$ matrix and J is a N dimensional vector. We can diagonalize A by $A = R^{-1}DR$, and we can integrate (2.1.4) for each eigen vector after diagonalization. Thus we get the coefficient similar with $(2\pi/a)^{1/2}$:

$$\left(\frac{(2\pi)^N}{\prod_i d_i}\right)^{\frac{1}{2}} = \left(\frac{(2\pi)^N}{\det A}\right)^{\frac{1}{2}} \tag{2.1.5}$$

where d_i are eigen values for A. Finally we evaluate the integrala (2.1.4) as

$$\int \cdots \int_{-\infty}^{+\infty} dx_1 \cdots dx_N e^{-\frac{1}{2}x\cdot A\cdot x + J\cdot x} = \left(\frac{(2\pi)^N}{\det A}\right)^{\frac{1}{2}} e^{\frac{1}{2}JA^{-1}J} \tag{2.1.6}$$

where the coefficient before $e^{\frac{1}{2}JA^{-1}J}$ is vacuum energy because we can denote by $Z(0)$, the funtional integral without external sources. However, when we apply this method to path integral, we will encounter mathimatical rigor problem while our physicists are not interested in such problems, and instead we pay more attention to the validity of a theory.

Since the functional integral (2.1.1) is not convergent when ψ goes large and $Z(J)$ fluctuates a lot, we should regularize it first. The easiest way is invented by Feynman, although it is not a rigorous way, that changing m^2 to $m^2 - i\epsilon$, where $\epsilon \to 0^+$. In this way, the exponent has a minus term proportional to ϵ,i.e.

$$e^{i\int d^4x\left(\frac{1}{2}(\partial_\mu\psi)^2-\frac{1}{2}m^2\psi^2-V(\psi)+J\psi\right)} \to e^{-\epsilon\int\psi^2\cdots} \tag{2.1.7}$$

which results an convergent integral. For harmonic oscillator potential, neglecting the bound condition, we have

$$\begin{aligned} Z(J) &= \int \mathcal{D}\psi e^{i\int d^4x\left(\frac{1}{2}(\partial_\mu\psi)^2-\frac{1}{2}m^2\psi^2+J\psi\right)} \\ &= \int \mathcal{D}\psi e^{i\int d^4x\left(\frac{1}{2}\psi[-(\partial^2+m^2)]\psi+J\psi\right)} \end{aligned} \tag{2.1.8}$$

And then we should find inverse for operator of $-(\partial^2+m^2)$, just like that in (2.1.6). As an analogy with the discrete condition

$$A_{ij}(A^{-1})_{jk} = \delta_{ik} \tag{2.1.9}$$

we can set the inverse of $-(\partial^2+m^2)$ as $D(x,y)$ which satisfies

$$-(\partial_x^2+m^2)D(x,y) = \delta^{(4)}(x-y) \tag{2.1.10}$$

The solution must have the form of $D(x-y)$, which is invariant under translation of $x-y \to x'-y'$ and remaining $x-y = x'-y'$, because both operators, $-(\partial_x^2+m^2)$ and $\delta^{(4)}(x-y)$, manifest such symmetry. Since $\delta^{(4)}(x-y) = \int \frac{d^4k}{(2\pi)^4}e^{ik(x-y)}$ and $-(\partial^2+m^2)e^{ik(x-y)} = (k^2+m^2)e^{ik(x-y)}$, the exact solution can be derived by Fourier transformation, namely

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2-m^2+i\epsilon} \tag{2.1.11}$$

We should pay attention the infinitesimal imaginary part in the denominator introduced by Feynman can effectively avoid pole problem in integral. $D_F(x-y)$ is called Feynman propagator which is analogous with Green function in electrodynamics, and we will add another subscript F to $D(x-y)$ to denote this kind of propagator(In appendix, we will meet other kinds of propagators). It is very important to do physics among all things I learn. By the way, I want to give an advice to all of you: *Understand physics before evaluating*, in Germany, it is *Erst der Denken Dann der Integral*.

So confronting this integral, we will not calculate it at first but look for its physical meaning since the integral indeed is not difficult to calculate by set up a contour and integrate it around the pole. This will be left as an excercise(for ones unfamiliar with this, see appendix). Now we consider the integral in another way. The numerator in (2.1.11) is a plane wave and the waves are summed by the weight $\frac{1}{k^2-m^2}$. Obviously, the term with $k^2 \approx m^2$ contributes most to this summation. Thus we name this condition $k^2 = m^2$ as mass shell condition(or on-shell condition). We can probe the physical meaning of this condition by the way as follows. For the de Broglie wave of a particle, $E = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$, moreover, we have Einstein relation, $E^2 = \vec{p}^2 + m^2$. We find the mass shell condition is indeed Einstein relation which means the mass m in the lagrangian of field is actually the mass of particles. For $\vec{p} = 0$, we have $E = m$, energy for a rest particle.

The propagator in a light cone discribes the propagation towards future as shown in Figure 2.1.1. However, since the Heisenberg uncertainty relation, the coordinate and momentum are uncertain at the vertex of light cone, this means that there is a leakage outside the light cone. Indeed we can evaluate the propagator $D(x-y)$ with $x^0 = y^0$ as a nonzero result(for subtle calculation, see Appendix). However, this does not contradict with casuality, because casuality is defined as $[\psi(x), \psi(y)] = 0$ for spacelike internal of x and y. In other words, nonzero $D(x-y)$ in spacelike condition does not yield observable physical result.

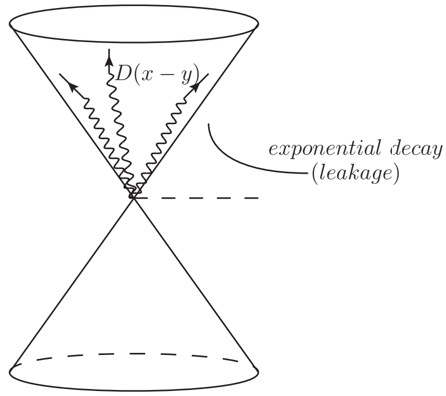


Figure 2.1.1: Leakage outside the light cone

Moreover, I will define $W(J)$ as

$$\begin{aligned} Z(J) &= Z(0)e^{iW(J)} \\ &= C\exp\left(-\frac{i}{2}\int d^4xd^4yJ(x)D(x-y)J(y)\right) \end{aligned} \tag{2.1.12}$$

where we should notice that the minus sign is very important. Someone might ask a question that why in (2.1.8) we only have one integrated virable, here we have two? The reason is that when we write

$$JA^{-1}J = \sum_{ij} J_i(A^{-1})_{ij}J_j \tag{2.1.13}$$

i, j are two index to be summed corresponding to dx and dy in (2.1.12). $C \sim \left(\frac{(2\pi)^N}{\det A}\right)^{\frac{1}{2}} = Z(0)$ and the term $e^{iW(J)} = \exp\left(-\frac{i}{2}\int d^4xd^4yJ(x)D(x-y)J(y)\right)$ are sum of all disconnected Feynman diagrams while if we add nontrivial potential term in $W(J)$, it becomes sum of connected diagrams. With Fourier transformation, we can rewrite (2.1.12) as

$$W(J) = -\frac{1}{2}\int d^4xd^4yJ(x)D(x-y)J(y) = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}J^*(k)\frac{1}{k^2-m^2+i\epsilon}J(k) \tag{2.1.14}$$

2.2 Exploit Freedom of Choosing $J(x)$

How to choose $J(x)$ means how to disturb the vacuum and we can set up it by experimental physics. Now, I will illustrate an example to show how we do this.

In Figure 2.2.1, we can see a visual interpretation of $W(J)$. J_1 is accelerator and J_2 is detector. All paths from accelerator to detector make contributions under the modulation of $\frac{1}{k^2-m^2+i\epsilon}$ because of the integral over k in (2.1.14). Among them, only the ones near the pole of $\frac{1}{k^2-m^2+i\epsilon}$ dominate the total summation. It turns out the mass shell condition. By the way, the response theory and response function in condense matter physics is related with our propagator.

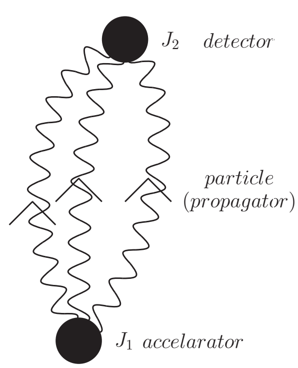


Figure 2.2.1: visual interpretation of $W(J)$

For example, we can build a couple of J_1 and J_2 as follows:

$$\begin{aligned} J(x) &= J_1(x) + J_2(x) \\ J_1(x) &= \delta^{(3)}(\vec{x} - \vec{x}_1) \\ J_2(x) &= \delta^{(3)}(\vec{x} - \vec{x}_2) \end{aligned} \tag{2.2.1}$$

which are independent on time. The picture is shown in Figure 2.2.2.

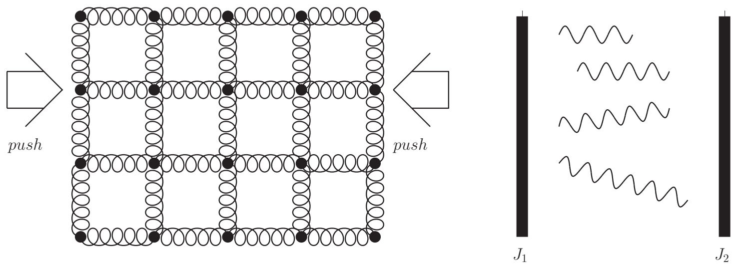


Figure 2.2.2: An example of sources

The right one is the discription of J and the left is the analogy with string mattress where we push on both sides with heavy masses. Our goal is to oberve how J_1influence J_2 but not that with itself. In otherwords, our interst is the interaction between the two external sources. This is analogous with observing the wave propagating on the mattress between the two heavy masses.

That is

$$\begin{aligned} W(J) &= -\frac{1}{2}\int d^4x d^4y (J_1+J_2)(x) D(x-y)(J_1+J_2)(y) \\ &= -\frac{1}{2}\int d^4x d^4y \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2-m^2+i\epsilon}(\delta^{(3)}(\vec{x}-\vec{x}_1)+\delta^{(3)}(\vec{x}-\vec{x}_2))(\delta^{(3)}(\vec{y}-\vec{x}_1)+\delta^{(3)}(\vec{y}-\vec{x}_2)) \\ &= -\frac{1}{2}\int dx^0 \int dk^0 \frac{dy^0}{2\pi} e^{ik^0(x^0-y^0)} \int d^3x d^3y \int \frac{d^3k}{(2\pi)^3}\frac{e^{-i\vec{k}\cdot(\vec{x}-\vec{y})}}{k^2-m^2+i\epsilon}(\delta^{(3)}(\vec{x}-\vec{x}_1)\delta^{(3)}(\vec{y}-\vec{x}_2)+\delta^{(3)}(\vec{y}-\vec{x}_1)\delta^{(3)}(\vec{x}-\vec{x}_2)) \\ &= \int dx^0 \int \frac{d^3k}{(2\pi)^3}\frac{e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2+m^2} \end{aligned} \tag{2.2.2}$$

where we have omitted the self-interaction terms and where we should notice we have left a integral over dx^0 which is time.

2.3 Field, Force, and Particle

According to (2.2.2) and definition of $W(J)$, we can immediately get the correspondence between $W(J)$ and energy E.

We know

$$\begin{aligned} Z(J) &\equiv \langle 0, J|e^{-iHT}|0, J\rangle \\ &= e^{-iET} \\ &\equiv e^{iW(J)} \end{aligned} \tag{2.3.1}$$

where E is the energy due to the presence of the sources. Thus we have

$$ET = -W(J) = -\int dx^0 \int \frac{d^3k}{(2\pi)^3}\frac{e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2+m^2} \tag{2.3.2}$$

If we set $\int dx^0$ as T, we get (setting $|\vec{x}_1 - \vec{x}_2| = r$)

$$\begin{aligned} E &= \int \frac{d^3k}{(2\pi)^3} \frac{e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2+m^2} \\ &= \int \frac{dk d\cos\theta d\phi k^2}{(2\pi)^3} \frac{e^{-ikr\cos\theta}}{k^2+m^2} \\ &= i\int \frac{dk}{(2\pi)^2 r} \frac{ke^{-ikr}}{k^2+m^2} \\ &= -\frac{1}{4\pi r} e^{-mr} \end{aligned} \tag{2.3.3}$$

This is the well-known Yukawa potential! That the energy is negative suggests the interaction between J_1 and J_2 is attraction and when $r \to \infty$, the energy will decay to zero in a exponential form. Since the exponent is $-mr$, we know the range of force caused by this energy is $1/m$. For $m = 0$ particle, such as photon, the force range is infinite, which means that electromagnetic force is long range force. And the potential reduces to $-\frac{1}{4\pi r}$, consistent with Coulomb's law.

Here we should pay more attention the relations between field, particle an force. Traditionally, force is absolutely different from particle. In history, we only have the connection between force and field as Faraday find electromagnetic field to discribe the electromagnetic force and that between field and particle as Einstein find electromagnetic field can be discribed as photon, in a view of particle. However, when Yukawa found this significant potential, our comprehension about particle and field changed revolutionarily. Since the force range is limited by the mass of the particle discribed by field, we can interprete force as the exchange of particle between two sources. In addtion, from this point of view, we can predict the mass of the interaction particle as long as we know the force range. It is in this way that Yukawa has succeeded in predicting the mass of pion.

There are two kind of particles with zero mass, namely, photon and graviton. Thus as is known to all, only electromagnetic force and gravitational force are long range forces among all interactions we know today. Moreover, massless particle has a deep connection with gauge invariance. OK, one may ask a question that why we must have $1/r$ law with massless particle? Is there some deep reasons? We can proceed this discussion first in dimensinal analysis:

$$(m=0) \to E \sim \int d^3k \frac{e^{ikr}}{k^2} : \left[\frac{k^3}{k^2}\right] \sim [k] \sim \frac{1}{r} \tag{2.3.4}$$

Then to find the origin of $1/k^2$. It is from the $(\partial\psi)^2$ term in lagrangian. Then how can we have this term? It is because the rotational invariance of the system(at least this reason). What about the origin of k^3? It is from the spatial dimension-3. Both of these two aspects result in $1/r$ law. In other words, if the spatial dimension is not 3 but other value, we de facto have other forms of potential.

Appendix: Casuality

$D_F(x-y)$ can be evaluated in integral along contour on complex plane. There are two pole points on the complex plane:$\omega_\pm = \pm\sqrt{\vec{k}^2 + m^2 - i\epsilon}$(setting $\omega = \omega_+$) for k^0 as shown in Figure A.2.1.

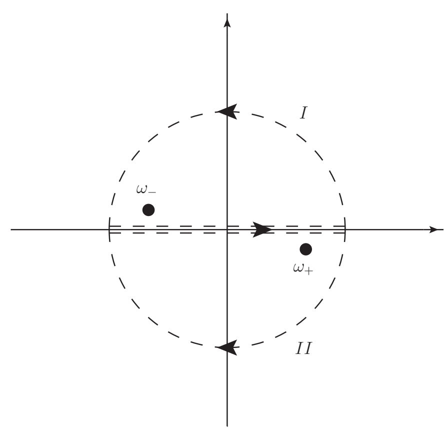


Figure A.2.1: Contour for integral

In this figure, we have two contours, contour I is for $x^0 > y^0$ and II for $x^0 < y^0$. For $x^0 > y^0$, the integral can be calculated as

$$\begin{aligned}
D_F(x-y) &= \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2 + i\epsilon} \\
&= \int \frac{d^3k}{(2\pi)^4} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \int_{-\infty}^{+\infty} dk^0 \frac{e^{ik^0(x^0-y^0)}}{(k^0)^2 - (\vec{k}^2 + m^2 - i\epsilon)} \\
&= \int \frac{d^3k}{(2\pi)^4} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \times 2\pi i \mathrm{Res}\left[\frac{e^{ik^0(x^0-y^0)}}{(k^0)^2 - \omega_\pm^2}\right]_{k^0=\omega_-} \\
&= \int \frac{d^3k}{(2\pi)^4} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \times 2\pi i \frac{e^{-i\omega(x^0-y^0)}}{-2\omega} \\
&= -i \int \frac{d^3k}{2\omega(2\pi)^3} e^{-i\vec{k}\cdot(\vec{x}-\vec{y}) - i\omega(x^0-y^0)}
\end{aligned} \tag{A.2.1}$$

For $y^0 < x^0$, we can get similar result. The final result for $D(x-y)$ is

$$D_F(x-y) = -i \int \frac{d^3k}{2\omega(2\pi)^3} e^{-i\vec{k}\cdot(\vec{x}-\vec{y}) - i\omega|x^0-y^0|} = -i \int \frac{d^3k}{2\omega(2\pi)^3} \left[e^{-ik(x-y)}\theta(x^0-y^0) + e^{ik(x-y)}\theta(y^0-x^0)\right] \tag{A.2.2}$$

where we have considered the freedom of flipping $\vec{k}$ to $-\vec{k}$ withour any change of result. When we calculate $D(x-y)$ in spacelike condition, we can set $x^0 = y^0$ and thus (A.2.2) becomes

$$\begin{aligned}
D_F(x-y) &= -i \int \frac{d^3k}{2\omega(2\pi)^3} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \\
&= -i \int \frac{k^2 dk d\cos\theta d\phi}{2(2\pi)^3} \frac{e^{-ikr\cos\theta}}{\sqrt{k^2+m^2}} \\
&= \int_0^\infty \frac{kdk}{2r(2\pi)^2} \frac{e^{-ikr} - e^{ikr}}{\sqrt{k^2+m^2}} \\
&= \int_{-\infty}^{+\infty} \frac{dk}{2r(2\pi)^2} \frac{ke^{-ikr}}{\sqrt{k^2+m^2}}
\end{aligned} \tag{A.2.3}$$

This integral can be evaluated as $\sim e^{-mr}$ when $r = |\vec{x} - \vec{y}| \to \infty$(see Peskin Page 27-28 despite the convergence does not seem to be proved rigorously). This seems to break casuality, since propagator is amplitude transmission and should be valid only in the light cone(Figure 2.1.1) whereas when in spacelike interval, there is a leakage in the form of approximately exponential decay. However, casuality is actually remained in quantum field theory because this is just a misunderstanding of casuality. The microscopic casuality of a local field should be expressed as $[\psi(x), \psi(y)] = 0$ in spacelike condition of x and y. This means the measurement of $\psi(x)$ and $\psi(y)$ cannnot be influenced when we exchange the order of measurement, i.e. the measurement of one point cannot influence the measurement of another spacelike point. The calculation can be found in Peskin, chapter 2. Here we just cite the result(fitting with A. Zee's notation):

$$\begin{aligned}
[\psi(x), \psi(y)] &= \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_p}} \int \frac{d^3q}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_q}} \left[(a_p e^{-ipx} + a_p^\dagger e^{ipx}), (a_q e^{-iqy} + a_q^\dagger e^{iqy})\right] \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \left(e^{-ip(x-y)} - e^{ip(x-y)}\right) \\
&= i\left(D_F(x-y) - D_F(y-x)\right)
\end{aligned} \tag{A.2.4}$$

where we have used $\psi(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_p}} (a_p e^{-ipx} + a_p^\dagger e^{ipx})$ and $[a_p, a_q^\dagger] = \delta^{(3)}(p-q)$. If $x^0 = y^0$, (A.2.4) is obviously vanishing and thus casuality is remained. For spacelike condition, one can make a Lorentz transformation with $[\psi(x), \psi(y)] \to [\psi(x'), \psi(y')]_{x'^0 = y'^0}$ and noticing (A.2.4) is Lorentz invariance and thus (A.2.4) always vanishes for spacelike interval. It is the guarantee of casuality.

Moreover, we can set up a more exact expression for why $[\psi(x), \psi(y)] = 0$ is the casuality(Ref. Quntum Field Theory, L. Brown). We can calculate the following expected value:

$$\langle 0, -\infty, J | \psi(x) | 0, -\infty, J \rangle \tag{A.2.5}$$

which is the expected value for a field in the initial vacuum with a source J(this is different from our ordinary calculation for scattering cross section, where the ket should be $|0, +\infty, J\rangle$). This physical interpretation is that the amplitude of existing a local field at x when there is a source in the vacuum, and it has nothing to do with some propagation from one place to another place. In other words, this expected value will tell us how sources influence the field. In order to calculate this expected value, we should first use Lagrangian-Euler equation to write down the equation of motion for $\psi(x)$ with nonvanishing $J(x)$

$$(\partial^2 + m^2)\psi(x) = J(x) \tag{A.2.6}$$

Then we have retarded Green function $D_R(x-y)$ satisfying

$$(\partial_x^2 + m^2)D_R(x-y) = -\delta^{(4)}(x-y) \tag{A.2.7}$$

where $x^0 > y^0$. Thus we get the solution for $\psi(x)$ as

$$\psi(x) = \psi_0(x) - \int d^4y D_R(x-y)J(y) \tag{A.2.8}$$

where $\psi_0(x)$ satisfies $(\partial^2 + m^2)\psi_0(x) = 0$. Then the calculation is straight forward:

$$\begin{aligned}
\langle 0, -\infty, J|\psi(x)|0, -\infty, J\rangle &= \sum_n \langle 0, -\infty, J|n, +\infty, J\rangle\langle n, +\infty, J|\psi(x)|0, -\infty, J\rangle \\
&= \sum_n \langle n, +\infty, J|0, -\infty, J\rangle^* \langle n, +\infty, J|\psi(x)|0, -\infty, J\rangle \\
&= \langle 0, +\infty, J|0, -\infty, J\rangle^* \langle 0, +\infty, J|\psi_0(x) - \int d^4y D_R(x-y)J(y)|0, -\infty, J\rangle \\
&= -\langle 0, +\infty, J|0, -\infty, J\rangle^* \langle 0, +\infty, J| \int d^4y D_R(x-y)J(y)|0, -\infty, J\rangle
\end{aligned} \tag{A.2.9}$$

where $\langle 0, +\infty, J|0, -\infty, J\rangle = Z(J)$ can be expanded as series of J and $\langle n, +\infty, J|\psi_0(x)|0, -\infty, J\rangle$ is zero when substituting the free field conditon. The expected value above can be dipicted in the diagram in Figure A.2.2.

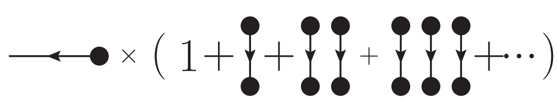
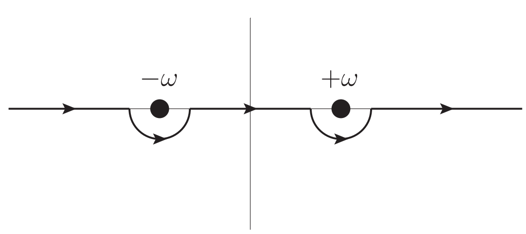


Figure A.2.2: Diagram for $\langle 0, -\infty, J|\psi(x)|0, -\infty, J\rangle$

In definition of $D_R(x-y)$ we have set that $x^0 > y^0$ and if not $D_R(x-y) = 0$. We can follow the method of (2.1.11) and get

$$D_R(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2 - m^2} \tag{A.2.10}$$

where we should notice that the integral path is below



Thus we can calculate the integral exactly. Choosing the contour as the upper semi-plane, we get

$$\begin{aligned}
D_R(x-y) &= \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2-m^2} \\
&= \int \frac{d^3k}{(2\pi)^4} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \int_{-\infty}^{+\infty} dk^0 \frac{e^{ik^0(x^0-y^0)}}{(k^0)^2-(\vec{k}^2+m^2)} \\
&= \int \frac{d^3k}{(2\pi)^4} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \times 2\pi i \mathrm{Res}\left[\frac{e^{ik^0(x^0-y^0)}}{(k^0)^2-\omega^2}\right]_{k^0=\pm\omega} \\
&= \int \frac{d^3k}{(2\pi)^4} e^{-i\vec{k}\cdot(\vec{x}-\vec{y})} \times 2\pi i \left(\frac{e^{-i\omega(x^0-y^0)}}{-2\omega} + \frac{e^{i\omega(x^0-y^0)}}{2\omega}\right) \\
&= -i \int \frac{d^3k}{2\omega(2\pi)^3} \left(e^{-ik(x-y)} - e^{ik(x-y)}\right) \\
&= -i[\psi(x),\psi(y)]\theta(x^0-y^0)
\end{aligned} \tag{A.2.11}$$

Thus (A.2.9) becomes

$$\langle 0,-\infty,J|\psi(x)|0,-\infty,J\rangle = i\int d^4y\theta(x^0-y^0)\langle 0,+\infty,J|0,-\infty,J\rangle^* \langle 0,+\infty,J|[\psi(x),\psi(y)]J(y)|0,-\infty,J\rangle \tag{A.2.12}$$

Thus we can exactly get the idea that, only when $[\psi(x),\psi(y)]$ does not vanish, i.e. not spacelike condition, the expected value of $\psi(x)$ has a nonzero result. This result can be interpreted in Figure A.2.3 where the vertex of light cone is source and only the timelike region remains a nonzero field. This means the casuality in fact survives.

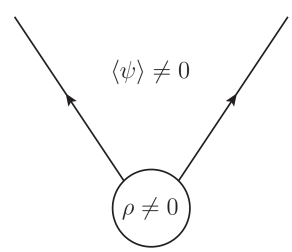


Figure A.2.3: Light cone of source

Lecture Notes of Quantum Field Theory

Prof. Anthony Zee. (recorded by Gao Ping)

July 4, 2012

Lecture 3

3.1 Deep Fact about Nature

Today we will tell you deep fact about nature between like objects which have same charge, and are indeed different. We know Newton's force is attractive whereas Coulomb's force is repulsive. We may ask why they behave like this. Many years later since they are discovered, we can answer part of this question with QED, quntum electrodynamics. QED is involved massless photon, which is related to gauge invariant. However, since people have spent decades to understand this and there are lots of subtlety in gauge theory, we will not talk about it untill later lectures. Today, we will just use Sidney Coleman's method to bypass discussing gauge invariance.

Its inspiration is from experimental physics. We know mass of photon m_γ is zero. But experimental physicists cannot tell you this exact result and instead they can only give a upper bound of m_γ, namely $m_\gamma <$ (upper bound) even though this bound is vanishing with our development in precision. Thus in theoretical calculation, we can set a nonzero mass m and in the last step of our calculation, we apply the limit $m \to 0$ to get the right answer if the result will not blow up.

OK, we can use this method at once. We know scalar field ϕ which is deifined by property of scalar under Lorentz transformation. And we also know there are some quantities transform like vector under Lorentz transformation, namely vector fields. Thus the way to build a vector field is to add Lorentz indices to scalar field, that is, ϕ_μ, where $\mu = 0, 1, 2, 3$. In electromagnetism we well know the vecter potential $A_\mu(x)$ is a local vector field, where I should emphasize that the coordinates $x = (t, \vec{x})$ is only the labels of the dynamics viriables A_μ. Then we can write the electric and magnetic field in tensor way:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.1.1}$$

where we know $F^{0i} \to \{\vec{E}\}$ and $F^{ij} \to \{\vec{B}\}$. In classicla electrodynamics, we well know that the lagrangian can be writtten down as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{3.1.2}$$

I should stress the importance of the minus sign before $\frac{1}{4}$. We know for scalar, the sign is read as

$$\mathcal{L} = +\frac{1}{2}(\partial^\mu \phi \partial_\mu \phi - m^2 \phi^2) \tag{3.1.3}$$

where the positive sign is indeed compatible with the minus one in (3.1.2) though it seems not. We can easily verify this by checking the time derivative terms. Since there is no $\partial A^0/\partial t$ term in (3.1.2) because of the antisymmetric property of $F^{\mu\nu}$, we know the time derivative term is

$$-\frac{1}{4} F_{0i} F^{0i} = \frac{1}{4} F_{0i} F_{0i} = \frac{\partial A^i}{\partial t} \frac{\partial A^i}{\partial t} \tag{3.1.4}$$

where we have lower down the superscripts $0i$ to subscripts using the metric $\eta_{\mu\nu} = diag(+, -, -, -)$. We see this positive sign, representing kinetic term, is compatible with that in (3.1.3). In fact, we will later recognize that 'sign' will determine whether the force is attractive or repulsive.

Now we are going to add the mass term. We write the lagrangian in (3.1.2) as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu + J^\mu A_\mu \tag{3.1.5}$$

where we should notice the positive sign before mass term is also compatible with the definition of lagrangian since it yields $-A^iA^i$ term, namely $-V$ in the lagrangian definition $L = T - V$. And J^μ is the source or say the current, which is assumed as conserved:

$$\partial_\mu J^\mu = 0 \tag{3.1.6}$$

Indeed, this conservation of current is from the gauge invariance which we will not discuss here. Then the lagrangian (3.1.5) can be written as

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{2}m^2 A_\mu A^\mu + J^\mu A_\mu \\
\text{(part integral)} &= +\frac{1}{2}A_\nu(\partial^2 A^\nu - \partial^\nu\partial_\mu A^\mu) + \frac{1}{2}m^2 A_\mu A^\mu + J^\mu A_\mu \\
&= \frac{1}{2}A_\mu\left[(\partial^2 + m^2)\eta^{\mu\nu} - \partial^\mu\partial^\nu\right]A_\nu + J^\mu A_\mu
\end{aligned} \tag{3.1.7}$$

Repeating the method in last class, we should find the propagator of this operator. We have

$$\left[(\partial^2 + m^2)\eta^{\mu\nu} - \partial^\mu\partial^\nu\right]D_{\nu\lambda}(x) = \delta^\mu_\nu \delta^{(4)}(x) \tag{3.1.8}$$

where we should notice that there is an extra delta symbel to count the Lorentz indices and the Feynman propagator also has Lorentz indices added. Through Fourier transformation, we can find the solution for propagator in the equation of

$$\left[(-k^2 + m^2)\eta^{\mu\nu} + k^\mu k^\nu\right]D_{\nu\lambda}(k) = \delta^\mu_\nu \tag{3.1.9}$$

To get the solution, we should first notice that $D_{\mu\nu}(k)$ carries two indices and thus it is a Lorentz tensor. And then all Lorentz quantities in hand with two indices are $\eta_{\mu\nu}$ and $k_\mu k_\nu$. Hence we can make the ansatz:

$$D_{\nu\lambda}(k) = a\eta_{\nu\lambda} + bk_\nu k_\lambda \tag{3.1.10}$$

Substituting this into (3.1.9) we can easily solve that

$$\begin{cases} a = -\frac{1}{k^2 - m^2} \\ b = \frac{1}{k^2 + m^2}\frac{1}{m^2} \end{cases} \tag{3.1.11}$$

which is left as an exercise. Thus we have the propagator:

$$D_{\nu\lambda}(k) = -\frac{1}{k^2 - m^2}\left(\eta_{\nu\lambda} - \frac{1}{m^2}k_\nu k_\lambda\right) \tag{3.1.12}$$

which is just as the one in scalar field in last class except for a few indices.

Then we can write down $W(J)$ as

$$\begin{aligned}
W(J) &= -\frac{1}{2}\int d^4x d^4y J^*_\mu(x) D^{\mu\nu}(x-y) J_\nu(y) \\
&= -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J^{*\mu}(k) D_{\mu\nu}(k) J^\nu(k) \\
&= -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} J^{*\mu}(k)\left[-\frac{1}{k^2 - m^2}\left(\eta_{\mu\nu} - \frac{1}{m^2}k_\mu k_\nu\right)\right] J^\nu(k)
\end{aligned} \tag{3.1.13}$$

where the details are left as an exercise. We can see this result is also very similar with the one in last class except for a few indices. Now we take the limit $m \to 0$. However, we may immediately encounter a terrible problem that the propagator will blow up since it has the $\frac{1}{m^2}$ term. Fortunately, we have a trick that since current is conserved, $\partial_\mu J^\mu(x) = 0$, we have $k_\mu J^\mu(k) = 0$. Thus the second term in propagator indeed vanishes with $J^\nu(k)$. Thus we get the answer:

$$W(J) = +\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{J^{*\mu}(k) J_\mu(k)}{k^2 - m^2} \tag{3.1.14}$$

Now, we should pay extreme attention to the plus sign before the integral. We set $J^\mu(x) = \delta^\mu_0 \delta^{(3)}(\vec{x} - \vec{x}_1) + \delta^\mu_0 \delta^{(3)}(\vec{x} - \vec{x}_2)$, namely a couple of two same still charges located at $\vec{x}_1$ and $\vec{x}_2$, the one we have set in last class.

Thus we only have nonvanishing component of J^μ as $J^0=\rho$, the charge density. Then ignoring self-interaction, we can rewrite (3.1.14) as

$$\begin{aligned} W(J) &= +\frac{1}{2}\int d^4xd^4yJ^{*\mu}(x)\frac{e^{ik(x-y)}}{k^2-m^2}J_\mu(y) \\ &= -\int dx^0\int\frac{d^3k}{(2\pi)^3}\frac{e^{-i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{\vec{k}^2+m^2} \end{aligned} \tag{3.1.15}$$

Then following the same method in last class, we get

$$E=\frac{1}{4\pi r} \tag{3.1.16}$$

which is very important! It shows that same charge in electrodynamics will yield repulsive force, whereas in scalar condition, we have the attractive force! Indeed just the repulsive force from electrodynamics and attractive nuclear force carried by pion(a composite scalar) cancelling out results in the stable nucleus.

3.2 Bypass Maxwell

From above we know the form of propagator has the following structure:

	propagator(momentum space)
spin 0	$\frac{1}{k^2-m^2}$
spin 1	$\frac{1}{k^2-m^2}\left(-\eta_{\nu\lambda}+\frac{1}{m^2}k_\nu k_\lambda\right)$
spin 2	$\frac{1}{k^2-m^2}(???)$

The third line is what we can guess. Before we handle the third line, we will use anthoer method to deduce the propagator of spin 1 particle without using Maxwell's theory. In other words, we will recover the result in subchapter 3.1 in a more convenient and physical way. First, we we can rewrite the second line in the table as

$$D_{\nu\lambda}(k)=\frac{-G_{\nu\lambda}}{k^2-m^2} \tag{3.2.1}$$

Then our central goal is to comprehend the physical meaning of $G_{\nu\lambda}$. At first, we can recall in electromagnetism we have three polarization states of a massive photon(To be more rigorous, one may say we only have two polarization states for a realistic photon; but we know there are three oscillation modes of a planar wave in solid object: two transverse waves and one longitudal wave). Then we will have three vectors to represent these three polarization states:

$$\begin{aligned} \epsilon_\mu^{(1)} &= (0,1,0,0) \\ \epsilon_\mu^{(2)} &= (0,0,1,0) \\ \epsilon_\mu^{(3)} &= (0,0,0,1) \end{aligned} \tag{3.2.2}$$

which are polarized along three spatial axes in Minkovski frame. Since we can find a rest frame of a massive particle, in this frame, we know

$$k^\mu=(m,0,0,0) \tag{3.2.3}$$

Then we immediately get

$$k^\mu\cdot\epsilon_\mu^{(a)}=0 \qquad \text{(for any } a\text{)} \tag{3.2.4}$$

in rest frame. Moreover, this product is Lorentz invariant, namely in any frame, $\epsilon_\mu^{(a)}$ transforms jus\t like a vector and thus is compatible with k^μ to be consistent with (3.2.4). Then we should notice Figure 3.2.1, where a source emits a particle to another with the momentum $\vec{k}$ and polarizaiton label (a).

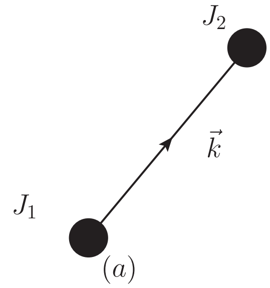


Figure 3.2.1: The propagation from one source to another

Then we will argue that the amplitude of producing a field in J_1 and absorbing a field in J_2 are both propotional with $\epsilon_\mu^{(a)}$. We can easily accept this by recall a planar wave can be written down as

$$A_\mu(x) = e^{ikx}\epsilon_\mu^{(a)} \tag{3.2.5}$$

(where we have indeed plugged in Lorentz gauge). Thus creating such a field from J_1, the amplitude should propotional to $\epsilon_\mu^{(a)}$ and so for destroying a field in J_2. And we will not distinguish these three polarizations and thus we get the propagator $D_{\mu\nu}$ should be propotional to the summation below

$$\sum_a \epsilon_\mu^{(a)}\epsilon_\nu^{(a)} = A\eta_{\mu\nu} + Bk_\mu k_\nu \tag{3.2.6}$$

where the reason of the form set at the right hand is the same as before for (3.1.10). Then from (3.2.5) we know

$$\partial_\mu A^\mu(x) = 0 \tag{3.2.7}$$

which is indeed Lorentz gauge(we should notice this condition is plugged in by hand not deduced rigorously). Thus from (3.2.7), we can easily obtain

$$k^\mu \epsilon_\mu^{(a)} = 0 \tag{3.2.8}$$

namely

$$k^\mu(A\eta_{\mu\nu} + Bk_\mu k_\nu) = 0 \to k_\nu(A + Bk^2) = 0 \to A = -Bk^2 = -Bm^2 \tag{3.2.9}$$

where we have used the mass shell condition. Then substiting (3.2.2) and setting $\mu = \nu = 1$, we can get the value of A and B:

$$A = -1, B = \frac{1}{m^2} \tag{3.2.10}$$

Then we have

$$G_{\mu\nu} \propto \sum_a \epsilon_\mu^{(a)}\epsilon_\nu^{(a)} = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{m^2} \tag{3.2.11}$$

Comparing this result with (3.1.12), we know

$$\sum_a \epsilon_\mu^{(a)}(k)\epsilon_\nu^{(a)}(k) = -G_{\mu\nu}(k) \tag{3.2.12}$$

Thus we have indeed bypassed Maxwell and constructed the propagator. Then we will try to bypass Einstein.

3.3 Bypass Einstein

Now we can use the method above to construct the propagator of graviton, which is a spin 2 particle. Before we construct it, we should first answer a question that why the graviton is spin 2 particle?

In electromagnetism the field couples the source with one Lorentz index, namely J^μ. And the gravitational field couples to mass, or energy momentum tensor density $T^{\mu\nu}$, just shown in Einstein equation. Since there are two indices in $T^{\mu\nu}$, we can easily guess it represents a spin 2 particle. Then we show an inspiring example. For a box containing lots of particles moving along som direction, the charge density is transformed as

$$\rho_c' = \frac{\rho_c}{\sqrt{1-v^2}} \tag{3.3.1}$$

since we know the volume is squeezed by the factor $\sqrt{1-v^2}$. But for mass density, it transforms as

$$\rho_m = \frac{\rho_m}{1-v^2} \tag{3.3.2}$$

where that the mass is amplified by the factor of $\frac{1}{\sqrt{1-v^2}}$ and the volume is squeezed by the factor $\sqrt{1-v^2}$ totally contribute a factor of $\frac{1}{1-v^2}$. This the different between vectors and tensors: tensors transform twice than vectors do. Thus a tensor field discribe spin 2 particle. Indeed, the rigorous proof can be in this way: we can construct a vector field by two Weyl spinors, namely two spin $\frac{1}{2}$ particles can be composited as a spin 1 particle(just like the composition of angular momentum in quantum mechanics); and then we can construct a spin n particle by composition of $(2n+1)$ spin $\frac{1}{2}$ particles; since one Lorentz index represents a spin 1 particle(from 2 spinors), we can safely say two Lorentz indices represent a spin 2 particle(from 4 spinors).

Then we can write down the interaction term in gravity

$$h_{\mu\nu}T^{\mu\nu} \tag{3.3.3}$$

where $h_{\mu\nu}$ is the gravitational field and $T^{\mu\nu}$ is the source or say conserved current. Since $T^{\mu\nu}$ is symmetric, we have $h_{\mu\nu} = h_{\nu\mu}$. This is just a generalization of spin 1 case and I remember C. N. Yang have ever said that a good way to do research is generalization. Thus we will analyse the polarization just like the process before. As there are $2j+1$ polarization for a spin j particle, thus we have totally 5 polarization for graviton, namely

$$h_{\mu\nu} \propto \epsilon^{(a)}_{\mu\nu} \qquad (a = 1,2,3,4,5) \tag{3.3.4}$$

From current conservation(conservation of energy and momentum), we have

$$k^\mu \epsilon^{(a)}_{\mu\nu} = 0 \tag{3.3.5}$$

where there are 4 equations for $\epsilon^{(a)}_{\mu\nu}$. Let us count the degree of freedom for the symmetric tensor $\epsilon^{(a)}_{\mu\nu}$: $\frac{4\times5}{2} = 10$. And as the extra 4 equations in (3.3.5), we have 6 degrees of freedom. This seems contradictrary with the number of total 5 polarization of graviton. In fact, we have missed one equation:

$$\eta^{\mu\nu}\epsilon^{(a)}_{\mu\nu} = 0 \tag{3.3.6}$$

This equation yields the traceless condition for $h_{\mu\nu}$ and its physical meaning is to remove the hidden spin 0 particle in the representation of tensor for spin 2 particle. We can show this in a more obvious way. $h_{\mu\nu}$ can be decomposed into two parts, one is traceless and the other is not:

$$h_{\mu\nu} = \mathrm{Tr}(h)\eta_{\mu\nu} + h'_{\mu\nu} \tag{3.3.7}$$

where $h'_{\mu\nu}$ is a traceless matrix and $\mathrm{Tr}(h)\eta_{\mu\nu}$ is actually a scalar since it is a Lorentz invariance. Thus we get the compatible result for degree of freedom.

Since everything is ready, we can follow the method in last subchapter to construct propagator. That is, we should calculate

$$\sum_{a=1}^{5}\epsilon^{(a)}_{\mu\nu}\epsilon^{(a)}_{\lambda\sigma} = (\text{linear combination of } k_\mu \text{and } \eta_{\mu\nu}) = AG_{\mu\lambda}G_{\nu\sigma} + BG_{\mu\sigma}G_{\nu\lambda} + CG_{\mu\nu}G_{\lambda\sigma} \tag{3.3.8}$$

where we have rewritten the linear combination of $k_\mu$and $\eta_{\mu\nu}$ as the linear combination of $G_{\mu\nu}$s. Using (3.3.5) and (3.3.6) we can solve the coefficient factors and finally we use the normalization condition of $\epsilon_{\mu\nu}$: $\sum_a \epsilon^{(a)}_{12}\epsilon^{(a)}_{12} = 1$(this equation is always positive) in rest frame to fix the result(left as exercise). At last we get the result:

$$\sum_{a=1}^{5}\epsilon^{(a)}_{\mu\nu}(k)\epsilon^{(a)}_{\lambda\sigma}(k) = (G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda}) - \frac{2}{3}G_{\mu\nu}G_{\lambda\sigma} \tag{3.3.9}$$

Then the propagator is

$$D_{\mu\nu,\lambda\sigma}(k) = \frac{\sum_{a=1}^{5}\epsilon^{(a)}_{\mu\nu}(k)\epsilon^{(a)}_{\lambda\sigma}(k)}{k^2-m^2} = \frac{(G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda}) - \frac{2}{3}G_{\mu\nu}G_{\lambda\sigma}}{k^2-m^2} \tag{3.3.10}$$

3.4 Why We Fall

We can use the same steps to calculate $W(T)$:

$$
\begin{aligned}
W(T) &= -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} T^{*\mu\nu}(k) D_{\mu\nu,\lambda\sigma}(k) T^{\lambda\sigma}(k) \\
&= -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} T^{*\mu\nu}(k) \frac{(G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda}) - \frac{2}{3}G_{\mu\nu}G_{\lambda\sigma}}{k^2 - m^2} T^{\lambda\sigma}(k) \\
&\text{(using } k^\mu T_{\mu\nu} = 0 \text{ and taking the limit } m \to 0 \text{, we thus have } G_{\mu\nu} \to \eta_{\mu\nu}) \\
&= -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \frac{2T^{*\mu\nu}T_{\mu\nu} - \frac{2}{3}T^{*\mu\mu}T_{\nu\nu}}{k^2 - m^2} \\
&\text{(particle in rest frame, } T^{\mu\nu} = T^{00}\delta^\mu_0\delta^\nu_0) \\
&= -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \frac{(2 - \frac{2}{3})T^{*00}T_{00}}{k^2 - m^2} < 0
\end{aligned}
\tag{3.4.1}
$$

The sign of negative is the same as what we have got from scalar field. This means for like charges, masses they attract each other! This is why we fall. And we can completely answer the question in the beginning of this lecture: why is Newton's force attractive whereas Coulomb's force repulsive?

3.5 Deep Fact in universe

Recalling we have learnt the relation between field, particle and force in last class, we know exchanging of spin 0 particles we get attractive force, exchanging of spin 1 particles we get repulsive force and exchanging of spin 2 particles we get attractive force again. In cosmology and very beginning of universe, spin 2 particle, namely graviton plays the most important role. Thar is, any irregularity in universe will grow and dense region will become more denser. This is the reason of formation of structures of galaxies. For our daily life, spin 1 particle photon and relative electromagnetic force dominate the most laws. For the stable structure of nucleus, we know the attractive force from exchange of spin 0 particle, pion, balances the repulsive force from exchange of spin 1 particle, photon. For star burning, we have nulear force, this is the topic of weak interaction and we will discuss later. In conlusion, the interplay between spin 0,1 and 2 forms our world.

Appendix:Deducing (3.3.9)

At first we have the symmetry of exchang of $\mu \leftrightarrow \nu$ for $\sum_{a=1}^5 \epsilon^{(a)}_{\mu\nu}(k)\epsilon^{(a)}_{\lambda\sigma}(k)$ as the polarization tensor $\epsilon^{(a)}_{\mu\nu}$ is symmetric. Then we have

$$
AG_{\mu\lambda}G_{\nu\sigma} + BG_{\mu\sigma}G_{\nu\lambda} + CG_{\mu\nu}G_{\lambda\sigma} = \sum_{a=1}^{5} \epsilon^{(a)}_{\mu\nu}(k)\epsilon^{(a)}_{\lambda\sigma}(k) = BG_{\mu\lambda}G_{\nu\sigma} + AG_{\mu\sigma}G_{\nu\lambda} + CG_{\nu\mu}G_{\sigma\lambda} \tag{A.3.1}
$$

However, $G_{\mu\nu}$ is also symmetric, thus we immediately get $A = B$. Then as $\eta^{\lambda\sigma}\epsilon_{\lambda\sigma} = 0$, we have

$$
\begin{aligned}
\eta^{\mu\nu}\sum_{a=1}^{5} \epsilon^{(a)}_{\mu\nu}(k)\epsilon^{(a)}_{\lambda\sigma}(k) &= A(\delta^\nu_\lambda - \frac{k^\nu k_\lambda}{m^2})(\eta_{\nu\sigma} - \frac{k_\nu k_\sigma}{m^2}) + A(...) + C(...) \\
&= A(\eta_{\lambda\sigma} - \frac{k_\lambda k_\sigma}{m^2}) + A(\eta_{\lambda\sigma} - \frac{k_\lambda k_\sigma}{m^2}) + C(\eta^{\mu\nu}\eta_{\mu\nu} - \frac{k^2}{m^2})(\eta_{\lambda\sigma} - \frac{k_\lambda k_\sigma}{m^2}) \\
&= (2A + 3C)(\eta_{\lambda\sigma} - \frac{k_\lambda k_\sigma}{m^2}) = 0 \\
&\to C = -\frac{2}{3}A
\end{aligned}
\tag{A.3.2}
$$

where we have used mass shell condition $k^2 = m^2$. Then we use the normalization condition, setting $\mu = \lambda = 1$ and $\nu = \sigma = 2$ in rest frame, and get

$$
A(G_{11}G_{12} + G_{12}G_{12}) + CG_{12}G_{12} = 0 \tag{A.3.3}
$$

Since the nonvanishing $G_{\mu\nu}$s are only diagonal elements, we get $A = 1$. Finally we get the result:

$$\sum_{a=1}^{5} \epsilon_{\mu\nu}^{(a)}(k)\epsilon_{\lambda\sigma}^{(a)}(k) = (G_{\mu\lambda}G_{\nu\sigma} + G_{\mu\sigma}G_{\nu\lambda}) - \frac{2}{3}G_{\mu\nu}G_{\lambda\sigma} \tag{A.3.4}$$

Lecture Notes of Quantum Field Theory

Prof. Anthony Zee. (recorded by Gao Ping)

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Lecture 4

4.1 Why Inverse Square?

As discussed in last class, we know in the universe, all interaction is based on exchange of particles with different spin: spin 0 and spin 2 for attraction while spin 1 for repulsion for like charges. All above originates from the perturbation of vacuum which is basically second order perturbation theory. The second order means we add the perturbation term as $J\psi$ in lagrangian and finally integrate out vacuum part and leave the perturbation part with present sources in the form of $\int d^4xJ^*(x)D(x-y)J(y)$. This form of J square shows it is basically second order perturbation theory.

However, in quantum mechanics, our perturbation for a spin 0 particle, which denotes attraction, just make the vacuum(ground state) energy go down. Let us probe how this happens. For a quantum system, we have a ground state where the energy is 0 and some another state where the energy is w without perturbation, and then the Hamiltonian is

$$H = \begin{pmatrix} w & v \\ v & 0 \end{pmatrix} \tag{4.1.1}$$

where the matrix elements $v \ll w$ denotes our perturbation. The determinant of Hamiltonian is $\det H = -v^2 < 0$, this minus sign shows that after diagonalization, since the energy of excited state will just shift a little, the energy of ground state will be negative(we can calculate the ground state will be shifted to $E_0 \sim -\frac{v^2}{w} < 0$). It is the shifted negative energy demonstrate that the force by perturbation is attractive. This is physics, and this is why spin 0 denotes attraction.

In quantum field theory, the potential is plotted in Figure 4.1. We see in the region of $r > \frac{1}{m}$, the potential goes to zero; and in the region of $r < \frac{1}{m}$, the exponential decay part is not important, and thus we can set $m \sim 0$ and thus $V(r) \sim -\frac{1}{r}$. The minus sign shows the the interaction is attractive.

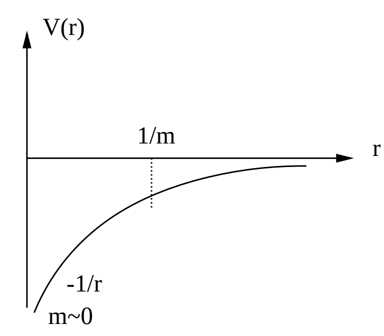


Figure 4.1.1: Potential of attraction

Moreover, we can see this in another angle. The ponential can be written down by

$$V(r) = \int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2 + m^2} \tag{4.1.2}$$

and when we constrict r in a small region, the momentum should be very large, and thus m can be ignored, thus the ponential becomes

$$V(r) = \int d^3k \frac{e^{i\vec{k}\cdot\vec{r}}}{\vec{k}^2} \sim -\frac{1}{r} \tag{4.1.3}$$

In this way, we obtain the relation between spin and attraction or repulsion. In this process, we add Lorentz indices to the sources from spin 0 to spin 1 ensuing to spin 2. Thus we get the form of

$$J_{\mu\lambda\ldots}\frac{A^{\mu\lambda\ldots\nu\sigma}}{k^2}J_{\nu\sigma} \tag{4.1.4}$$

We should notice that $A^{\mu\lambda\ldots\nu\sigma}$ with arbitrary number of indices must comes from a set of metric $\eta^{\mu\nu}$. And the vital origin of whether the force is attractive or repulsive depends on the sign of elements in metric, namely in Minkovski space, time is $+1$ while the space is -1. The opposite sign of space and time is the crucial reason of different type of interactions. I think this problem need to be further studied to try to answer why nature has such opppsite sign?

OK, in (4.1.1), this matrix is not only used in perturbation of quantum mechanics, but also the idea of well-known seesaw mechanism. We all know the mass of neutrino is very small, but how to explain such small quantity of mass? Gellman and some others developed so-called seesaw mechanism to solve this problem. According to violation of parity, we have only observed left-hand nuetrino while the right-hand one has never been discovered. It may not exist but also may have a very heavy mass and thus we cannot create it with today's technology of accelator. Seesaw mechanism is an assumption that mass eigen states are different from the weak interaction states, namely we have a mass matrix of the form of (4.1.1). And then we can solve its eigen values, denoting the mass of right-hand and left-hand neutrinos respectively. We see the two eigen values are

$$m_R = w + \frac{v^2}{w}, m_L = -\frac{v^2}{w} \tag{4.1.5}$$

where if w is very large, we will have a very small m_L, this is why we have such a little mass of left-hand neutrino. However, there is a problem that we encounter a negative mass in (4.1.5), how could it be like this? Well, in fact, all experimental results only show the square of particle mass, we never know whether it is negative or positive. So in theoretical physics, we can absorb this minus sign into fields as a phase factor. And thus we can get a satisfying result. About seesaw mechanism, there are some more introduction in Appendix.

4.2 Caution about Coleman's Trick

Since we know Coleman invented the trick to avoid the subtle gauge invariance, that is, using nonzero m until the final step setting $m \to 0$. However, I will point out that we should realize $m \neq 0$ and $m \to 0$ are really different in physics. This is about math. Although some famous physicist, such as Feynman, seems to use math as little as possible, math is very important in some sense(indeed Feynman's math is pretty well!). We all know some functions are discontinuous in some limit. So in this case the limit value cannot be equal to the function value at the limit point. In physics, the degree of freedom is one of such functions: spin 1 particles have 3 degree of freedom when $m \neq 0$ while they have only 2 when $m = 0$; spin 2 particles have 5 degree of freedom when $m \neq 0$ while they still have only 2 when $m = 0$(indeed all fields with zero mass only have two degree of freedom).

In math, this problem is caused by Casimir operators of Lorentz group(see Weinberg's book or mine for a simple introduction), while in physics, we will have a very simple explaination: zero mass particles cannot be brought to its rest frame, thus they only have two kinds polarization, namely two degree of freedom. How could this problem influence our physic? In statistical mechanics, we have equipartition of energy theorem: every degree of freedom of one particle occupies energy of $\frac{1}{2}kT$. For photons, with $m \neq 0$, we will have the total energy of $N \times \frac{1}{2}kT \times 3$, while with $m = 0$, we only have $N \times \frac{1}{2}kT \times 2$. This is a significant difference! About this problem, the key is that when $m \to 0$, the longitudal polarization decouples with all physical process.(see A. Zee's book of Chap II.7) Thus 'experimentalist could tell if the photon is truly massless rather than have a mass of a zillionth of an electron volt'. Moreover, when a photon goes nearby a heavy star, like the sun, the trajectory will be bent. But nobody can claim that the mass of graviton is exactly zero because of the precision of experiments although the degree of freedom will jump from 5 to 2 in $m \to 0$.

4.3 Recover Maxwell Lagrangian

Suppose we never heard of Maxwell , then how can we deduce the Maxwell lagrangian? We see in quantum mechanics, the vector potential has three components $\vec{A} = \{A_i, i = 1, 2, 3\}$ where $A^0 = 0$. Then in realistic situation, we will use the Klein-Gordon equation for massive photon

$$(\partial^2 + m^2)A_\mu = 0 \tag{4.3.1}$$

And since here we have four components, different with those in quantum mechanics, we should add an extra contraint. We are inspired by the situation in rest frame, that the momentum is $k^\mu = m(1, \vec{0})$. Then in order to fix $A^0 = 0$, we can plug in the condition:

$$k^\mu A_\mu = 0 \tag{4.3.2}$$

which is indeed the Fourier transformation of Lorentz gauge $\partial^\mu A_\mu = 0$. Combining these two equations, we can write both in a more compact way. Plugging (4.3.2) into (4.3.1), we get

$$(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu + m^2 A^\mu = 0 \tag{4.3.3}$$

We can product ∂_μ to this equation and obtain

$$(\partial^\nu\partial^2 - \partial^2\partial^\nu)A_\nu + m^2\partial_\mu A^\mu = 0 \to m^2\partial_\mu A^\mu = 0 \tag{4.3.4}$$

Since $m \neq 0$, we recover (4.3.2). Then using (4.3.4) and (4.3.3), we can also get

$$\eta^{\mu\nu}\partial^2 A_\nu + m^2 A^\mu = 0 \to (\partial^2 + m^2)A_\mu = 0 \tag{4.3.5}$$

which is same as (4.3.1). And then another clever young guy reconstruct the lagrangian from (4.3.5): multiplying A_μ to (4.3.3) and resulting in

$$A_\mu(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu + m^2 A_\mu A^\mu = 0 \tag{4.3.6}$$

which can be deduced from the lagrangian $\mathcal{L} = \frac{1}{2}A_\mu[(\eta^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)A_\nu + m^2 A^\mu]$ by Euler-Langrangian equation. And then we can integrate this lagrangian by part and define $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ to get the final lagrangian:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu \tag{4.3.7}$$

Setting $m \to 0$, we get the Maxwell lagrangian. By the way, I should mention one thing that not all physicists use GR from geometry, some use it from assumption of spin 2 particle and then get the relative geometric property. Among them, there are Feynman and Weinberg. Different physicists in fact have different taste, all of you should find what your taste is.

4.4 Brane World

In 4.1, we have conclude that the inverse square law originates from both rotation invariance and spatial dimension. But how about higher dimension? We can naturally realize that the power of law should be changed to be compatible with the spatial dimension.

Here we consider our 3-dimension world is a brane embedded in a higher dimensional world. The extra spatial dimension is $x^4, x^5, ..., x^{n+3}$. Totally we are living in a $(n+3+1)$ dimensional world. In this world, it is surely not our former inverse square law, but instead it becomes

$$V(r) \propto \int d^{3+n}k \frac{1}{\vec{k}^2} e^{i\vec{k}\cdot\vec{x}} \sim \frac{1}{r^{n+1}} \tag{4.4.1}$$

where the result can be easily got without any integral, just with dimension analysis: $[k^{n+3-2}] = [r^{-n-1}] \sim \frac{1}{r^{n+1}}$. Now comes a problem, how could this inverse $(n+1)$ order of power law compatible with our experiment? The key is that all particles should be restricted in the our daily $(3+1)$ dimensional world, except for graviton which can go through the extra n dimensions. We furthermore define the characteric length of n extra dimension is R that is different from that of $(3+1)$ world, r. Then when $r \gg R$, Newton's inverse square law is correct because in this case, the flux is almost restricted in $(3+1)$ world, just as the electromagnetic field in a wave guide is forced to propagate down the tube and thus the law reduces to $1/r$ (for details, see Appendix). But when $r \ll R$, the extra dimension is free, and thus the flux cannot know whether it flows to extra dimension or not. Then the new law of (4.4.1) makes sense. Up to now, all experiments only test the inverse square law in very large scale in universe, where the result is not contradict with our new law. We need more experiments to test whether in microscopic scale, there are large extra dimensions(we all should keep in mind those 'large' extra dimensions means only comparing with microscopic length scale).

4.5 Plank Mass

Now we come closer to gravity, especially for the Plank mass. Since we know Newton's gravity law demonstrates as

$$V(r) = \frac{Gm_1m_2}{r^2} \tag{4.5.1}$$

we can use the dimension analysis to see the dimension of constant G. With the natural units, namely $\hbar = c = 1$, we know

$$[E] = L^{-1}, [G] = \frac{1}{M^2} \tag{4.5.2}$$

This method is introduced by Plank, and is evaluated as his second importat work. With natural units, we can rewrite Newton constant as

$$G = \frac{1}{M_P^2} \tag{4.5.3}$$

where since G is very small, the mass M_P, called Plank mass, should be very large, and it reads$10^{19}m_p$, where m_p is the mass of proton. Then subsequently we will ask why Plank mass is so large comparing with mass of proton?(In other words, comparing with other kind of interaction, why gravity is so weak?) Making use of extra dimension, we can find an exactly possible solution. We can assume the true scale for gravity is $M_{TG} \ll M_P$, and then when $r \ll R$, the gravitational law is

$$V(r) = \frac{m_1m_2}{(M_{TG})^{n+2}}\frac{1}{r^{1+n}} \tag{4.5.4}$$

where the dimension of new $[G] = \frac{1}{M^{n+2}}$(this is easy to check). Then in $r \gg R$, as the flux is restricted on $(3+1)$ world our law becomes

$$V(r) = \frac{m_1m_2}{(M_{TG})^{n+2}}\frac{1}{R^n r} \propto \frac{1}{r} \tag{4.5.5}$$

Then we will find the relation between M_{TG} and M_P:

$$\frac{1}{M_P^2} = \frac{1}{(M_{TG})^{n+2}R^n} = \frac{1}{(M_{TG}R)^n M_{TG}^2} \tag{4.5.6}$$

where we have write it with the dimensionless quantity $(M_{TG}R)$. We can see, if $(M_{TG}R)$ is large enough,we can get a small M_{TG}, namely the true scale of gravity is not so large as we have always thought. However, this is only one kind of theory, which need to be verified in the future(maybe in LHC).

Appendix 1: Seesaw Mechanism

Here I will simply introduce more details in seesaw mechanism. In standard model, we set the mass of left-hand neutrino as zero. In Weinberg's year, this true because the low precision of measure for neutrino mass. However, since people find neutrino oscillation, we must set nuetrino should have nonzero mass. As is known to all, the mass of all fermions in standard model are given by Yukawa coupling with Higgs particle. Every fermion has its own Yukawa coupling constant y_f and then the mass of fermion is given by

$$m_f = \frac{y_f}{\sqrt{2}}v \tag{A.4.1.1}$$

where v is the expected value of vacuum. We see the Yukawa coupling is proprotional to the mass of fermion. Although it is named as 'standard' model, many people do not satisfy with the setting of mass of fermions. This causes two many degree of freedom of coefficients, that is, all Yukawa couplings are plugged in by hand and cannot be predicted by theory itself. We have to beg for experiments for these values. However, even we admit this assumption, there is another problem raised since the nonzero value of left hand neutrino mass. We well know, neutrino is about only 0.01eV of mass while other leptons are much heavier, such as the mass of electron is 0.5 MeV. We may at once ask a question: why the Yukawa coupling constant for neutrino is so small? Can we find another way to adjust the constant for left-hand neutrino to an $a$verage le$v$el? Here is the possible solution, seesaw mechanism.

In standard model, we can write down the kinetic term of left-hand neutrino:

$$i\nu^\dagger \bar{\sigma}^\mu \partial_\mu \nu \tag{A.4.1.2}$$

And since we do not have any mass term fot ν, we can add several terms for right-hand neutrino:

$$\mathcal{L}_{kin} = i\bar{\nu}^\dagger \bar{\sigma}^\mu \partial_\mu \bar{\nu} \tag{A.4.1.3(a)}$$

$$\mathcal{L}_{Yuk} = -\frac{y}{\sqrt{2}}(v+H)(\nu\bar{\nu} + \bar{\nu}^\dagger \nu^\dagger) \tag{A.4.1.3(b)}$$

where $\bar{\nu}$ is right-hand neutrino and I have chosen unitary gauge for Higgs scalar field(here H is Higgs). (A.4.1.3(b)) is the interaction term of right-hand neutrino with left-hand neutrino, and omitting the interaction with Higgs field H, we will have:

$$-\frac{y}{\sqrt{2}}v(\nu\bar{\nu} + \bar{\nu}^\dagger \nu^\dagger) = -m(\nu\bar{\nu} + \bar{\nu}^\dagger \nu^\dagger) \tag{A.4.1.4}$$

Then we set a mass term for right-hand neutrino

$$\mathcal{L}_{\bar{\nu}mass} = -\frac{1}{2}M(\bar{\nu}\bar{\nu} + \bar{\nu}^\dagger \bar{\nu}^\dagger) \tag{A.4.1.5}$$

Finally we get the total mass term for ν and $\bar{\nu}$

$$\mathcal{L}_{mass} = -\frac{1}{2}\begin{pmatrix} \nu & \bar{\nu} \end{pmatrix}\begin{pmatrix} 0 & m \\ m & M \end{pmatrix}\begin{pmatrix} \nu \\ \bar{\nu} \end{pmatrix} + h.c. \tag{A.4.1.6}$$

We can diagnalize the 2×2 mass matrix and get the two eigen values: M and $-m^2/M$, just as what we have mentioned in 4.1. Here we have two things to explain. One is all fields in (A.4.1.2)-(A.4.1.6) are weak interaction eigen states but not the mass eigen states. In experiments, physicists can only detect mass eigen states of particles since th outcoming particle must be on-shell(mass shell). Thus the eigen vectors for eigen values, linear combination of ν and $\bar{\nu}$, are outcoming states. The other is that we are surely recognize the smaller eigen value is negative, which corresponds to negative mass. In experiment, this is unsignificant, because we only get square of mass of one particle since we can only detect the final energy $E = \sqrt{p^2 + m^2}$, from which we are not able to tell the sign of mass. In theoretical physics, we can absorb this minus sign into eigen vectors. We see the eigen vectors are

$$\begin{cases} \nu_1 = \bar{\nu} + \frac{m}{M}\nu & \text{for mass of } M \\ \nu_2 = \frac{m}{M}\bar{\nu} - \nu & \text{for mass of } -m^2/M \end{cases} \tag{A.4.1.7}$$

Absorbing the minus sign we will get $\nu_2 = i(\frac{m}{M}\bar{\nu} - \nu)$. And substituting back into the kinetic term, we get

$$\begin{aligned} & i(\frac{m}{M}\nu_1^\dagger - i\nu_2^\dagger)\bar{\sigma}^\mu \partial_\mu (\frac{m}{M}\nu_1 + i\nu_2) + i(\nu_1^\dagger + i\frac{m}{M}\nu_2^\dagger)\bar{\sigma}^\mu \partial_\mu (\nu_1 - i\frac{m}{M}\nu_2) \\ = & \frac{m^2}{M^2}\left(i\nu_1^\dagger \bar{\sigma}^\mu \partial_\mu \nu_1 + i\nu_2^\dagger \bar{\sigma}^\mu \partial_\mu \nu_2\right) + \left(i\nu_1^\dagger \bar{\sigma}^\mu \partial_\mu \nu_1 + i\nu_2^\dagger \bar{\sigma}^\mu \partial_\mu \nu_2\right) \\ \to & i\nu_1^\dagger \bar{\sigma}^\mu \partial_\mu \nu_1 + i\nu_2^\dagger \bar{\sigma}^\mu \partial_\mu \nu_2 \quad \text{(ignoring } m^2/M^2 \text{ term)} \end{aligned} \tag{A.4.1.8}$$

Now we see after absorbing the minus sign into phase of fields, even the lagrangian never changes(except for indices)! This means the extra minus sign of mass term make no difference.

With this seesaw mechanism, on the one hand, we can set the m to a mass of electron and thus the Yukawa coupling constant of right-hand neutrino is at the same order of that of eletron; on the other hand, we can set M large enough to resolve the too small mass problem for left-hand neutrino.

Appendix 2:How the Law Reduces to $1/r$

Here we should deduce the $1/r$ law when $r \gg R$ using the original definition of $W(J)$. In $n+3+1$ dimension, we will have the propagator satisfying

$$-(\partial_x^2 + m^2)D(x,y) = \delta^{(n+4)}(x-y) \tag{A.4.2.1}$$

However, the solution is not as the from as used to be. We notice when $r \gg R$, extra dimensions are intensely squeezed and thus no nonzero flux will direct to extra dimensions, namely $\partial^a D(x-y) = 0$, when $a = 5, 6, ..., n+3$(I will use a to denote the indices for extra dimension and μ for $(3+1)$ indices). Thus we have

$$-(\partial_\mu^2 + m^2)D(x,y) = \delta^{(n+4)}(x-y) \tag{A.4.2.2}$$

Then the solution is

$$D(x-y)=\int\frac{d^{n+4}k}{(2\pi)^{n+4}}\frac{e^{ik(x-y)}}{k_\mu k^\mu-m^2+i\epsilon}\tag{A.4.2.3}$$

Then

$$\begin{aligned}
W(J)&=-\frac{1}{2}\int d^{n+4}xd^{n+4}yJ(x)D(x-y)J(y)\\
&=-\frac{1}{2}\int d^{n+4}xd^{n+4}y\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik^\mu(x-y)_\mu}}{k_\mu k^\mu-m^2+i\epsilon}\int\frac{d^{n+4}p}{(2\pi)^{n+4}}e^{ipx}J(p)\int\frac{d^{n+4}q}{(2\pi)^{n+4}}e^{iqy}J(q)\left(\int\frac{d^nk}{(2\pi)^n}e^{ik^a(x-y)_a}\right)\\
&=-\frac{1}{2}\int d^{n+4}xd^{n+4}y\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik^\mu(x-y)_\mu}}{k_\mu k^\mu-m^2+i\epsilon}\int\frac{d^{n+4}p}{(2\pi)^{n+4}}e^{ipx}J(p)\int\frac{d^{n+4}q}{(2\pi)^{n+4}}e^{iqy}J(q)\delta^{(n)}(x_a-y_a)\\
&=-\frac{1}{2}\int d^4xd^4y\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik^\mu(x-y)_\mu}}{k_\mu k^\mu-m^2+i\epsilon}\int\frac{d^4p}{(2\pi)^4}e^{ip^\mu x_\mu}\int\frac{d^4q}{(2\pi)^4}e^{iq^\mu y_\mu}\\
&\quad\times\left[\int d^nx\int\frac{d^np}{(2\pi)^n}e^{ip^ax_a}J(p)\int\frac{d^nq}{(2\pi)^n}e^{iq^ax_a}J(q)\right]\\
&=-\frac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k_\mu k^\mu-m^2+i\epsilon}\times\int\frac{d^np}{(2\pi)^n}J^*(k_\mu,p_a)J(k_\mu,p_a)
\end{aligned}\tag{A.4.2.4}$$

where we see the second part is a const without any singularity resulting a couple of new sources $J'^*(k_\mu)J'(k_\mu)$ after integration and which represents the extra contribution of sources from the extra dimension to total energy without any propagation. For the source like $J(x)=\left(\delta^{(3)}(x-x_1)+\delta^{(3)}(x-x_2)\right)\delta^{(n)}(x)$, we can exactly deduce the former expression for $W(J)$. Thus we have recovered the $1/r$ law.

Lecture Notes of Quantum Field Theory

Prof. Anthony Zee. (recorded by Gao Ping)

July 11, 2012

Lecture 5

5.1 Two Questions about $m \to 0$

In last class, we have met two questions about $m = 0$ case and they are essentially different from the coleman's trick: $m \to 0$. Since last weekend, some students sent e-mails to me to discuss this problem, here I will just make some comments on this issue.

According to equipartition of energy theorem, the energy per degree of freedom is averaged to $\frac{1}{2}kT$. Thus we can immediately know whether photon is massless or not will result very different results in specific heat, whose ratio should be $2:3$. Experimentalists can thus measure the specific heat to judge the mass of photon. However, they can only give an upper bound at most. The reason is that when $m \to 0$, the longitudal component will decouple to all physical process. This means we have to wait for infinite long time to get the thermal equilibrium of longitudal polarization photon with detector. Since the intensity of coupling is proportional to m(see Chap II.7 in my book), if m is small enough, the time will be still so long that no experimentalist can wait until thermal equilibrium. Then even if $m \neq 0$, experimentalists are only able to measure the specific heat of two transverse polarized components of equilibrium plus longitudal component of unequilibrium(whose specific heat is not defined). Thus, experimentalists can only give an upper bound for the mass of photon.

The other comment is about graviton. We know the light will be defected going nearby a massive star, such as the sun. The angle of defection is discontinuous changing from $m \neq 0$ to $m = 0$. In my book, Chap VIII.1, this difference seems to be measured and to be used to judge whether the graviton is tiny massive or essentially massless. However, this paradox was resolved by A. Vainshtein in1972. He found there is a distance scale

$$r_V = \left(\frac{GM}{m_G^4}\right)^{\frac{1}{5}} \tag{5.1.1}$$

in the gravitational field around a body of mass M. When the distance between the light and the body $r < r_V$, the $m \to 0$ case and $m = 0$ cases are same while $r > r_V$, the difference becomes visible. Since the the upperbound of today's result of m_G is very small, the scale above is larger than the size of solar system. Thus we cannot in practical give an answer whether $m_G = 0$. This case is similar with the the first case $m_\gamma = 0$, where we have a time scale of equilibrium of longitudal polarization, which must be longer than the time of our process of experiments. These scale problems are all originated from the interchange of two limits. For example, in the photon case, one limit is to take the mass to be zero, the other is to take the time of waiting for equilibrium to be infinity. When we take $m \to 0$ first, we will get the essential $m = 0$ result whereas when we take $t \to \infty$ first, we will get the $m \to 0$ result.

To be more precise, I can give a simple but not rigorous enough proof about decouple of longitudal component of photon. A massive particle moves along z axis, and its 4-monmentum is $k^\mu = (\omega, 0, 0, k)$, where $\omega^2 = k^2 + m^2$. We have the polarization condition: $\epsilon_\lambda k^\lambda = 0$ and normalization $\epsilon_\lambda \epsilon^\lambda = -1$. Then we care about $\epsilon^{(3)}_\lambda = (-k, 0, 0, \omega)/m$, and when $m \to 0$, $k^\lambda \to \omega(1, 0, 0, 1)$, we see

$$\epsilon^{(3)\lambda} = \eta^{\lambda\mu}\epsilon^{(3)}_\mu \to (-1, 0, 0, -1)\frac{\omega}{m} = -\frac{k^\lambda}{m} \tag{5.1.2}$$

As the amplitude of emitting a photon from a source is proprotional to $\epsilon^{(3)}_\lambda J^\lambda \to -\frac{k_\lambda}{m}J^\lambda \sim \partial_\lambda J^\lambda = 0$, we know this amplitude is zero when $m \to 0$. Of course, you may notice that this argument is not completely tenable because when $m \to 0$, the expression above will be the form of $\frac{0}{0}$ and we need to be more careful about this limit. However, I do not plan to introduce more details(see Chap II.7) here as more complicated contents should be covered. No matter what method we use, physics need to be correct regardless whether so does math.

By the way, I will mention an argument about Higgs. Higgs can decay into two photons through a W triangle(Figure 5.1.1). The decay rate was calculated independently during 1976-1980 by at least three groups and their results are all agreed. Nevertheless in 2011, T. T. Wu had published two papers to argue all these three groups are wrong. Soon afterwards, three other groups wrote papers to refute T. T. Wu. Their points is that whether the longitudal component of W boson in the triangle loop decouples when the mass of W goes to zero. I recommend a paper by Vainshtein, who has explain this problem very clearly in his paper(Ref. arXiv:1109.1785v3).

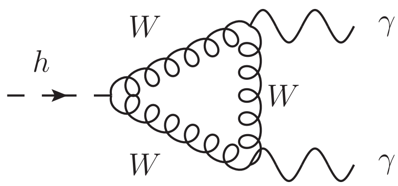


Figure 5.1.1: $h \to \gamma\gamma$

5.2 Feynman Diagram

Here I cannot derive all equations for you about Feynman diagram because of scarcity of time. I have to just tell you the brief idea and hope you to read Chap I.7 and I.8 in my book.

The functional integral

$$Z(J) = \int \mathcal{D}\phi e^{i\int d^4x \frac{1}{2}[(\partial\phi)^2 - m^2\phi^2] - \frac{\lambda}{4!}\phi^4 + J\phi} \tag{5.2.1}$$

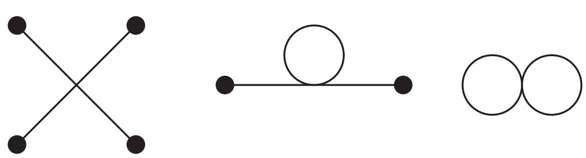
is anharmonic because of the quartic term. Its simple analogy of mattress is that the compression coefficient of the strings in the mattress is nonlinear($\propto (\Delta x)^3$). Our goal is to calculate (5.2.1) whereas it cannot be integrated in any brief ways. We have to assume λ is so small that we can expand it in series and integrate it term by term. Before we do this complicated functional integral, let us see a baby problem: calculate the following integral:

$$Z(J) = \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2q^2 - \frac{\lambda}{4!}q^4 + Jq} \tag{5.2.2}$$

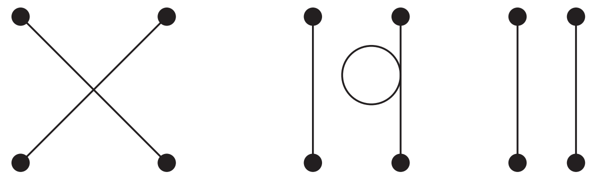
We expand it and do the integral:

$$\begin{aligned} Z(J) &= \int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2q^2+Jq}\left(1 - \frac{\lambda}{4!}q^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2 q^8 - ...\right) \\ &= \left(1 - \frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8 - ...\right)\int_{-\infty}^{+\infty} dq e^{-\frac{1}{2}m^2q^2+Jq} \\ &= \frac{\sqrt{2\pi}}{m}\left(1 - \frac{\lambda}{4!}\left(\frac{d}{dJ}\right)^4 + \frac{1}{2}\left(\frac{\lambda}{4!}\right)^2\left(\frac{d}{dJ}\right)^8 - ...\right)e^{\frac{J^2}{2m^2}} \\ &= \frac{\sqrt{2\pi}}{m}\left(1 - \frac{\lambda}{4!}\left(\left(\frac{J}{m^2}\right)^4 + \frac{6J^2}{m^6} + \frac{3}{m^4}\right) + ..\right)e^{\frac{J^2}{2m^2}} \end{aligned} \tag{5.2.3}$$

Then Feynman's idea is drawing a little diagram to keep track of each expanded term. For example, the terms above in bracket can be plotted as



Moreover, for the terms of order J^4, we will have the following diagrams:



Here we find some diagrams are connected while others are not. This difference is important though I have no time to remark more about it. Please read my book for relative contents.

Then we will do a child problem: integrating

$$Z(J) = \int_{-\infty}^{+\infty} dq_1 ... \int_{-\infty}^{+\infty} dq_N e^{-\frac{1}{2}qAq - \frac{\lambda}{4!}q^4 + Jq} \tag{5.2.4}$$

where q is a N dimensional vector and A is a $N \times N$ matrix. We can use the same trick of expanding the quartic term to get the result. The propagator is $(A^{-1})_{ij}$, which represents a propagation from j to i. Finally we can summarize all these calculation as Feynman rules for his well-known diagrams. The deducing process is subtle and I have to ask you again to read my book.

5.3 Canonical Quantization

In this subsection, I also do not plan to deduce the details. So you should read my book Chap I.8.

We all know in quantum mechanics, we have Heisenberg commutation relation:

$$[q, p] = i \tag{5.3.1}$$

and we also know how to deal with harmonic oscillator. However, in one point of view, field thory can be regarded as infinite harmonic oscillators. Then how to understand this?

The answer is very simple. We can use Fourier transformation from $(t, \vec{x})$ to $(\omega, \vec{k})$. Then the action becomes

$$\begin{aligned} S &\sim \int dt \int d^3x \frac{1}{2} \left[(\partial_0\phi)^2 - (\nabla\phi)^2 - m^2\phi^2\right] \\ &\to \int dt \int d^3x \frac{1}{2} \left[(\partial_0\phi)^2 - (\vec{k}^2 + m^2)\phi^2\right] \\ &\sim \int dt \int d^3x \frac{1}{2} \left[(\partial_0\phi)^2 - \omega_{\vec{k}}^2\phi^2\right] \end{aligned} \tag{5.3.2}$$

where $\omega_{\vec{k}}^2 = \vec{k}^2 + m^2$. We see the action is in the form of

$$\frac{1}{2}\left(\frac{dq(t)}{dt}\right)^2 - \frac{1}{2}\omega^2 q(t)^2 \tag{5.3.3}$$

which is the lagrangian of harmonic oscillator. We only need to change $q(t)$ into $\phi(t, \vec{k})$ and thus we find for each $\vec{k}$, there is a harmonic oscillator with the frequency of $\omega_{\vec{k}}$.

5.4 Zero Point Energy

Since each oscillator has a zero point energy of $\frac{1}{2}\hbar\omega$ because of uncertainty principle, our field also has a zero point energy

$$\langle 0|H|0\rangle = \int d^3k \frac{1}{2}\hbar\omega_{\vec{k}} \tag{5.4.1}$$

which was found in 1930. We may at once find this expression is divergent. Fortunately, we need not worry about this because all physical measurements are valid relative to "energy of vacuum" unless we handle with gravitational problem. Though the vacuum energy is infinite, we only need to correct Hamiltonian as $H - \langle 0|H|0\rangle$. This trick is similar with that used in renormalization as we will see later. In gravational theory, we have to count this divergent term because it represents cosmologic constant. Unfortunately, experimental value of cosmologic constant is very small rather such a divergent value. This problem is still not solved.

5.5 Casimir Effect & Vacuum Energy

We know the vacuum energy ϵ is not observable, nevertheless shifted vacuum energy $\Delta\epsilon$ is indeed observable. Thus we can disturb the vacuum to observe such shifted energy $\Delta\epsilon$. This effect was found by Casimir, a brilliant person, in 1948.

His idea is as follows. There are three perfectly conducting plates where between them the intervals are d and $L-d$ respectively ($L \gg d$) shown as Figure 5.5.1.

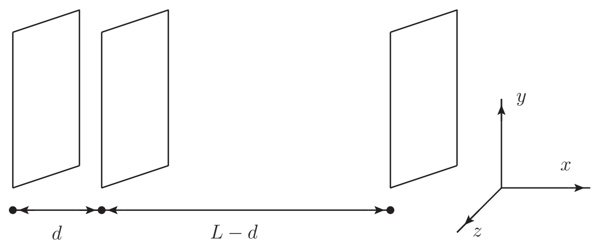


Figure 5.5.1: Casimir effect

Since the plates are perfectly conducting, the parallel component of fluctuated electric field between plates on the surface of each plate should be zero. In other words, the x component of wave number should be $\frac{\pi n}{d}$. Then the wave number is

$$\vec{k} = (\frac{\pi n}{d}, k_y, k_z) \tag{5.5.1}$$

Thus the total vacuum energy originated from fluctuation is

$$\epsilon = 2\sum_{n=1}^{\infty} \int dk_y dk_z \frac{1}{2}\hbar \left[\sqrt{\left(\frac{\pi n}{d}\right)^2 + k_y^2 + k_z^2} + \sqrt{\left(\frac{\pi n}{L-d}\right)^2 + k_y^2 + k_z^2}\right] \tag{5.5.2}$$

where 2 before the sum is for two polarization of photon. Then if we move the middle plate a little, the total energy will change more or less, which indicate there is a force between two plates, i.e.

$$\frac{\partial}{\partial d}\epsilon = (\text{Casimir Force}) \tag{5.5.3}$$

However, (5.5.2) is hard to calculate exactly. Precise calculation can be found in Kardar's paper. Here we only focus on physics and simplify the model as a scalar field only in $(1+1)$ dimension and admit the physical feature is schematically correct. I remember a Nobel Laureate, Sam Edwards, has said, when we encounter a unsolvable theoretical problem, we can remove the features to simplify this model until it becomes trivial, and then we add some feature to solve it. With this simplification, we do not need to do the integral and it becomes

$$\epsilon = \sum_{n=1}^{\infty} \frac{1}{2}\hbar \left(\frac{\pi n}{d} + \frac{\pi n}{L-d}\right) \tag{5.5.4}$$

We immediately find this equation is divergent. Then we should consider where the divergence comes from. We see we have assumed the plates are perfectly conducting so that the electron can move as quickly as possible on the surface of the plates to cancel the electric field in the body of plates. But the in reality, there are no such perfect conductors. When the frequency of electric field increase, the electrons cannot follow such high 'rhythm' and thus the plates become less conducting for those frequencies. Thus we should add a cutoff term to (5.5.4) by hand. We shift it as

$$\begin{aligned}\epsilon &= \sum_{n=1}^{\infty} \frac{1}{2}\hbar \left(\omega_n(d) e^{-\frac{a\omega_n(d)}{\pi}} + \omega_n(L-d) e^{-\frac{a\omega_n(L-d)}{\pi}}\right) \\ &= \sum_{n=1}^{\infty} \frac{\pi}{2} n\hbar \left(\frac{1}{d} e^{-\frac{an}{d}} + \frac{1}{L-d} e^{-\frac{an}{L-d}}\right)\end{aligned} \tag{5.5.5}$$

where the cutoff frequency is $\omega_* \sim 1/a$, when $\omega_n \ll \omega_*$, the cutoff term is order of 1; when $\omega_n \gg \omega_*$, the cutoff term is vanishing. Notice here we use the cutoff for ω_n but not for n because the cutoff n_* is relative to the interval d rather a constant. We define

$$f(d) = \frac{\pi}{2d} \sum_{n=1}^{\infty} n e^{-\frac{an}{d}} \tag{5.5.6}$$

and it can be calculated as

$$
\begin{aligned}
f(d) &= -\frac{\pi}{2}\frac{\partial}{\partial a}\sum_{n=1}^{\infty} e^{-\frac{an}{d}} \\
&= -\frac{\pi}{2}\frac{\partial}{\partial a}\frac{1}{1-e^{-a/d}} \\
&= \frac{\pi}{2d}\frac{e^{a/d}}{(e^{a/d}-1)^2} \\
(a \ll 1) &= \frac{d}{2\pi a^2} - \frac{\pi}{24d} + O(a^2)
\end{aligned}
\tag{5.5.7}
$$

Then the energy is

$$
\epsilon = \hbar[f(d) + f(L-d)] \tag{5.5.8}
$$

and the force is

$$
\begin{aligned}
F &= \hbar[f'(d) - f'(L-d)] \\
&= \hbar\left[\frac{1}{2\pi a^2} + \frac{\pi}{24d^2} - \frac{1}{2\pi a^2} - \frac{\pi}{24(L-d)^2} + O(a^2)\right] \\
&= \hbar\left[\frac{\pi}{24d^2} - \frac{\pi}{24(L-d)^2} + O(a^2)\right] \\
(a \to 0, L \gg d) &\sim \frac{\pi\hbar}{24d^2}
\end{aligned}
\tag{5.5.9}
$$

Since this result is positive, the force is attractive. Indeed, the d^{-2} part can be got easily from dimensional analysis.

Finally, I would like to make two comments about Casimir effect. The first is how do we know the Casimir force independent on a though we have calculated out the result independent on a. a is called regulator, and the process we add the terms of a is called regularization. In physics, we often use regularization, which I think is a terrible thing. And there is no explanation about why it must be independent on a. It could be. And if so, we may get some features about these metal through measuring the Casimir force. However, this issue is not interesting for fundamental physicist in spite of interest for condense matter physicists. Such different interest is similar with the different interest in universality versus specificity for scientists in various fields. The other comment is how do we know the result is independent on regularization schemes? I cannot give a rigorous proof. Nevertheless, I have used three kinds of regularization schemes in my book of second edition and have got the same result. Then, in a physical rigorous level, I say the force is independent on regularization scheme.

Lecture Notes of Quantum Field Theory

Prof. Anthony Zee. (recorded by Gao Ping)

July 13, 2012

Lecture 6

6.1 Symmetry

Today we talk about an important aspect of quantum field theory, symmetry. Symmetry means an action is invariant under some transformation, such as spacetime transformation(Lorentz transformation) corresponding to Lerentz symmetry. There is one of the greatest insight by Heisenberg, for finding the isospin symmetry, which is the beginning for speculating more internal symmetries. Indeed, even the standard model is based on the internal symmetry of $SU(3) \times SU(2) \times U(1)$. But what is isospin symmetry? Heisenberg found that the mass of proton and neutron are very near. Just from this feature, he postulated there is a internal symmetry which discribes the transformation between proton and neutron. This means if we ignore the presence of electromagnetic property(namely, proton carris a unit of positive charge while neutron is electrically neutral), proton and neutron should be classified into one type of particle. This idea is extremely bold! And it also has inspired lots of physicist in doing fundamental physicist.

OK, here we can write such an isopin symmetry in a simplified way. It is actually $SO(2)$ symmetry. Consider a lagrangian,

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\phi_1)^2 - m_1^2\phi_1^2\right] - \frac{\lambda_1}{4}\phi_1^4 + \frac{1}{2}\left[(\partial_\mu\phi_2)^2 - m_2^2\phi_2^2\right] - \frac{\lambda_1}{4}\phi_2^4 - \frac{\rho}{2}\phi_1^2\phi_2^2 \tag{6.1.1}$$

where we have 5 parameters: $m_1,\ m_2,\ \lambda_1,\ \lambda_2,\ \rho$. These parameters should be measured by experiments and rest of physics can be completely predicted with these parameters. In (6.1.1) we can at once find a discrete symmetry: under the following tranformation

$$\begin{aligned}\phi_1 &\to -\phi_1\\ \phi_2 &\to -\phi_2\end{aligned} \tag{6.1.2}$$

the lagrangian is invariant. After the first glimpse of symmetry we can start from (6.1.1) and build more symmetries in stages.

We can suppose $m_1 = m_2 = m$ and $\lambda_1 = \lambda_2 = \lambda$, and thus the lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 - m^2(\phi_1^2 + \phi_2^2)\right] - \frac{\lambda}{4}\phi_1^4 - \frac{\lambda}{4}\phi_2^4 - \frac{\rho}{2}\phi_1^2\phi_2^2 \tag{6.1.3}$$

which is invariant under the transformation of

$$\phi_1 \leftrightarrow \phi_2 \tag{6.1.4}$$

This is also a discrete symmetry. Furthermore, we can suppose $\rho = \lambda$, Then we can rewrite the lagrangian as

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2 - m^2(\phi_1^2 + \phi_2^2)\right] - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 \tag{6.1.5}$$

Here we must notice we encounter a new kind of symmetry-continuous symmetry! Consider the following transformation,

$$\begin{aligned}\phi_1 &\to \cos\theta\phi_1 + \sin\theta\phi_2\\ \phi_2 &\to -\sin\theta\phi_1 + \cos\theta\phi_2\end{aligned} \tag{6.1.6}$$

where θ is a continuous angle(this the reason why we call it as continuous symmetry). And we can easily find (6.1.5) is invariant under such transformation. Indeed, it is just like a rotation of a 2 dimensional vector $\vec{\phi} = (\phi_1, \phi_2)$. We see the transformation of rotation keeps the length of vector, i.e. $(\phi_1^2 + \phi_2^2)$, invariant. And we assume θ is independent of coordinate x. Thus in the same way, $(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2$ is also invariant. Finally we find this symmetry has nothing to do with spacetime and it is the internal symmetry which means we only transform between fields themselves but irrelavent with their coordinate labels. This is the isospin symmetry first discovered by Heisenberg.

We can set $\theta \to 0$ to represent the infinitesimal transformation('infinitesimal' means infinitesimal parameter change which only valid in continuous symmetry). In this way, (6.1.2) becomes

$$\begin{aligned}\delta\phi_1 &= \theta\phi_2 \\ \delta\phi_2 &= -\theta\phi_1\end{aligned} \tag{6.1.7}$$

Such infinitesimal transformation can be described by Lie algebra. With such infinitesimal transformation, we can rebuild the intact transformation by doing it for infinite times which is called exponential mapping.Maybe some of you are not familiar with such transformations. But that is not important. You need to read my book for details because you should know ignorance and stupidity are two differnet things.

OK, here we may be onfused by two ways to do physics. One is given a lagrangian, we try to find the symmetry. The other one is reversed: given a symmetry, we try to find a lagrangian just like if we want ϕ_1 and ϕ_2 in hand to satisfy $SO(2)$ symmetry, we can construct the lagrangian of (6.1.5). Some students will be confused about which way is correct for doing physics on earth. The answer is both. Some may also ask what is the big deal of the reversed method. That is because it is very profound no less than a new way of doing physics started by Einstein! One person who had understood this since the modern physicis developed is Yang. I recognized this in his little book. Also you can see my book *Fearful Symmetry* where I also cited this story. Very roughly speaking, in 19th century, most physics was done in the first way: many people including Maxwell tried their best and spent decades to find a lagrangian, and then many years later, people gruadually found there is some symmetry hidden in the lagrangian. However, in 20th centrury, started by Heisenberg, physicists build langrangian by assuming some symmetry at first. Indeed, this is the most prevalent way of modern physics. Among these development, Yang-Mills theory is the most significant one.

6.2 Symmetry and Conservation

It is a profound discovery that the symmetry is tightly relative to conservation quantity. I will not make comments on many subtle details but instead give you the most simple cases and proof.

From Euler-Lagrangian equation, we know

$$\frac{\delta\mathcal{L}}{\delta\phi_i} = \partial^\mu \frac{\delta\mathcal{L}}{\delta\partial^\mu\phi_i} \qquad (i = 1, 2) \tag{6.2.1}$$

Substitute (6.1.5) in this equation, we can find the equation of motion for field:

$$(\partial^2 + m^2)\phi_i = -\lambda\vec{\phi}^2\phi_i \tag{6.2.2}$$

where $\vec{\phi}^2 \equiv (\phi_1^2 + \phi_2^2)$. We should know this is for classical field theory, so we are allowed to use this classical equation. Furthermore, we can find there is a conserved current

$$J^\mu = i(\phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1) \tag{6.2.3}$$

Using (6.2.2) we can easily get

$$\partial_\mu J^\mu = i(\phi_1\partial^2\phi_2 - \phi_2\partial^2\phi_1) = i\left(\phi_1(-m^2 - \lambda\vec{\phi}^2)\phi_2 - \phi_2(-m^2 - \lambda\vec{\phi}^2)\phi_1\right) = 0 \tag{6.2.4}$$

Thus $J^\mu$is really a conserved current. In summary, symmetry restricts lagrangian from which we can deduce the conservation with equation of motion. We can define a conserved charge:

$$Q = \int d^3x j^0 \tag{6.2.5}$$

We can easily verify this charge is conserved:

$$\frac{d}{dt}Q = \int d^3x\partial_0 j^0 = -\int d^3x\partial_i j^i = -\int_{\text{surface}} d\sigma j^i = 0 \tag{6.2.6}$$

Then we illustrate an example to show how symmetry works. From $SO(2)$ to $SO(N)$, we only need to add indice for ϕ_a, where $a = 1, 2...N$. Then the $SO(N)$ transformation is read as

$$\phi_a \to R_{ab}\phi_b \tag{6.2.7}$$

where the repeated indices are summed. Setting $\vec{\phi} = (\phi_1, \phi_2, ..., \phi_N)$, we can write down the lagrangian as

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu \vec{\phi})^2 - m^2 \vec{\phi}^2\right] - \frac{\lambda}{4}(\vec{\phi}^2)^2 \tag{6.2.8}$$

With path integral formalism, the consequence of symmtry is easily to see. For Feynman propagator, it should be added two group indices:

$$D_{ab}(x) = \int \mathcal{D}\phi e^{iS} \phi_a(x)\phi_b(0) \tag{6.2.9}$$

Since we know S is invariant under transformation and so is $\mathcal{D}\phi$ as we assumed. We can apply a transformation to ϕ_a and ϕ_b to get

$$\begin{aligned} D'_{ab}(x) &= R\left(\int \mathcal{D}\phi e^{iS} \phi_a(x)\phi_b(0)\right) \\ &= \int \mathcal{D}\phi e^{iS} R_{ac}\phi_c(x) R_{bd}\phi_d(0) \\ &= \int \mathcal{D}\phi e^{iS} \phi'_a(x)\phi'_b(0) \\ (\mathcal{D}\phi \text{and } S \text{ are invariant}) &= \int \mathcal{D}\phi' e^{iS'} \phi'_a(x)\phi'_b(0) \\ (\text{as integral, renaming } \phi' \text{ as } \phi) &= \int \mathcal{D}\phi e^{iS} \phi_a(x)\phi_b(0) \\ &= D_{ab}(x) \end{aligned} \tag{6.2.10}$$

Thus we know the propagator is invariant. The only second order invariant tensor under $SO(N)$ is δ_{ab}. Thus we can at once evaluate $D_{ab}(x)$ as

$$D_{ab}(x) = \delta_{ab} D(x) \tag{6.2.11}$$

We see how powerful of the method of using symmetry.

6.3 A Little Contents about Lie Group

In Lie group, we can write a transformation as a exponential form:

$$R = e^{i\theta \cdot T} = e^{i\sum_A \theta^A T^A} \tag{6.3.1}$$

where θ^As are group parameters just like the θ in $SO(2)$ and T^As are matrices called generators of Lie group. In quantum mechanics, we know the rotation in 3 dimensional space can be discribed as $SO(3)$, which can be write in the form of (6.3.1), i.e.

$$R = \exp i(\theta_x J_x + \theta_y J_y + \theta_z J_z) \tag{6.3.2}$$

where the angular momentum J_is are generators for $SO(3)$. Furthermore, for $SO(N)$, we will have the transformation as $N \times N$ matrices: R_{ab}, where $a, b = 1, ..., N$. For infinitesimal transformation we have

$$\phi_a \to R_{ab}\phi_b = (I + i\theta^A T^A)_{ab}\phi_b \tag{6.3.3}$$

and thus

$$\delta\phi_a = i\theta^A T^A_{ab}\phi_b \tag{6.3.4}$$

We may ask how many generators we can have. Different groups have different number of generators. For $SO(N)$, we know the $R^{\mathrm{T}}R = 1$, namely for infinitesimal transformation,

$$(I + i\theta^A \left(T^{\mathrm{T}}\right)^A)_{ab}(I + i\theta^A T^A)_{bc} = I_{ac} + i\theta^A(T^{\mathrm{T}} + T)^A_{ac} + O(\theta^2) = I_{ac} \tag{6.3.5}$$

from which we know the generator should be antisymmetric:

$$T^{\mathrm{T}} + T = 0 \tag{6.3.6}$$

And since R is a real matrix, T must be pure imaginary. Finally we know the degree of freedom for T is $\sum_{n=1}^{N-1} n = \frac{N(N-1)}{2}$. In other words, we have $\frac{N(N-1)}{2}$ number of generators: $A = 1, ..., \frac{N(N-1)}{2}$.

6.4 Noether Theorem

From 6.2, we see some indication of symmetry of conservation. Indeed, this is dictated by Noether Theorem: if the lagrangian has a continuous global symmetry, there is a classical conserved current correspondingly. This theorem is profound but proved simplily. One should be amazed by such a wonderful theorem. The proof is very short.

The variation of lagrangian is zero:

$$0 = \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi_a}\delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta(\partial_\mu\phi_a) \tag{6.4.1}$$

On one hand, global symmetry dictates the infinitesimal transformation is independent of coordinates, namely

$$\delta(\partial_\mu\phi_a) = i\theta^A T^A_{ab}\partial_\mu\phi_b = \partial_\mu i\theta^A T^A_{ab}\phi_b = \partial_\mu\delta\phi_a \tag{6.4.2}$$

where θ^A is not a function of x. On the other hand, we have Euler-Lagrangian equation, namely

$$\frac{\delta\mathcal{L}}{\delta\phi_a} = \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a} \tag{6.4.3}$$

Combining these two parts together, we have

$$\begin{aligned} 0 = \delta\mathcal{L} &= \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\phi_a + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\partial_\mu\delta\phi_a \\ &= \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\phi_a\right) \end{aligned} \tag{6.4.4}$$

Thus we find a conserved current

$$J^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\phi_a \tag{6.4.5}$$

This theorem is very highly valued by Einstein: it is a spiritual formula!

Moreover, I should add a remark about this. Indeed, we need not have $\delta\mathcal{L} = 0$, but instead we could relax the restriction to require $\delta\mathcal{L} = \partial_\mu K^\mu$. In this case, we can rewrite our conserved current as

$$J^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}\delta\phi_a - K^\mu \tag{6.4.6}$$

Then we can easily verify (6.2.3) actually satisfies (6.4.5):

$$\begin{aligned} J^\mu &= \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_1}\delta\phi_1 + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_2}\delta\phi_2 \\ &= \partial^\mu\phi_1(\theta\phi_2) + \partial^\mu\phi_2(-\theta\phi_1) \\ &= -\theta(\phi_1\partial^\mu\phi_2 - \phi_2\partial^\mu\phi_1) \end{aligned} \tag{6.4.7}$$

which is consistent with (6.2.3) except for some unsignificant coeffiecient. As an excercise, you could calculate the conserved current for energy-momentum conservation. It originates from the invariant of translation of $x^\mu \to x^\mu + a^\mu$. Here we will use (6.4.6), because $\mathcal{L}'^\mu(x') = \mathcal{L}(x) + a^\mu\partial_\mu\mathcal{L}(x)$ and thus $K^\mu = a^\mu\mathcal{L}$. We will have

$$J^\mu \equiv P^\mu = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_a}a_\nu\partial^\nu\phi_a - a^\mu\mathcal{L} \tag{6.4.8}$$

You can expand it to write in concrete form.

6.5 Dirac Equation

I am sorry that I have bypassed Chap I.11. I hope you can read it by yourself. Here we will go into Chap II.1 and II.2 for Dirac equation.

First, we have had a relativistic equation Klein-Gordon equation in hand. However, it is very different from Schroodinger equation for it is second order equation whereas Schoodinger's is first order. How could we find a first order relativistic equation? Dirac found his equation. If we read the original paper by Dirac, we would find he indeed guessed out this equation rather than by deduction. I remember Yang has said a sentence, the most beautiful physics is found by guess. So physics is never a logical deduction.

Dirac wrote down his linear equation in the form of

$$(a^\mu \partial_\mu + b)\psi = 0 \tag{6.5.1}$$

where a^μ and b are coefficients independent on coordinates. One may immediately argue this equation is not Lorentz invariant since a^μ is not a Lorentz vector and thus throw this equation into trash can. However, Dirac never give up, he continued manipulating his equation. Of course, finally we must find this equation is Lorentz invariant although the operator $(a^\mu \partial_\mu + b)$ is not because after including the transformation of ψ, we can find an identity for a_μ and thus the Lorentz invariance is satisfied(see Chap II.1 in my book).

We multiply (6.5.1) to the left with $(a^\mu \partial_\mu - b)$ to get

$$\begin{aligned} 0 &= (a^\mu \partial_\mu - b)(a^\mu \partial_\mu + b)\psi \\ &= (a_\mu a_\nu \partial^\mu \partial^\nu - b^2)\psi \end{aligned} \tag{6.5.2}$$

since $\partial_\mu \partial_\nu$ is commutable, we rewrite (6.5.2) as

$$\left(\frac{1}{2}(a_\mu a_\nu + a_\nu a_\mu)\partial^\mu \partial^\nu - b^2\right)\psi = \left(\frac{1}{2}\{a_\mu, a_\nu\}\partial^\mu \partial^\nu - b^2\right)\psi \tag{6.5.3}$$

As a relativistic equation, it should also satisfy Klein-Gordon equation which is indeed the mass-energy relation. Thus we should postulate

$$\begin{cases} b^2 = m^2 \\ \{a_\mu, a_\nu\} = -2\eta_{\mu\nu} \end{cases} \tag{6.5.4}$$

The rest work is to find such b and a_μ to satisfy (6.5.4). This is not a very easy work and it is proved this can be realized at least with 4 by 4 matrix for a_μ in $3+1$ dimensional spacetime. In modern notation, we find $a^\mu = i\gamma^\mu$and $b = m$, thuse we have the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{6.5.5}$$

where $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. From this we can easily write down:$(\gamma^0)^2 = 1$, $(\gamma^i)^2 = -1$, $\gamma^i\gamma^j = -\gamma^j\gamma^i$ for different i, j.

This is just the Clifford algebra in math. We can get the main idea for this in the simplified example in $1+1$ dimension. Here we have only two gamma matrix: $(\gamma^0)^2 = 1$, $(\gamma^1)^2 = -1$ and $\gamma^0\gamma^1 = -\gamma^1\gamma^0$. We can find them effortlessly:

$$\gamma^0 = \tau_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \qquad \gamma^1 = i\tau_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \tag{6.5.6}$$

Thus we can use the Pauli matrix which is nonrelativistic to represent the $1+1$ dimensional relativistic Dirac equation.

In $3+1$ dimension, γ^μ can be found as

$$\gamma^\mu = \begin{pmatrix} & \sigma^\mu \\ \bar{\sigma}^\mu & \end{pmatrix} \tag{6.5.7}$$

where $\sigma^\mu = (I, \sigma^i)$ and $\bar{\sigma}^\mu = (-I, \sigma^i)$.

Appendix: Quantum Conserved Current

In the contents above, we have discussed the classical conseved current using Euler-Lagrangian equation. However, in quantum field theory, we cannot use this classical equation. Then how could we revise the conserved current law? Here I according to the Srednicki's book to give a glimpse on how conserved current works in quantum field theory.

First we use the canonical quantization where all fields are treated as operators rather than that in path integral the fields in the lagrangian in exponent are only complex(or grassman number's) functions. Then we may write the variation of action as

$$\begin{aligned}\frac{\delta S}{\delta\phi_a(x)} &= \int d^4y \frac{\delta\mathcal{L}(y)}{\delta\phi_a(x)} \\ &= \int d^4y \left[\frac{\partial\mathcal{L}(y)}{\partial\phi_b(x)}\frac{\delta\phi_b(y)}{\delta\phi_a(x)} + \frac{\partial\mathcal{L}(y)}{\partial(\partial_\mu\phi_b(x))}\frac{\delta(\partial_\mu\phi_b(y))}{\delta\phi_a(x)}\right] \\ &= \int d^4y \left[\frac{\partial\mathcal{L}(y)}{\partial\phi_b(x)}\delta_{ba}\partial_\mu\delta^4(y-x) + \frac{\partial\mathcal{L}(y)}{\partial(\partial_\mu\phi_b(x))}\delta_{ba}\partial_\mu\delta^4(y-x)\right] \\ &= \frac{\partial\mathcal{L}(x)}{\partial\phi_a(x)} - \partial_\mu\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi_a(x))} \end{aligned} \tag{A.6.1}$$

In addition we have

$$\delta\mathcal{L}(x) = \frac{\partial\mathcal{L}(x)}{\partial\phi_b(x)}\delta\phi_b(x) + \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi_b(x))}\partial_\mu\delta\phi_b(x) \tag{A.6.2}$$

Then combining both we have

$$\delta\mathcal{L}(x) = \partial_\mu\left(\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi_a(x))}\delta\phi_a(x)\right) + \frac{\delta S}{\delta\phi_a(x)}\delta\phi_a(x) \tag{A.6.3}$$

In classical case, $\delta\mathcal{L} = \delta S = 0$, we recover the classical conserved current

$$j^\mu = \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi_a(x))}\delta\phi_a(x) \tag{A.6.4}$$

Now in quantum field theory, we never have the $\delta S = 0$. Nonetheless, we can still define the current as (A.6.4) and then let us probe what would happen to j^μ.

Though we never have $\delta S = 0$, we have another invariant, namely $\delta Z(J) = 0$, where $Z(J)$ is the path integral. We see

$$\begin{aligned} 0 &= \delta Z(J) \\ &= i\int \mathcal{D}\phi e^{i[S+\int d^4y J_b\phi_b]}\left(\frac{\delta S}{\delta\phi_a(x)} + J_a(x)\right)\delta\phi_a(x) \end{aligned} \tag{A.6.5}$$

Then we set $J_a(x) = 0$, we get

$$i\langle 0|T\frac{\delta S}{\delta\phi_a(x)}\delta\phi_a(x)|0\rangle = 0 \tag{A.6.6}$$

Using (A.6.3) and set $\delta\mathcal{L} = 0$, we can get

$$\partial_\mu\langle 0|Tj^\mu(x)|0\rangle = 0 \tag{A.6.7}$$

This is the quantum current conservation where we find the result is changed as in the sense of vacuum expected value.

Furthermore, now we take n functional derivatives with respect to $J_{a_j}(x_j)$ and finally set $J_a = 0$ in (A.6.5), we can immediately get

$$\int \mathcal{D}\phi e^{iS}\left[i\frac{\delta S}{\delta\phi_a(x)}\phi_{a_1}(x_1)...\phi_{a_n}(x_n) + \sum_{j=1}^n \phi_{a_1}(x_1)...\delta_{aa_j}\delta^4(x-x_j)...\phi_{a_n}(x_n)\right]\delta\phi_a(x) = 0 \tag{A.6.6}$$

Since $\delta\phi_a$ is arbitrary, we can drop it out, we have

$$i\langle 0|T\frac{\delta S}{\delta\phi_a(x)}\phi_{a_1}(x_1)...\phi_{a_n}(x_n)|0\rangle + \sum_{j=1}^n\langle 0|T\phi_{a_1}(x_1)...\delta_{aa_j}\delta^4(x-x_j)...\phi_{a_n}(x_n)|0\rangle = 0 \tag{A.6.7}$$

which is called Schinger-Dyson equation. If we substitute (A.6.3) and set $\delta\mathcal{L} = 0$, we can get the revised version of (A.6.7)

$$0 = \partial_\mu\langle 0|Tj^\mu(x)\phi_{a_1}(x_1)...\phi_{a_n}(x_n)|0\rangle + i\sum_{j=1}^n\langle 0|T\phi_{a_1}(x_1)...\delta\phi_{a_j}\delta^4(x-x_j)...\phi_{a_n}(x_n)|0\rangle = 0 \tag{A.6.8}$$

where we have seen the current is conserved with presense of other fields except for some terms including $\delta^4(x-x_j)$. These terms are called contact terms. However, these terms will not contribute to S matrix and thus in S matrix level the current is conserved. For more details, please see Ward-Identity in Srednicki's book.

Lecture Notes of Quantum Field Theory

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Lecture 7

7.1 Review of Dirac Equation

last week, we had encountered the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{7.1.1}$$

To fix the vector γ^μ, we should multiply this equation by $(i\gamma^\mu\partial_\mu + m)$ and simplify it as

$$(-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi = 0 \tag{7.1.2}$$

If in Dirac equation γ^μ satisfies$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, we can recover Klein-Gordon equation effortlessly, that is

$$0 = (-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu - m^2)\psi = (-\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu - m^2)\psi = (-2\eta^{\mu\nu}\partial_\mu\partial_\nu - m^2)\psi \to (\partial^2 + m^2)\psi = 0 \tag{7.1.3}$$

From the necessary condition $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, we know γ^μ must be matrices, rather than numbers. Sylvester is the founder of matrix(some other data shows the founder is Cayley). It was invented in middle period in 19th century, and was applied in physics later in beginning of 20th century.

The restriction of γ^μ is just the Clifford algebra in math. If $\mu \neq \nu$, $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$; if $\mu = \nu$, $(\gamma^0)^2 = 1$ while $(\gamma^i)^2 = -1$ for $i = 1, 2, 3$. Of course, we can extend this argument to arbitrary D spatial dimension and thus $d = D + 1$ spacetime dimension. For example, in $1 + 1$ dimensional spacetime, we have $(\gamma^0)^2 = 1$ and $(\gamma^1)^2 = -1$, and $\gamma^1\gamma^0 = -\gamma^0\gamma^1$. To express γ, we can just use Pauli matrices:

$$\gamma^0 = \tau_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},\ \gamma^1 = i\tau_2 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \tag{7.1.4}$$

You can verify (7.1.4) satisfying the relations above.

In $(3 + 1)$ dimension, $D = 3$, there are four gamma matrices, namely γ^0, γ^1, γ^2 and γ^3. It is proved that we cannot realize them by 2×2 matrices. Thus Dirac extended them to 4×4 matrices. We will see this was a great step in the history of physics. Dirac found his matrices as

$$\gamma^0 = \begin{pmatrix} I & \\ & -I \end{pmatrix},\ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{7.1.5}$$

where I is the 2 by 2 unit matrix. To denote these matrices in a more convenient way, Dirac introduced his Dirac product notation:

$$\gamma^0 = I \otimes \tau_3,\ \gamma^i = \sigma^i \otimes i\tau_2 \tag{7.1.6}$$

where we should notice this is indeed the direct product(with a little insignificant discrepancy about usual notation in matrix textbook) in matrix calculation in which we should know the rule is

$$(A \otimes B)_{ij,kl} = A_{ik}B_{jl} = \begin{pmatrix} Ab_{11} & Ab_{12} & \cdots & Ab_{1n} \\ Ab_{21} & Ab_{22} & \cdots & Ab_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ Ab_{n1} & Ab_{n2} & \cdots & Ab_{nn} \end{pmatrix}, \quad (A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{7.1.7}$$

And I have mixed two conventions τ_i and σ_i both for Pauli matrices in order to distinguish the different position in the Dirac product. With this convention, we can easily calculate some product of gamma matrices. For example,

$$\begin{aligned}\{\gamma^i, \gamma^j\} &= (\sigma^i \otimes i\tau_2)(\sigma^j \otimes i\tau_2) + (i \leftrightarrow j) \\ &= (\sigma^i \sigma^j) \otimes (-1) + (i \leftrightarrow j) \\ &= -\{\sigma^i, \sigma^j\} \otimes I \\ &= \begin{cases} 0 & \text{if } i \neq j \\ -2 & \text{if } i = j \end{cases} = 2\eta^{ij} \end{aligned} \tag{7.1.8}$$

Furthermore, Feynman invented his notation for γ^μ. In Fourier transformation of Dirac equation, we have

$$(\gamma^\mu p_\mu - m)\psi(p) = 0 \tag{7.1.9}$$

He introduced a notation for simplicity for writing:

$$\gamma_\mu p^\mu \equiv \not{p} \tag{7.1.10}$$

Let us manipulate Dirac equation in more details. In rest frame since $p^\mu = (m, 0, 0, 0)$, we can write the Dirac equation in momentum space as

$$(m\gamma^0 - m)\psi = 0 \rightarrow (\gamma^0 - 1)\psi = 0 \tag{7.1.10}$$

where we can solve this equation and immediately find only the first two components of ψ is nonzero, namely,

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \psi = 0 \rightarrow \psi(p)|_{rest} = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix} \tag{7.1.11}$$

This means only two components of the Dirac equation are physical and the other two are projected out by Dirac equation. Indeed, we all know electron only has two degrees of freedom, just like what we have found in (7.1.11) here. Comparing with the Klein-Gordon equation, the situation is almost the same. For the motion of equation of a massive spin 0 and 1 particle, Klein-Gordon equation $(\partial^2 + m^2)\phi(x) = 0$ projects out the components in momentum space which does not satisfy mass shell condition: $k^2 = m^2$. So we come to unified understanding of equation of motion: They just project out the unphysical components. For massive spin 1 field A^μ, which has 4 components, the equation $\partial_\mu A^\mu = 0$ just erases out one component and then results in a field of three degrees of freedom. For massless one, we should even project one more component.

In math, this is the property of projection operator, $P^2 = P$, where P is the projection operator. Here $(\gamma^0 - 1)$ plays the role of projecting, so square of $(\gamma^0 - 1)$ must be proportional to itself:

$$(\gamma^0 - 1)(\gamma^0 - 1) = (\gamma^0)^2 - 2\gamma^0 + 1 = -2(\gamma^0 - 1) \propto (\gamma^0 - 1) \tag{7.1.12}$$

We see it really projects out some unphysical components.

7.2 How Lorentz Invariant?

In the first glimpse of Dirac equation, we must naively think it breaks Lorentz invariance, because in $(a^\mu \partial_\mu + b)\psi = 0$, we have dictated a direction in spacetime, namely a^μ which is invariant under Lorentz transformation. However, Dirac escaped this problem just setting a^μ a matrix rather than a number. Since a direction cannot be assigned by a matrix, we remain Lorentz symmetry in principle.

In order to prove Lorentz invariance, we should do some preparations. At first, we should know the field $\psi(x)$ changes to $S(\Lambda)\psi(x)$ under Lorentz transformation, where $S(\Lambda)$ is a 4 by 4 matrix. In order to write $S(\Lambda)$ in a more exact way, we should understand gamma matrices in more details. We well know gamma matrices are 4 by 4 matrices, which totally have 16 number of degrees of freedom(that is, totally 16 independent matrices for all 4 by 4 matrices). 4 of them denotes our γ^μ. Here some students might confused by so much '4' in the above sentences. We should distinguish them that on one hand, '4 by 4' of γ^μ means they are originated from two Pauli matrices, which is 2 by 2 and thus this 4 is indeed '$4 = 2 + 2$'; on the other hand, '4 of them' means the Lorentz indices of γ^μ are 4 in total, and thus here 4 is form $3 + 1$. These '4's are just coincidence. We see the iterative construction for gamma matrices, such as $\gamma^0 = I \otimes \tau_3$, how can we get the result in higher dimensions, such as $5+1$ and $9+1$(string theory!)?

I will leave this as an exercise of some advanced students. Furthermore, what about odd spacetime dimension, e.g. $2+1$? This problem was studied by Chern-Simons. You can read some references for this issue.

Now we will construct all 16 independent matrices with γ^μ in hand. Those are $\gamma^\mu\gamma^\nu$, $\gamma^\mu\gamma^\nu\gamma^\lambda$ and $\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma$ for $\mu\neq\nu\neq\lambda\neq\sigma$. First of all, we construct γ^5, which is defined as

$$\begin{aligned}\gamma^5 &= i\gamma^0\gamma^1\gamma^2\gamma^3\\ &= i(I\otimes\tau_3)(\sigma^1\otimes i\tau_2)(\sigma^2\otimes i\tau_2)(\sigma^3\otimes i\tau_2)\\ &= i^4(I\otimes\tau_3)(\sigma^1\sigma^2\sigma^3\otimes\tau_2)\\ &= i(I\otimes\tau_3)(I\otimes\tau_2)\\ &= i(I\otimes\tau_3\tau_2)\\ &= I\otimes\tau_1 = \begin{pmatrix} & I\\ I & \end{pmatrix}\end{aligned} \tag{7.2.1}$$

We can easily check γ^5 satisfying following relations:

$$\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5,\ (\gamma^5)^2 = 1 \tag{7.2.2}$$

By the way, I will leave another exercise to find γ^5 in odd dimensional spacetime. We then construct $\gamma^\mu\gamma^\nu\gamma^\lambda$ by γ^5. As three multiplied gamma matrices are equivalent with five ones because the identical gamma matrices can cancel out since $(\gamma^0)^2=1$ and $(\gamma^i)^2=-1$. Thus we can write down another four independent matrices:

$$\gamma^\mu\gamma^5 \sim \gamma^\mu\gamma^\nu \tag{7.2.3}$$

Finally, we should construct matrices like $\gamma^\mu\gamma^\nu$. But here I will just shift it a little as for

$$\gamma^\mu\gamma^\nu = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu] = \eta^{\mu\nu} + \frac{1}{2}[\gamma^\mu,\gamma^\nu] \tag{7.2.4}$$

Since $\eta^{\mu\nu}$ can be one of the independent 4 by 4 matrix, we can denote $\frac{i}{2}[\gamma^\mu,\gamma^\nu]$ for $\gamma^\mu\gamma^\nu$ and rename it as

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu] = -\sigma^{\nu\mu} \tag{7.2.5}$$

OK, let us count the total number of matrices:

$$1(I) + 4(\gamma^\mu) + 6(\sigma^{\mu\nu}) + 4(\gamma^\mu\gamma^5) + 1(\gamma^5) = 16 \tag{7.2.6}$$

which is just the degree of freedom for 4 by 4 matrices.

If we extend $3+1$ to $D+1$ spacetime, we will have totally $\frac{D(D+1)}{2}$ independent $\sigma^{\mu\nu}$ matrices, among which σ^{0i} of D and σ^{ij} of $\frac{D(D-1)}{2}$. For Lorentz transformation in $3+1$, we also have D operators for boost in D different spatial directions and $\frac{D(D-1)}{2}$ rotations in different $i-j$ planes. They just correspond to each other. Some students might be confused by the number of rotations in $3+1$ dimensional spacetime. In this dimension, we have 3 rotations, each along one directions. We can also express them as rotation in some $i-j$ plane. Since the total spatial dimension is 3, these two representations are both OK, e.g. a rotation along z axis equivalent with a rotation in $x-y$ plane. However, in higher dimension, we cannot find a single orthogonal axis to a two dimensional plane. So in that case, speaking a rotation along some axis is confusing. Hence we should use the notations such as J_{xy}, J_{yz} and J_{zx} for angular momentum for correct demonstration in $3+1$ dimension.

Let us count $\sigma^{\mu\nu}$ in precise way.

$$\begin{aligned}\sigma^{ij} &= \frac{i}{2}[\gamma^i,\gamma^j]\\ &= \frac{i}{2}[\sigma^i\otimes i\tau_2,\sigma^j\otimes i\tau_2]\\ &= \frac{i}{2}[\sigma^i,\sigma^j]\otimes(-I)\\ &= \epsilon^{ijk}\sigma^k\otimes I\end{aligned} \tag{7.2.7}$$

Specially,

$$\sigma^{12} = \begin{pmatrix}\sigma^3 & \\ & \sigma^3\end{pmatrix} \tag{7.2.8}$$

For $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$, the transformation of $\psi \to e^{i\theta\sigma^{12}}\psi$ is evaluated as

$$\begin{cases} \phi \to e^{i\theta\sigma_3}\phi \\ \chi \to e^{i\theta\sigma_3}\chi \end{cases} \tag{7.2.9}$$

where we see it is just the rotation transformation along z axis(or say, in $x-y$ plane) in quantum mechanics and $\sigma^{\mu\nu}$ can be regarded as two Pauli matrices 'added together'. For boost,

$$\begin{aligned} \sigma^{0i} &= \frac{i}{2}[\gamma^0, \gamma^i] \\ &= \frac{i}{2}[I \otimes \tau_3, \sigma^i \otimes i\tau_2] \\ &= \frac{i}{2}\sigma^i \otimes [\tau_3, i\tau_2] \\ &= i\sigma^i \otimes \tau_1 = i\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \end{aligned} \tag{7.2.10}$$

A boost in x direction reads as $\psi \to e^{i\theta\sigma^{01}}\psi$ which shows the mix between ϕ and χ since (7.2.10) is not diagonal. Furthermore, if we notice the i before (7.2.10), we will find the boost transformation is not unitary. It is actually the case that we should remember we always use $\sinh\theta$ and $\cosh\theta$ to represent a boost while $\sin\theta$ and $\cos\theta$ are used for rotation which is unitary.

In addition, in the slow electrons we can set $\chi \sim 0$ because in rest frame we merely have one nonzero component for ϕ in (7.1.11). This is the non-relativistic approximation for Dirac equation.

With all preparation ready, let us prove the Lorentz invariance in Dirac equation. Under Lorentz transformation, $\psi(x)$ changes to $\psi'(x')$ according to

$$\psi'(x') = S(\Lambda)\psi(x) = e^{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}}\psi(x) \tag{7.2.11}$$

where ω_{0i} represents 3 boost angles and ω_{ij} represents 3 rotation angles, which summed to be 6 free parameter of Lorentz group. We should first calculate

$$\begin{aligned} [\sigma^{\mu\nu}, \gamma^\lambda] &= [\frac{i}{2}[\gamma^\mu, \gamma^\nu], \gamma^\lambda] \\ &= \frac{i}{2}\left(\gamma^\mu[\gamma^\nu, \gamma^\lambda] + [\gamma^\mu, \gamma^\lambda]\gamma^\nu - \gamma^\nu[\gamma^\mu, \gamma^\lambda] - [\gamma^\nu, \gamma^\lambda]\gamma^\mu\right) \\ &= \frac{i}{2}\left(-2\gamma^\mu\gamma^\lambda\gamma^\nu - 2\gamma^\lambda\gamma^\mu\gamma^\nu + 2\gamma^\nu\gamma^\lambda\gamma^\mu + 2\gamma^\lambda\gamma^\nu\gamma^\mu\right) \\ &= i\left(-\{\gamma^\mu, \gamma^\lambda\}\gamma^\nu + \{\gamma^\nu, \gamma^\lambda\}\gamma^\mu\right) \\ &= 2i(\gamma^\mu\eta^{\nu\lambda} - \gamma^\nu\eta^{\mu\lambda}) \end{aligned} \tag{7.2.12}$$

if $\lambda \neq \mu \neq \nu$ the expression results in zero. We then calculate $S\gamma^\lambda S^{-1}$ in infinitesimal transformation:

$$\begin{aligned} S\gamma^\lambda S^{-1} &= e^{-\frac{i}{4}\omega\sigma}\gamma^\lambda e^{\frac{i}{4}\omega\sigma} \\ &\sim \gamma^\lambda - \frac{i}{4}[\omega\sigma, \gamma^\lambda] \\ &= \gamma^\lambda + \gamma^\mu{\omega_\mu}^\lambda \\ &= {\Lambda^\lambda}_\mu\gamma^\mu \end{aligned} \tag{7.2.13}$$

Finally we can prove the Lorentz invariance of Dirac equation:

$$\begin{aligned} (i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \to (i\gamma^\mu\partial'_\mu - m)\psi'(x') &= (i\gamma^\mu{\Lambda^\nu}_\mu\partial_\nu - m)S(\Lambda)\psi(x) \\ &= S(S^{-1}i\gamma^\mu{\Lambda^\nu}_\mu S)\partial_\nu\psi(x) - Sm\psi(x) \\ &= Si\gamma^\nu\partial_\nu\psi(x) - Sm\psi(x) \\ &= S(\Lambda)(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \end{aligned} \tag{7.2.14}$$

7.3 Dirac Bilinears

In quantum mechanics, we have bilinear invariance $\psi^\dagger\psi$ for some unitary transformation. But this is not correct for QFT, because the γ^0 is Hermite while γ^i is anti-Hermite. In a compact expression, we can write this as

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \tag{7.3.1}$$

We can check this by $(\gamma^0)^\dagger = (\gamma^0)^3 = \gamma^0$ and $(\gamma^i)^\dagger = \gamma^0\gamma^i\gamma^0 = -\gamma^0\gamma^0\gamma^i = -\gamma^i$. Then

$$\begin{aligned}
(\sigma^{\mu\nu})^\dagger &= \left(\frac{i}{2}[\gamma^\mu,\gamma^\nu]\right)^\dagger \\
&= -\frac{i}{2}[\gamma^{\dagger\nu},\gamma^{\dagger\mu}] \\
&= -\frac{i}{2}\gamma^0[\gamma^\nu,\gamma^\mu]\gamma^0 \\
&= \gamma^0\sigma^{\mu\nu}\gamma^0
\end{aligned} \tag{7.3.2}$$

and

$$\begin{aligned}
S^\dagger &= \left(e^{i\omega\sigma}\right)^\dagger \\
&= \left(\sum_n \frac{i^n}{n!}(\omega\sigma)^n\right)^\dagger \\
&= \gamma^0 e^{-i\omega\sigma}\gamma^0
\end{aligned} \tag{7.3.3}$$

Thus we can check $\psi^\dagger\psi$ is not invariance:

$$\psi^\dagger\psi \to \psi^\dagger S^\dagger S\psi = \psi^\dagger\gamma^0 e^{-i\omega\sigma}\gamma^0 e^{i\omega\sigma}\psi \neq \psi^\dagger\psi \tag{7.3.4}$$

Nevertheless, we can define a new conjugate of ψ as $\bar{\psi}$ to remain $\bar{\psi}\psi$ invariant, namely

$$\bar{\psi} = \psi^\dagger\gamma^0 \tag{7.3.5}$$

From (7.3.4), we can easily get the invariant property of $\bar{\psi}\psi$:

$$\bar{\psi}\psi = \psi^\dagger\gamma^0\psi \to \psi^\dagger S^\dagger\gamma^0 S\psi = \psi^\dagger\gamma^0 e^{-i\omega\sigma}\gamma^0\gamma^0 e^{i\omega\sigma}\psi = \psi^\dagger\gamma^0\psi = \bar{\psi}\psi \tag{7.3.6}$$

Appendix: γ^μ in Higher Dimensions

Here we use the iterative construction to write down the gamma matrices γ^μ in higher dimensions. Our start point is Dirac equation,

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{A.7.1}$$

and thus we can multiply it with $(i\gamma^\mu\partial_\mu + m)$ and get

$$(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu + m^2)\psi = 0 \tag{A.7.2}$$

In order to recover Klein-Gordon equation, we must have

$$\{\gamma^\mu,\gamma^\nu\} = 2\eta^{\mu\nu} \tag{A.7.3}$$

where $\eta^{\mu\nu} = diag(1,-1,...,-1)$ where there are D number of -1. Here we constrain our D as an odd number. In $(3+1)$ dimension, we have

$$\gamma^0 = I\otimes\tau_3,\ \gamma^i = \sigma^i\otimes i\tau_2 \tag{A.7.4}$$

Observing the calculation of $\{\gamma^i,\gamma^j\}$, we may get inspired:

$$\{\gamma^i,\gamma^j\} = (\sigma^i\otimes i\tau_2)(\sigma^j\otimes i\tau_2) + (i\leftrightarrow j) = \{\sigma^i,\sigma^j\}\otimes(-I) = 2\delta^{ij}I\otimes(-I) \tag{A.7.5}$$

where the central step is $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$ and rest part resulting in $\tau_2\tau_2 = 1$. Then in higher dimension, we can simulate this construction and assume

$$\gamma^i = iA^i \otimes \tau_2 \tag{A.7.6}$$

Substituting this equation into $\{\gamma^i, \gamma^j\}$, we have

$$\begin{aligned}\{\gamma^i, \gamma^j\} &= (-1)(A^i \otimes \tau_2)(A^j \otimes \tau_2) + (i \leftrightarrow j) \\ &= (-1)\{A^i, A^j\} \otimes I\end{aligned} \tag{A.7.7}$$

Our goal is to find $\{A^i, A^j\} = 2\delta^{ij} \otimes \underbrace{I \otimes ... \otimes I}_{\frac{D-1}{2} \text{ of } I}$ for $i = 1, ..., D$.

This can be reached since we know in 3 spatial dimension, we have $\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}$, $\{\gamma^5, \gamma^\mu\} = 0$ and $(\gamma^5)^2 = 1$, with these 5 gamma matrices we can build A^is for 5 spatial dimension just multiplying some is to γ^μ or γ^5. Then we can do the same process to 5 spatial dimension and continue to get A^is for 7 spatial dimension. This is the iterative process, and thus we can get the gamma matrices for any D of odd number. But there is still one problem, how to construct the new γ^0? This is also very easy, since we can observe

$$\{\gamma^0, \gamma^j\} = (I \otimes \tau_3)(\sigma^j \otimes i\tau_2) + (\sigma^j \otimes i\tau_2)(I \otimes \tau_3) = \sigma^i \otimes i\{\tau_2, \tau_3\} = 0 \tag{A.7.9}$$

from which we find the central step is $\{\tau_2, \tau_3\} = 0$. Then we can assume $\gamma^0 = B \otimes \tau_3$. We have

$$\begin{aligned}\{\gamma^0, \gamma^j\} &= (B \otimes \tau_3)(A \otimes \tau_2) + (A \otimes \tau_2)(B \otimes \tau_3) \\ &= (BA) \otimes \tau_2\tau_3 + AB \otimes \tau_3\tau_2\end{aligned} \tag{A.7.10}$$

If we have $AB = BA$, we can get $\{\tau_2, \tau_3\} = 0$ and yield zero. The unique choice is $B = \alpha I \otimes ... \otimes I$. As $(\gamma^0)^2 = 1$ we find $\alpha = 1$, thus we have

$$\gamma^0 = \underbrace{I \otimes ... \otimes I}_{\frac{D-1}{2} \text{ of } I} \otimes \tau_3 \tag{A.7.11}$$

Thus all steps in principle are finished. Let us calculate some in iterative way. For $D = 5$, we denote $\gamma^\mu_{D=5}(\mu = 0, ..., 5)$ and γ_5^F for gamma matrices as well gamma five. Since we have assume $\gamma_5^i = iA_5^i \otimes \tau_2$, we can set

$$A_5^1 = \gamma_3^0,\ A_5^2 = -i\gamma_3^1,\ A_5^3 = -i\gamma_3^2,\ A_5^4 = -i\gamma_3^3,\ A_5^5 = \gamma_3^F \tag{A.7.12}$$

where γ_3^F is gamma five for $D = 3$. Then we can easily check $\{\gamma_5^i, \gamma_5^j\} = -2\delta^{ij}$. Writing γ_5^μ in Dirac product way, we have

$$\left\{\begin{array}{c}\gamma_5^0 = I \otimes I \otimes \tau_3 \\ \gamma_5^1 = iI \otimes \tau_3 \otimes \tau_2 \\ \gamma_5^{2,3,4} = \sigma^i \otimes \tau_2 \otimes \tau_2\ (i = 1, 2, 3) \\ \gamma_5^5 = iI \otimes \tau_1 \otimes \tau_2\end{array}\right. \tag{A.7.13}$$

Then we should calculate $\gamma_5^F = i\gamma_5^0 ... \gamma_5^5$.

$$\begin{aligned}\gamma_5^F &= i\gamma_5^0 ... \gamma_5^5 \\ &= i(I \otimes I \otimes \tau_3)(i\gamma_3^0 \otimes \tau_2)(\gamma_3^1 \otimes \tau_2)(\gamma_3^2 \otimes \tau_2)(\gamma_3^3 \otimes \tau_2)(i\gamma_3^F \otimes \tau_2) \\ &= -(I \otimes I \otimes \tau_3)\left((i\gamma_3^0\gamma_3^1\gamma_3^2\gamma_3^3\gamma_3^F) \otimes \tau_2\right) \\ &= -(I \otimes I \otimes \tau_3)\left((\gamma_3^F\gamma_3^F) \otimes \tau_2\right) \\ &= -(I \otimes I) \otimes (\tau_3\tau_2) \\ &= i(I \otimes I) \otimes \tau_1\end{aligned} \tag{A.7.14}$$

where $(\gamma_5^F)^2 = -1$, which is different from that in $D = 3$.

Then for $D = 7$, with the same method, we have $\gamma_7^i = iA_7^i \otimes \tau_2$, we can set

$$A_7^1 = \gamma_5^0,\ A_7^2 = -i\gamma_5^1,\ A_7^3 = -i\gamma_5^2,\ A_7^4 = -i\gamma_5^3,\ A_7^5 = -i\gamma_5^4,\ A_7^6 = -i\gamma_5^5,\ A_7^7 = -i\gamma_5^F, \tag{A.7.15}$$

where γ_5^F is gamma five for $D=5$. Then we can easily check $\{\gamma_7^i,\gamma_7^j\}=-2\delta^{ij}$. Writing γ_7^μ in Dirac product way, we have

$$\begin{cases} \gamma_7^0 = I\otimes I\otimes I\otimes\tau_3 \\ \gamma_7^1 = iI\otimes I\otimes\tau_3\otimes\tau_2 \\ \gamma_7^2 = iI\otimes\tau_3\otimes\tau_2\otimes\tau_2 \\ \gamma_7^{3,4,5} = \sigma^i\otimes\tau_2\otimes\tau_2\otimes\tau_2\ (i=1,2,3) \\ \gamma_7^6 = iI\otimes\tau_1\otimes\tau_2\otimes\tau_2 \\ \gamma_7^7 = iI\otimes I\otimes\tau_1\otimes\tau_2 \end{cases} \tag{A.7.16}$$

Then we should calculate $\gamma_7^F = i\gamma_7^0...\gamma_7^7$.

$$\begin{aligned} \gamma_7^F &= i\gamma_7^0...\gamma_7^5 \\ &= i(I\otimes I\otimes I\otimes\tau_3)(i\gamma_5^0\otimes\tau_2)(\gamma_5^1\otimes\tau_2)(\gamma_5^2\otimes\tau_2)(\gamma_5^3\otimes\tau_2)(\gamma_5^4\otimes\tau_2)(\gamma_5^5\otimes\tau_2)(\gamma_5^F\otimes\tau_2) \\ &= i(I\otimes I\otimes I\otimes\tau_3)\left((i\gamma_5^0\gamma_5^1\gamma_5^2\gamma_5^3\gamma_5^4\gamma_5^5\gamma_5^F)\otimes\tau_2\right) \\ &= i(I\otimes I\otimes I\otimes\tau_3)\left((\gamma_3^F\gamma_3^F)\otimes\tau_2\right) \\ &= -i(I\otimes I\otimes I)\otimes(\tau_3\tau_2) \\ &= -(I\otimes I\otimes I)\otimes\tau_1 \end{aligned} \tag{A.7.17}$$

where $(\gamma_5^F)^2=1$, which is different from that in $D=5$ but the same with $D=3$. Thus we have encounter a circulation for $(\gamma_5^F)^2=1$. Then to iterate for $D=9$, we can just follow the way we deducing for $D=5$, and set

$$A_9^1=\gamma_7^0,\ A_9^i=-i\gamma_7^{i-1}(i=2,..,8),\ A_9^9=\gamma_7^F \tag{A.7.18}$$

This results

$$\begin{cases} \gamma_9^0 = \underbrace{I\otimes..\otimes I}_{4}\otimes\tau_3,\ \gamma_9^j = i\underbrace{I\otimes..\otimes I}_{4-j}\otimes\tau_3\otimes\underbrace{\tau_2\otimes...\otimes\tau_2}_{j}(j=1,2,3) \\ \gamma_9^k = \sigma^{k-3}\otimes\underbrace{\tau_2\otimes...\otimes\tau_2}_{4}(k=4,5,6),\ \gamma_9^l = i\underbrace{I\otimes..\otimes I}_{l-6}\otimes\tau_1\otimes\underbrace{\tau_2\otimes...\otimes\tau_2}_{10-l}(l=7,8,9) \end{cases} \tag{A.7.19}$$

And

$$\gamma_9^F = i\underbrace{I\otimes..\otimes I}_{4}\otimes\tau_1 \tag{A.7.20}$$

Indeed, we can conclude the gamma matrices for arbitrary high odd spatial dimension D as

$$\begin{cases} \gamma_D^0 = \underbrace{I\otimes..\otimes I}_{(D-1)/2}\otimes\tau_3,\ \gamma_D^j = i\underbrace{I\otimes..\otimes I}_{(D-1)/2-j}\otimes\tau_3\otimes\underbrace{\tau_2\otimes...\otimes\tau_2}_{j}(j=1,...,\frac{D-3}{2}) \\ \gamma_D^k = \sigma^{k-(D-3)/2}\otimes\underbrace{\tau_2\otimes...\otimes\tau_2}_{(D-1)/2}(k=\frac{D-1}{2},\frac{D+1}{2},\frac{D+3}{2}),\ \gamma_D^l = i\underbrace{I\otimes..\otimes I}_{l-(D+3)/2}\otimes\tau_1\otimes\underbrace{\tau_2\otimes...\otimes\tau_2}_{D+1-l}(l=\frac{D+5}{2},...,D) \\ \gamma_D^F = \delta_D\underbrace{I\otimes..\otimes I}_{(D-1)/2}\otimes\tau_1,\ \delta_D = \begin{cases} -1 & \frac{D-1}{2}\ \text{odd} \\ i & \frac{D-1}{2}\ \text{even} \end{cases} \end{cases} \tag{A.7.21}$$

Lecture Notes of Quantum Field Theory

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Lecture 8

8.1 Parity

Today I will talk about some important contents which can be found in any quantum field theory textbooks. So I am not going to go to many details but instead just introduce some key concepts for. Please read my book for relative chapters.

At first, we will talk about parity which is very important especially in China(for many ancient artworks are designed to be left-right symmetric or asymmetric). The parity transformation in physics is

$$x^\mu \to x'^\mu = (x^0, -\vec{x}) \tag{8.1.1}$$

We have Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$, and multiplying γ^0 to the left of it, we will then get

$$(i\gamma^\mu\partial'_\mu - m)\gamma^0\psi = 0 \tag{8.1.2}$$

We see the new field $\gamma^0\psi$ is the one with opposite parity. With this, we can define a parity transformation for ψ:

$$\psi(x) \to \psi'(x') = \eta\gamma^0\psi(x) \tag{8.1.3}$$

where η is an arbitrary phase.

$\bar{\psi}\psi$, under parity transformation, is invariant because

$$\bar{\psi}\psi \to \bar{\psi}\gamma^0\gamma^0\psi = \bar{\psi}\psi \tag{8.1.4}$$

while $\bar{\psi}\gamma^5\psi$ results in an extra minus sign:

$$\bar{\psi}\gamma^5\psi \to \bar{\psi}\gamma^0\gamma^5\gamma^0\psi = -\bar{\psi}\gamma^5\psi \tag{8.1.5}$$

Furthermore, since we have worked out all 16 number of 4 by 4 Dirac matrices, we can write down 16 number of currents with these matrices shown as follows:

	$\bar{\psi}\psi$	$\bar{\psi}\gamma_\mu\psi$	$\bar{\psi}\sigma_{\mu\nu}\psi$	$\bar{\psi}\gamma_\mu\gamma^5\psi$	$\bar{\psi}\gamma^5\psi$
name	scalar	vector	tensor	pseudo-vector/axial vector	pseudo-scalar
abbr.	S	V	T	A	P

Table 8.1.1: All currents with Dirac matrices

These currents transform in ordinary way under Lorentz transformation, such as $\bar{\psi}\sigma_{\mu\nu}\psi \to \Lambda^\tau_\mu\Lambda^\rho_\nu\bar{\psi}\sigma_{\tau\rho}\psi$, since we know $S^{-1}\gamma_\mu S \to \Lambda^\nu_\mu\gamma_\nu$ in last lecture. But in parity transformation, they behave in different way, such as

$$\begin{cases} \bar{\psi}\sigma_{ij}\psi \to \bar{\psi}\gamma^0\sigma_{ij}\gamma^0\psi = \bar{\psi}\sigma_{ij}\psi \\ \bar{\psi}\sigma_{0i}\psi \to \bar{\psi}\gamma^0\sigma_{0j}\gamma^0\psi = -\bar{\psi}\sigma_{ij}\psi \\ \bar{\psi}\sigma_{00}\psi \to \bar{\psi}\gamma^0\sigma_{00}\gamma^0\psi = \bar{\psi}\sigma_{00}\psi \end{cases} \tag{8.1.6}$$

where all components are invariant except for ones with one spatial index resulting an extra minus sign. And for $\bar{\psi}\gamma_\mu\gamma^5\psi$, it transforms as

$$\begin{cases} \bar{\psi}\gamma_0\gamma^5\psi \to -\bar{\psi}\gamma_0\gamma^5\psi \\ \bar{\psi}\gamma_i\gamma^5\psi \to \bar{\psi}\gamma_0\gamma^5\psi \end{cases} \tag{8.1.7}$$

which endows the name of pseudo-vector to $\bar{\psi}\gamma_0\gamma^5\psi$ which transforms just like the well-know pseudo-vector, angular momentum.

8.2 The Dirac Lagrangian

Here we can construct Dirac lagrangian from his equation using Euler-Lagrangian equation and we get

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{8.2.1}$$

From this lagrangian, we can also get the equation of $\bar{\psi}$ using the Euler-Lagrangian equation for ψ:

$$\frac{\delta \mathcal{L}}{\delta \psi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \psi} = 0 \tag{8.2.2}$$

in which we should notice to applying part integral to change the lagrangian as $\mathcal{L} = \bar{\psi}(-i\gamma^\mu \overleftarrow{\partial_\mu} - m)\psi$ first, and then we get

$$i\partial_\mu(\bar{\psi}\gamma^\mu) + m\bar{\psi} = 0 \tag{8.2.3}$$

I will come up with two comments. One is for the confusion about lagrangian. Some students may think since we have Dirac equation, the lagrangian (8.2.1) must be zero. This thought confound the relation between the lagrangian and Dirac equation: we indeed use variation principle of action to get Dirac equation from lagrangian, namely the fields in lagrangian are just numbers rather than the solution of Dirac equation. The other one is some one might feel the difference in treating ψ and $\bar{\psi}$. We can adjust this by defining $\partial_\mu = \frac{1}{2}(\overrightarrow{\partial_\mu} + \overleftarrow{\partial_\mu})$.

8.3 Slow and Fast Electrons

If we have a γ^μ, somebody else may use another $\tilde{\gamma}^\mu$ which differ from γ^μ just for a similarity transformation, $\tilde{\gamma}^\mu = W^{-1}\gamma^\mu W$. We will find this new gamma matrix also obey the definition of γ^μ, that is,

$$\{\tilde{\gamma}^\mu, \tilde{\gamma}^\nu\} = 2\eta^{\mu\nu} \tag{8.3.1}$$

However no matter what basis we use, the physics is independent on our basis choice as long as (8.3.1) is satisfied. So later we will use Weyl basis to get a new expression for γ^μ. That is

$$\gamma^0 = \begin{pmatrix} & I \\ I & \end{pmatrix} = I \otimes \tau_1,\ \gamma^i \text{ unchanged},\ \gamma^5 = \begin{pmatrix} -I & \\ & I \end{pmatrix} = -(I \otimes \tau_3) \tag{8.3.2}$$

For slow electrons, we use Dirac basis and know

$$(\gamma^\mu p_\mu - m)\psi(p) = 0 \tag{8.3.3}$$

which means in low energy condition, since $\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$, $\chi \to 0$ as we discussed in last lecture. For fast ones, we can safely set $m = 0$ and get

$$\gamma^\mu p_\mu \psi(p) = 0 \tag{8.3.4}$$

Multiplying on the left by γ^5, we find $\gamma^5\psi(p)$ also satisfies this equation: $\gamma^\mu p_\mu \gamma^5 \psi(p) = 0$. Since $(\gamma^5)^2 = 1$, we can construct two projection operators: $P_L = \frac{1-\gamma^5}{2}$ and $P_R = \frac{1+\gamma^5}{2}$ which satisfy $P_L^2 = P_L$, $P_L P_R = 0$ and $P_R^2 = P_R$, namely two orthogonal projection operators. Using these operators, we can decompose ψ into left part and right part, both of which are solutions for massless Dirac equation. In Weyl basis, we can define these two parts very convenient because

$$P_L = \begin{pmatrix} I & \\ & 0 \end{pmatrix},\ P_R = \begin{pmatrix} 0 & \\ & I \end{pmatrix} \tag{8.3.5}$$

and thus $\psi_L = P_L\psi$ only has the first two components while $\psi_R = P_R\psi$ only has the last two components. Note that $\gamma^5\psi_L = -\psi_L$ and $\gamma^5\psi_R = \psi_R$. We can correspond left and right parts to spin clockwise and anticlockwise around the direction of motion. We can see this property easily by assuming the electron is moving along the z axis and applying a rotation along z axis. Thus the field will transform as

$$\psi_{L,R} \to e^{-\frac{i}{4}\omega\sigma^{12}}\psi_{L,R} \tag{8.3.6}$$

Then we can write down exactly that

$$\psi_L = \begin{pmatrix} \phi \\ 0 \end{pmatrix},\ \psi_R = \begin{pmatrix} 0 \\ \chi \end{pmatrix},\ \sigma^{12} = \begin{pmatrix} \sigma^3 & \\ & \sigma^3 \end{pmatrix} \tag{8.3.7}$$

Thus we know

$$\begin{cases} \psi_L \to \begin{pmatrix} e^{-\frac{i}{4}\omega\sigma^3}\phi \\ 0 \end{pmatrix} \\ \psi_R \to \begin{pmatrix} 0 \\ e^{-\frac{i}{4}\omega\sigma^3}\chi \end{pmatrix} \end{cases} \tag{8.3.8}$$

In Dirac equation for massless left-hand electron, we know

$$0 = \gamma^\mu p_\mu \psi_L = \begin{pmatrix} & \omega(1-\sigma^3) \\ \omega(1+\sigma^3) & \end{pmatrix} \begin{pmatrix} \phi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega \begin{pmatrix} 2 & \\ & 0 \end{pmatrix} \phi \end{pmatrix} \to \phi = \begin{pmatrix} 0 \\ a \end{pmatrix} \tag{8.3.9}$$

In the same way, we get $\chi = \begin{pmatrix} b \\ 0 \end{pmatrix}$. Substituting these back to (8.3.8), we at once get

$$\begin{cases} \psi_L \to e^{+\frac{i}{4}\omega} \begin{pmatrix} \phi \\ 0 \end{pmatrix} = e^{+\frac{i}{4}\omega}\psi_L \\ \psi_R \to e^{-\frac{i}{4}\omega} \begin{pmatrix} 0 \\ \chi \end{pmatrix} = e^{-\frac{i}{4}\omega}\psi_R \end{cases} \tag{8.3.10}$$

which shows that $\psi_{L,R}$ really represent spinning clockwise or anticlockwise around the direction of motion. We call the projection of spin along the direction of motion of massless electrons as helicity which is like the double helex in DNA structure. Helicity can be defined in formula as $h = \vec{P}\cdot\vec{S}/|\vec{P}|$. Here for ψ_L, $h = -\frac{1}{2}$ as well for ψ_R, $h = \frac{1}{2}$.

We see in Weyl basis, our discussion for fast electrons is more convenient because it can be decomposed as left-hand and right-hand ones which are just the first two components and last two components of Dirac field respectively. This attributes to the diagonal property of γ^5. Of course, for slow electrons, we had better use Dirac basis for only one nonzero component left in Dirac field ψ. However, no matter which one we choose, the physics is unchanged.

8.4 Handedness

Since we can decompose the Dirac field into two parts

$$\psi = P_L\psi + P_R\psi \tag{8.4.1}$$

Then for lagrangian, we have

$$\begin{aligned} \mathcal{L} &= \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \\ &= \bar{\psi}_L i\gamma^\mu\partial_\mu\psi_L + \bar{\psi}_R i\gamma^\mu\partial_\mu\psi_R - m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L) \end{aligned} \tag{8.4.2}$$

where we find the left-hand part couples to left-hand part in kinetic part(the same for right-hand part), but in mass term, they couple to each other. Thus we know mass is a thing which connects left and right. If the particle is massless, the left-hand and right-hand parts decouples, thus we can precisely say whether a particle is left-hand or right-hand. This is consistent with the concept that we cannot go to a rest frame of a massless particle. Because, in a rest frame of a particle, we cannot say the helicity as for $\vec{p} = 0$, namely there is no meaning to define left and right.

For $m \neq 0$, we have a $U(1)$ symmetry for ψ. It transforms as

$$\psi \to e^{i\theta}\psi \tag{8.4.3}$$

For left and right-hand respectively, we have

$$\psi_{L,R} \to e^{i\theta}\psi_{L,R} \tag{8.4.4}$$

Recalling Noether theorem, the conserved current is

$$J^\mu = \bar{\psi}\gamma^\mu\psi \tag{8.4.5}$$

But we have an extra symmetry for $m = 0$. That is under the transformation of $\psi_L \to e^{-i\phi}\psi_L$ and $\psi_R \to e^{i\phi}\psi_R$, the lagrangian is invariant. We can write this transformation is a neat way: $\psi \to e^{i\phi\gamma^5}\psi$. Then we can get an extra conserved current:

$$J^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi \tag{8.4.5}$$

which is an axial vector.

That Lee and Yang in 1956 found that parity is violated in weak interaction, was eventually realized. In modern language, this means in weak interaction terms, only left-hand field appears. The reason for this is not understood so far as well we can only write down such a parity violation term. For example, in standard model, we well know there is only left neutrino. Sudarshan and Marshak, two unfamous physicists, invented V-A theory to describe the parity violation. They idea is subtracting J^μ by $J^{5\mu}$:

$$J^\mu_V - J^\mu_A = \bar{\psi}\gamma^\mu(1-\gamma^5)\psi \propto \bar{\psi}_L\gamma^\mu\psi_L \tag{8.4.6}$$

where there are only left hand interaction terms. Thus we understand parity violation term is really $\bar{\psi}\gamma^\mu\gamma^5\psi$, the axial vector.

8.5 Interactions

Since we have various current in Table 8.1.1, we can construct couplings with other corresponding fields. For example, we have $\bar{\psi}\gamma^\mu\psi A_\mu$ for quantum electrodynamics and $\bar{\psi}\psi\phi$ for scalar quantum electrodynamics.

However, I should point out that only when we have both vector and axial vector terms, the parity of interaction can be violated. This means the interaction term should be in the form of

$$+g\bar{\psi}\gamma^\mu\psi\phi + g'\bar{\psi}\gamma^\mu\gamma^5\psi\phi \tag{8.5.1}$$

If we only have one, such as $g = 0$, parity is also OK, because we can set ϕ transforms as $\phi \to -\phi$ under parity transformation to remain $g'\bar{\psi}\gamma^\mu\gamma^5\psi\phi$ an invariance. So as shown in V-A theory, when vector current and axial vector current both appears in lagrangian, we will have the parity violation.

With interactions between electromagnetic field and Dirac field, we reach the lagrangian of quantum electrodynamics:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\mu^2 A_\mu A^\mu + eA_\mu\bar{\psi}\gamma^\mu\psi \tag{8.5.2}$$

where we should set $\mu = 0$ for massless photon. We then can rewrite it as

$$\mathcal{L} = \bar{\psi}i\gamma^\mu(\partial_\mu - ieA_\mu)\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{8.5.2}$$

and we can define $D_\mu \equiv \partial_\mu - ieA_\mu$. This definition is very important, and it also appears in GR and Yang-Mills theory!

8.6 Concept of Charge

One may ask a good question: how to define the sign of charge?

In Dirac equation

$$[i\gamma^\mu(\partial_\mu - ieA_\mu) - m]\psi = 0 \tag{8.6.1}$$

can we flip the sign of e? This was down by Dirac and with this work, he found antimatter! We can write the conjugate one of (8.6.1):

$$[-i\gamma^{*\mu}(\partial_\mu + ieA_\mu) - m]\psi^* = 0 \tag{8.6.2}$$

Then we find if $\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}$, we have

$$\{-\gamma^{*\mu}, -\gamma^{*\nu}\} = \eta^{\mu\nu} \tag{8.6.3}$$

which means new $-\gamma^{*\mu}$ also satisfies Clifford algebra and whose difference from γ^μ is just a similarity transformation. We then can find the transformation as

$$-\gamma^{*\mu} = (C\gamma^0)^{-1}\gamma^\mu(C\gamma^0) \tag{8.6.4}$$

where we can always find such a matrix C. If we define $\psi_c \equiv C\gamma^0\psi^*$, the Dirac equation becomes

$$[i\gamma^\mu(\partial_\mu + ieA_\mu) - m]\psi_c = 0 \tag{8.6.2}$$

Notice in this equation we have found another field which has the same mass with ψ but opposite charge! This is antimatter! For electron, ψ_c was called positron.

All above comes from the conjugate transformation in math. Mathematicians are great sometimes. If they had not invented $\sqrt{-1}$ and then to develop conjugate transformation, one could have classical mechanics, could have SR and GR, but could not have quantum mechanics!

As an exercise, one could find for a left-hand Dirac field ψ, ψ_c is just right-hand one and vice versa. Here we will use the exact expression: $C = \gamma^2\gamma^0$.

For more details, you should read my book.

8.7 Majorana Neutrino

In Dirac equation, we have

$$i\gamma^\mu \partial_\mu \psi = m\psi \tag{8.7.1}$$

But an Italian genius Majorana promoted it as

$$i\gamma^\mu \partial_\mu \psi = m\psi_c \tag{8.7.2}$$

Thus ψ cannot have a electric charge. This is called Majorana fermion as which neutrino is prevalently regarded. Not only appearing in high energy physics, Majorana fermions can be applied even in condensed matter physics, such as topological insulation). From (8.7.2), we can show

$$i\gamma^\mu \partial_\mu \psi_c = i\gamma^\mu C\gamma^0 \partial_\mu \psi^* = iC\gamma^0 (C\gamma^0)^{-1} \gamma^\mu C\gamma^0 \partial_\mu \psi^* = -iC\gamma^0 \gamma^{*\mu} \partial_\mu \psi^* = mC\gamma^0 \psi_c^* = m\psi \tag{8.7.3}$$

Then we apply $i\gamma^\mu \partial_\mu$ to the left of (8.7.2), we have

$$i\gamma^\mu \partial_\mu \left(i\gamma^\mu \partial_\mu \psi \right) = i\gamma^\mu \partial_\mu m\psi \to -\partial^2 \psi = m^2 \psi \tag{8.7.4}$$

which is exact Klein-Gordon equation. From this we know the mass m in Majorana equation is indeed a mass. Then from (8.7.2) we can construct the lagrangian of Majorana fermion including mass term:

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - \frac{1}{2} m(\psi^{\mathrm{T}} C\psi + \bar{\psi}^{\mathrm{T}} C\bar{\psi}) \tag{8.7.5}$$

Here one may find $\psi^{\mathrm{T}} C\psi = C_{\alpha\beta}\psi_\alpha \psi_\beta$. Since $C_{\alpha\beta} = -C_{\alpha\beta}$, one may say this mass term is zero. Fortunately, the field ψ is not a complex number but instead a Grassman number which satisfies the property: $\psi_\alpha \psi_\beta = -\psi_\beta \psi_\alpha$. If $\psi = \psi_c$, we call it a Majorana spinor. However, we should notice there is no $U(1)$ symmetry in (8.7.5).

8.8 Time Reversal

Here I have no enough time to make a precise comment on time reversal. So I want all of you to read my book. In Newton's law, time reversal invariance remains, because

$$m\frac{d^2}{dt^2} x = F \tag{8.8.1}$$

is invariant under the transformation $t \to -t$. One important aspect should be emphasized that time reversal transformation is not Hermite but instead anti-Hermite. We can understand this in Schrodinger equation:

$$i\frac{\partial \psi}{\partial t} = H\psi \tag{8.8.2}$$

where when we transform t to $-t$, we must change i to $-i$ to keep the equation invariant.

Finally, I will tell you a conclusion about CPT. Recalling all transformation above, we have learnt charge conjugation, parity transformation and time reversal transformation. Also, we have constructed some lagrangian where charge, parity or time reversal is broken. But the key is no matter how we break these properties, CPT multiplied together cannot be broken as long as Lorentz symmetry is OK.

Lecture 9: Group theory derivative of Dirac equation

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1 The representation of rotation group

$SO(3)$ can be generated by a exponential mapping from the representation of $so(3)$ by $e^{i\vec{\theta}\cdot\vec{J}}$, where $\vec{J}$ is the generator of $so(3)$ algebra which satisfies the following condition:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{1}$$

In the 3-dimension space, $\vec{J}$ can be written as:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{2}$$

while the other J_i by cyclic permutation. Note that as each J_i is hermitian, they generate a unitary representation of $SO(3)$.

2 Representation of the algebra

We characterize the states by quantum number angular momentum j which is an integer. $j = 0, 1, 2, ...$ For each j, another quantum number $m = -j, -j+1, ..., j-1, j$ which is the angular momentum along z direction. There are $2j+1$ states for each j.

3 Relativistic physics

To obtain the representation of Lorentz group $SO(3,1)$, we consider the generators in 4-dimension space, and add one row and one column of zeros to the matrices of J_i as the new rotation generators. The reason for this is that the rotation generator does not operate on the time dimension. For Lorentz boost, we start from from one such transformation along x direction:

$$t' = cosh\phi t + sinh\phi x \tag{3}$$
$$x' = cosh\phi x + sinh\phi t \tag{4}$$

which for an infinitesimal boost reads:

$$t' = t + \phi x \tag{5}$$
$$x' = x + \phi t \tag{6}$$

If we write it in the way corresponding to the general form $e^{i\phi K_1}$, it reads:

$$e^{i\phi K_1} \doteq I + \begin{pmatrix} 0 & \phi & 0 & 0 \\ \phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

From this equation we find the explicit form of K_1:

$$iK_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

while the others by cyclic permutation. By brutal calculation, we can obtain the algebra of $so(3,1)$:

$$[J_i, K_j] = i\epsilon_{ijk}K_k \tag{9}$$

This relation means that under rotation $\vec{K}$ transforms like a vector. As the algebra must close, we should also consider the commutator $[\vec{K},\vec{K}]$:

$$[K_i, K_j] = -i\epsilon_{ijk}J_k \tag{10}$$

This sign originates from the space-time sign of $SO(3,1)$. The Lie algebra of $SO(4)$ does not have this sign. $i\vec{K}$ is real and symmetric, hence hermitian and $\vec{J}$ is hermitian. Then one thing we need to note is that in this representation, rotation $e^{i\vec{\phi}\cdot\vec{J}}$ is unitary but boost $e^{i\vec{\phi}\cdot\vec{K}}$ is not.

One important observation: Let us define $J_{\pm i} = (J_i \pm iK_i)/2$, $(i = 1,2,3)$, check:

$$[J_{+i}, J_{-i}] = 0 \tag{11}$$

Formally, it means that $so(4)$ splits into 2 separate pieces. Further more, check:

$$[J_{+i}, J_{+j}] = i\epsilon_{ijk}J_{+k} \tag{12}$$

$$[J_{-i}, J_{-j}] = i\epsilon_{ijk}J_{-k} \tag{13}$$

This means that actually $so(4)$ splits in the way of $su(2) \otimes su(2)$. It seems that nature has been kind to us theorist for in higher dimension space this reduction is invalid. And we have a rich understanding of $su(2)$ already.

4 Representation of Lorentz algebra and Dirac equation

From the above analysis, the representation of Lorentz algebra is characterized by 2 numbers (j_+, j_-). A few simplest representations are listed here:

j_+, j_-	dim	
$(0,0)$	1	trivial
$(1/2,0)$	2	
$(0,1/2)$	2	
$(1,0)$	3	
$(0,1)$	3	
$(1/2,1/2)$	4	vector

The representation dimension can be obtained from formula $dim = (2j_+ + 1)(2j_- + 1)$ for each (j_+, j_-) By definition, the Lorentz algebra naturally has a 4-dim representation, one can prove that it is just the representation of $(1/2,1/2)$. Dirac spinor is also a 4-dim representation, as there are no more 4-dim representations, where is Dirac representation in this table? Let us look at the representation of $(1/2,0)$ in the space of ψ_α, $\alpha = 1,2$. Then $J_{+i} = \frac{1}{2}(J_i + iK_i)]$ acting on ψ_α is represented by $\frac{1}{2}\sigma_i$ while $J_{-i} = \frac{1}{2}(J_i - iK_i)$ acting on ψ_α is represented by 0.(Not responding to an infinitesimal transformation)

Solve for representation of J_i and iK_i,

$$J_i = \frac{1}{2}\sigma_i \tag{14}$$

$$iK_i = \frac{1}{2}\sigma_i \tag{15}$$

ψ_α transforms under rotation and boost:

$$e^{i\vec{\theta}\cdot\vec{J}} = e^{i\vec{\theta}\cdot\vec{\sigma}/2} \tag{16}$$

$$e^{i\vec{\phi}\cdot\vec{k}} = e^{\vec{\phi}\cdot\vec{\sigma}/2} \tag{17}$$

This two component spinor is just the Weyl spinor which has been studied in the previous course. And if we consider the $(0,1/2)$ representation, we write the basis as $\bar{\chi}^{\dot{\alpha}}$, $\dot{\alpha} = 1,2$ by Van der Waerden notation. Then we can find the representation as :

$$J_i = \frac{1}{2}\sigma_i \tag{18}$$

$$iK_i = -\frac{1}{2}\sigma_i \tag{19}$$

And then why this kind of representation was not used in physics in 1930s? Later we will see that it violates parity conservation. By a space inversion: $\vec{x} \to -\vec{x}$, and $\vec{p} \to -\vec{p}$. The generators transform in the way that : $\vec{J} \to \vec{J}$ and $\vec{K} \to -\vec{K}$. Hence $J_{+i} \leftrightarrow J_{-i}$. Under parity, the 2 pieces J_{+i} and J_{-i}

exchange: $rep.(1/2,0) \leftrightarrow (0,1/2)$ If we want to have parity, we are forced to use the representation of $(1/2,0)\oplus(0,1/2)$ which stick the two Weyl representation together. By the way, it is a historical irony that in 1956, Lee and Yang discovered parity violation.

To summarize, the representation $(1/2,0)\oplus(0,1/2)$ can be written as:

$$\vec{J} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{20}$$

$$i\vec{K} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix} \tag{21}$$

The sign difference in $i\vec{K}$ corresponds to the secret of space and time. Now with this deeper understanding, Dirac's equation should pop out from group theory instead of inspired guessing. As we know, in the rest frame the electron has two degrees of freedom which means that we have to project 4 to 2 components. Parity forces us to have 4 components but only 2 are physical. Generally, the projection operator can be written as $P = \frac{1}{2}(1-\gamma_0)$ where γ_0 is some unknown matrix with the constrain $\gamma_0^2 = I$ which corresponds to the property of projection operator $P^2 = P$. This equation can be written as:

$$(\gamma^0 - 1)\psi_r = 0 \tag{22}$$

where

$$\psi_r = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{23}$$

And by convention,

$$\gamma^0 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \tag{24}$$

Under Lorentz boost to the moving frame: $\psi(p) = e^{i\vec{\phi}\cdot\vec{K}}\psi_r$. Thus $(e^{-i\vec{\phi}\cdot\vec{K}}\gamma^0 e^{i\vec{\phi}\cdot\vec{K}} - I)\psi(p) = 0$. By brutal calculation, we can see that this is just the Dirac equation.

5 Homomorphism from $SL(2,\mathcal{C})$ to Lorentz group

Check the representation of $(1/2,0)$, the group elements $e^{i\vec{\theta}\cdot\vec{J}} = e^{i\vec{\theta}\cdot\vec{\sigma}/2}$ is special(determinant of which equals 1) and unitary and $e^{i\vec{\phi}\cdot\vec{K}} = e^{\vec{\phi}\cdot\vec{\sigma}/2}$ is special but not unitary. Every group element is special which belongs to $SL(2,\mathcal{C})$. The number of generators of $SL(2,\mathcal{C})$ and $SO(3,1)$ are the same. As we have known, $SO(3,1)$ has 6 generators. $SL(2,\mathcal{C})$ have 8 matrix elements and 2 constraints (real part and imagine part of det=1) and hence 6 generators too. The further relation can be seen explicitly by mathematic trick, define matrix X as:

$$X = x^0\sigma_0 - \vec{x}\cdot\vec{\sigma} = \begin{pmatrix} x^0 - x^3 & -(x^1 - ix^2) \\ -(x^1 + ix^2) & x^0 + x^3 \end{pmatrix} \tag{25}$$

This definition satisfies the condition $detX = (x^0)^2 - (\vec{x})^2$. Let L be an element of $SL(2,\mathcal{C})$ ($detL = 1$). Consider $X' = L^{-1}XL$, we have $detX' = detX = 1$. Thus if we also write X' in the same form as $X' = (x')^0\sigma_0 - \vec{x'}\cdot\vec{\sigma}$. Then $X \overset{L}{\Rightarrow} X'$ describes a Lorentz transformation. Hence we prove the homomorphism from $SL(2,\mathcal{C})$ to Lorentz group. But their topology property are not the same, as L and $-L$ corresponds to the same Lorentz transformation, $SL(2,\mathcal{C})$ double covers $SO(3,1)$ just like $SU(2)$ double covers $SO(3)$. Strictly speaking, spinors are representation of $SL(2,(C))$ rather than $SO(3,1)$. (Read p532 of 2^{nd} ed).